



Università degli Studi di Cagliari

Facoltà di Ingegneria e Architettura

Dipartimento di Ingegneria Civile, Ambientale e Architettura

**Magistrale Ingegneria Civile 2 anno a.a. 2021/22**

# Instabilità delle strutture e calcolo a rottura

> **Lezione 15**

lo snap-through dell'arco a tre cerniere  
(von Mises truss)

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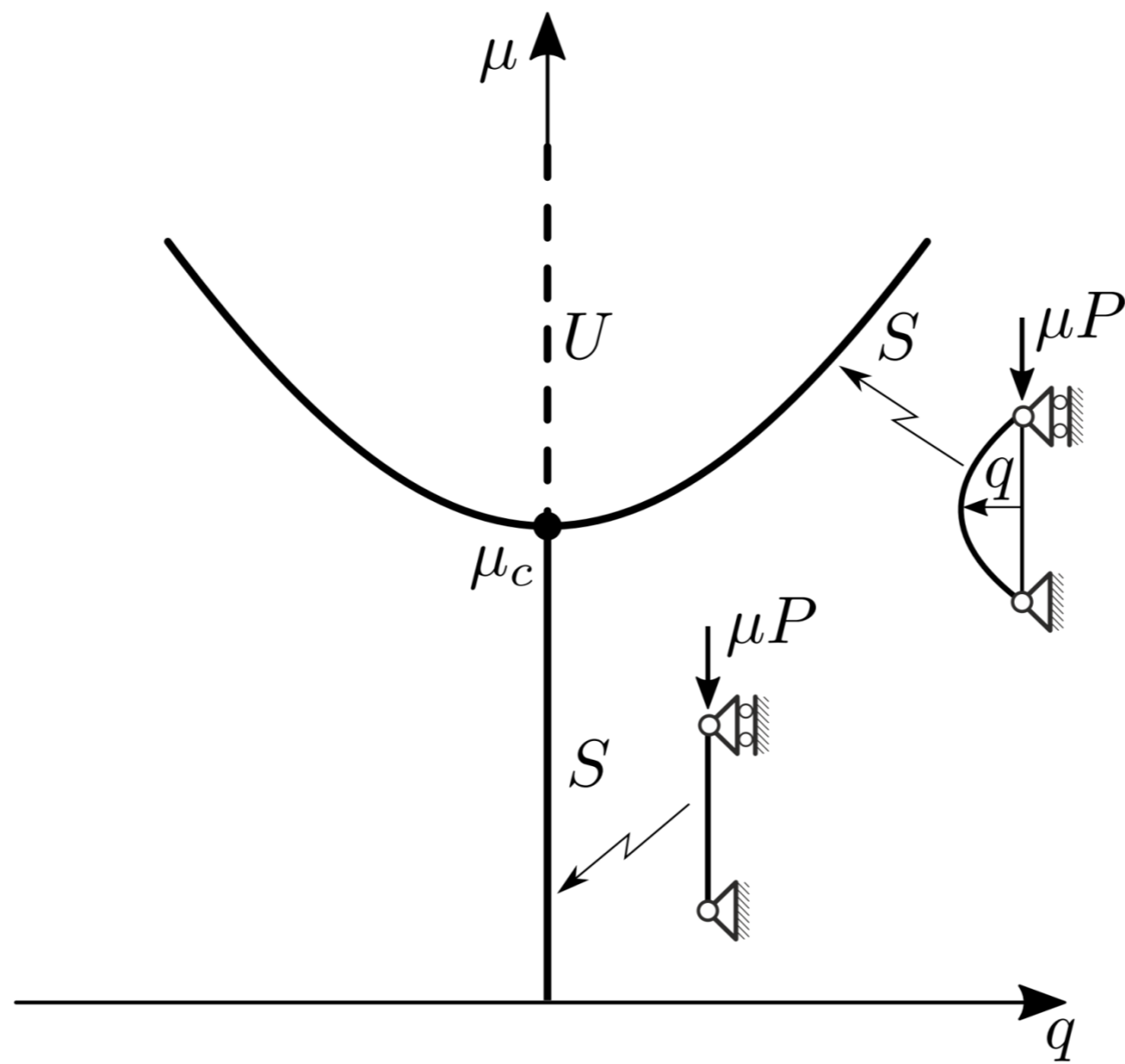


Diagramma di biforcazione della trave di Eulero;  $S$  stabile,  $U$  instabile.

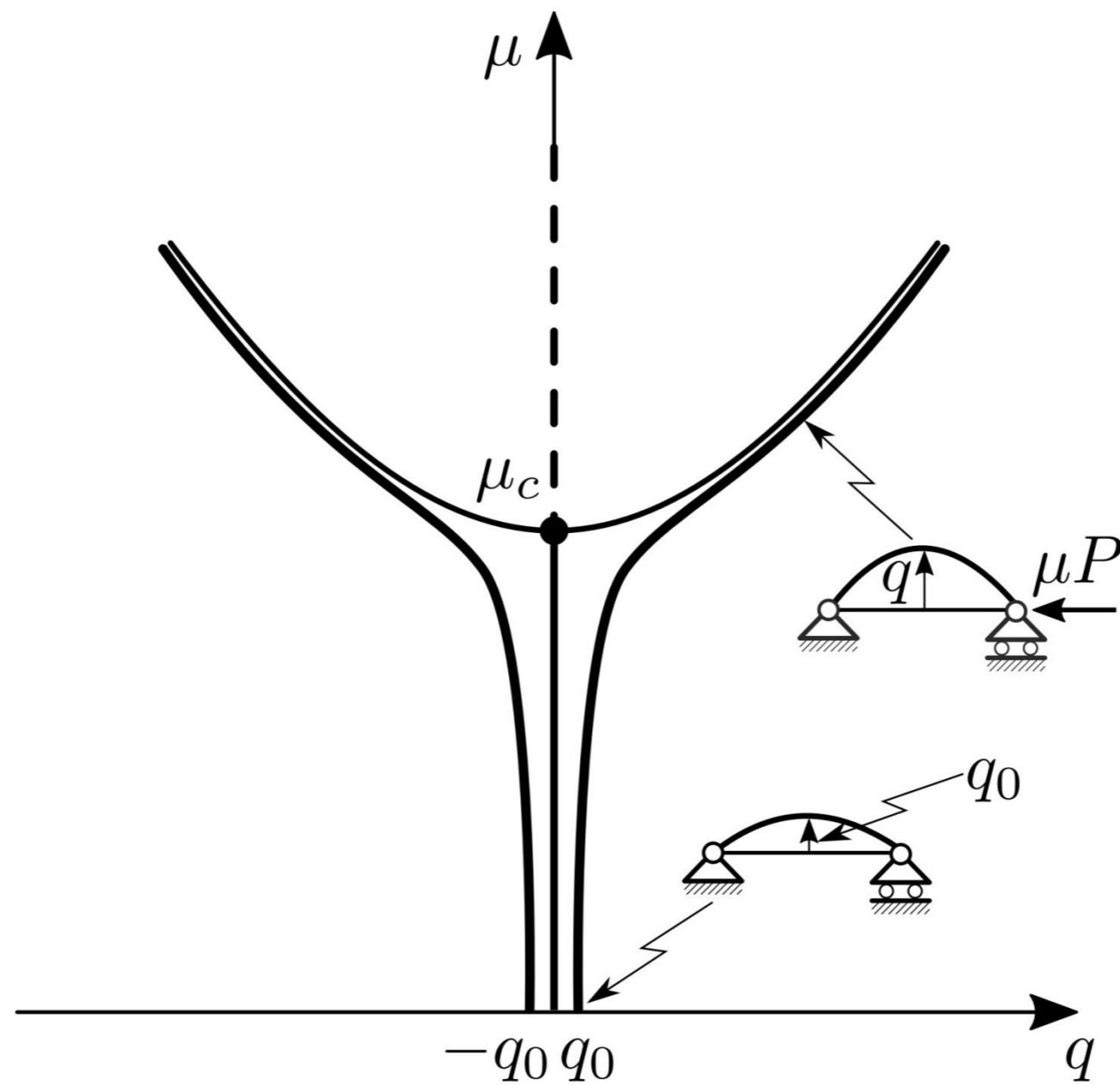
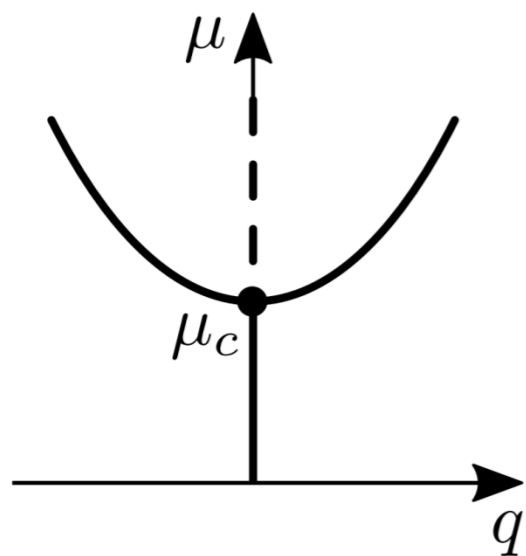
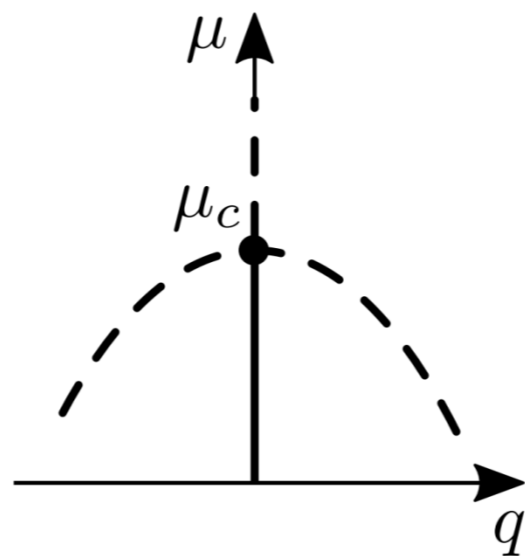


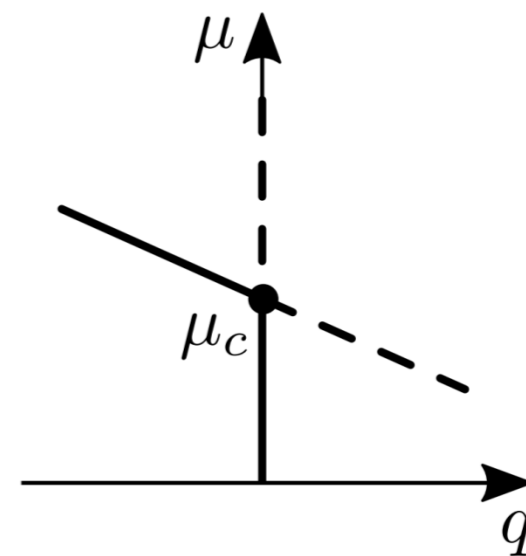
Diagramma di biforcazione della trave di Eulero imperfetta.



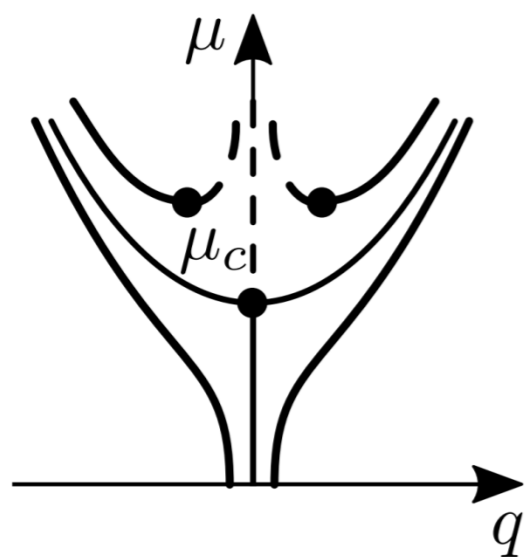
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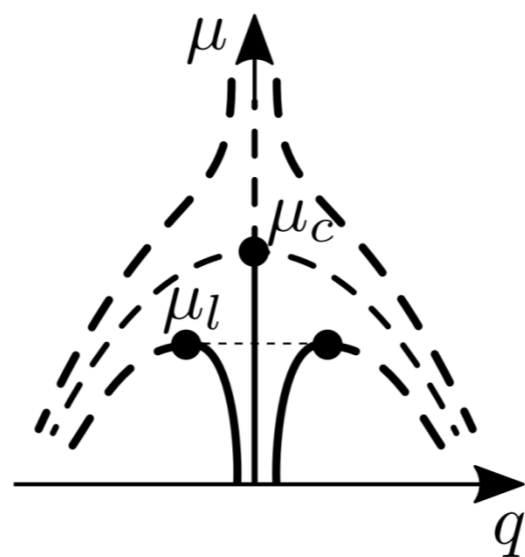
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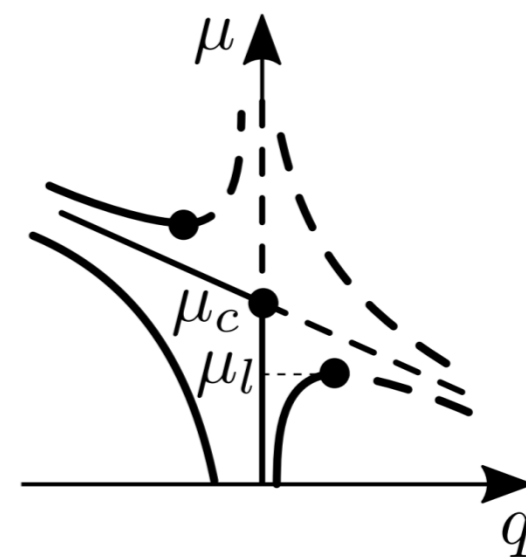
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(d)



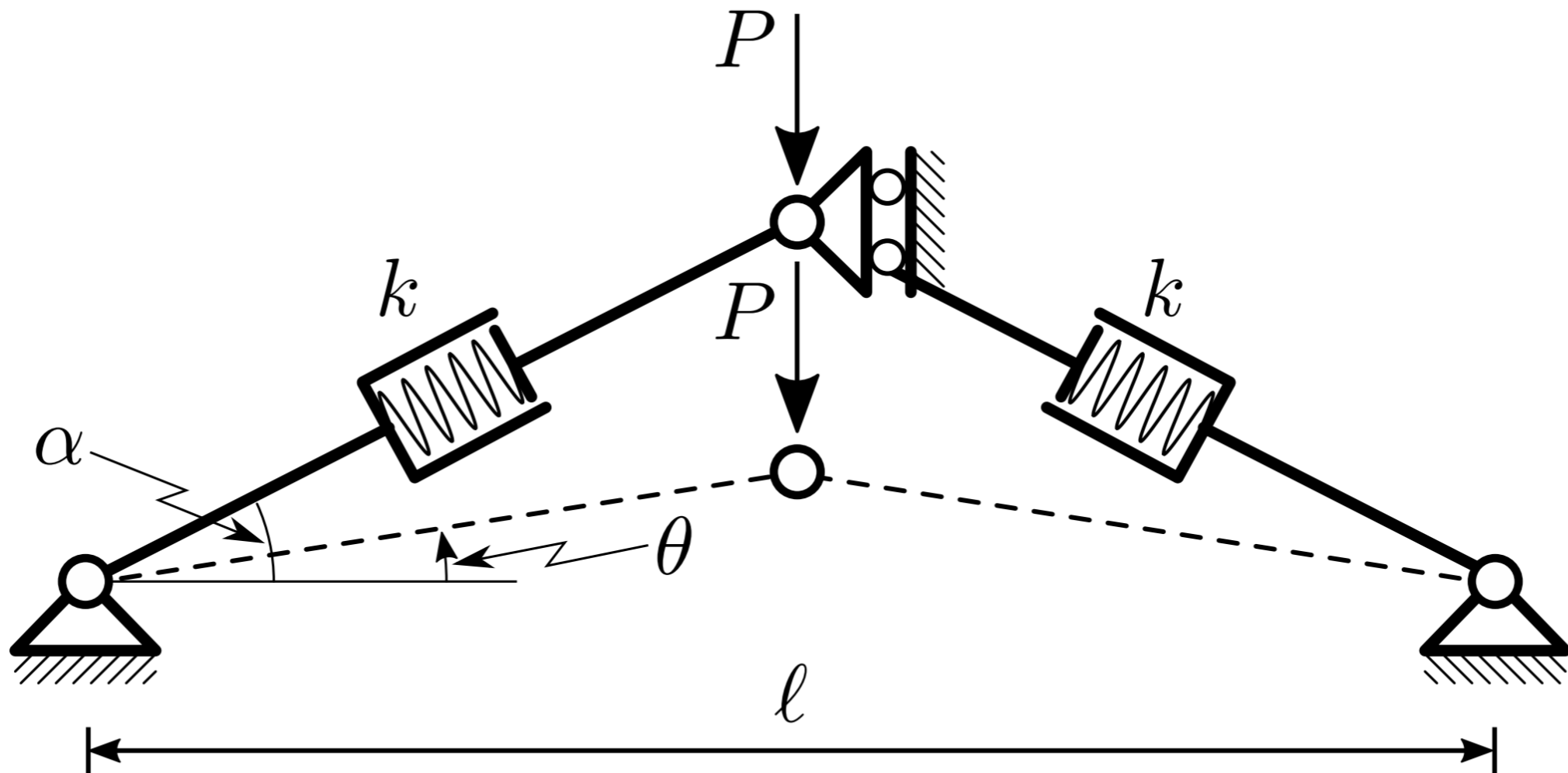
(e)



(f)

**Figura** : Biforcazioni statiche: (a) forza stabile; (b) forza instabile; (c) transcritica; (d) forza stabile imperfetta; (e) forza instabile imperfetta; (e) transcritica imperfetta.

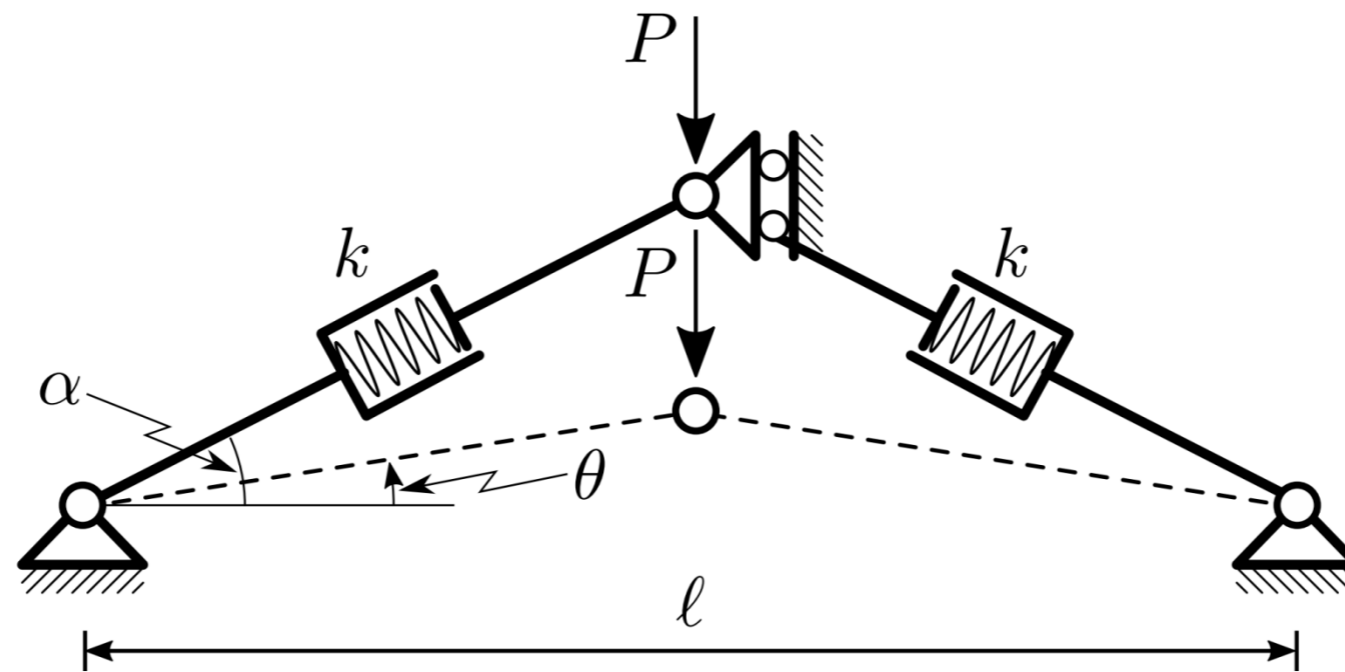
# Lo snap-through dell'arco a tre cerniere (detto von Mises truss)





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Si considera un esempio paradigmatico di struttura perfetta, che subisce instabilità per raggiungimento di un carico limite (*biforcazione a piega*). Questo è rappresentato dall'*arco a tre cerniere ribassato*, simmetrico e costretto a deformarsi simmetricamente (Fig. ) L'arco è costituito da due aste identiche, flessionalmente rigide, ma assialmente elastiche, di rigidezza  $k$ , inizialmente inclinate sull'orizzontale di un angolo  $\alpha$ , soggetto ad una forza verticale di intensità  $P$ , che induce compressione. Si assume quale parametro lagrangiano l'angolo  $\theta$  che, nella configurazione variata, le aste formano con l'orizzontale



**Figura** Arco a tre cerniere simmetrico, ribassato, vincolato a deformarsi simmetricamente.

# L'analisi esatta

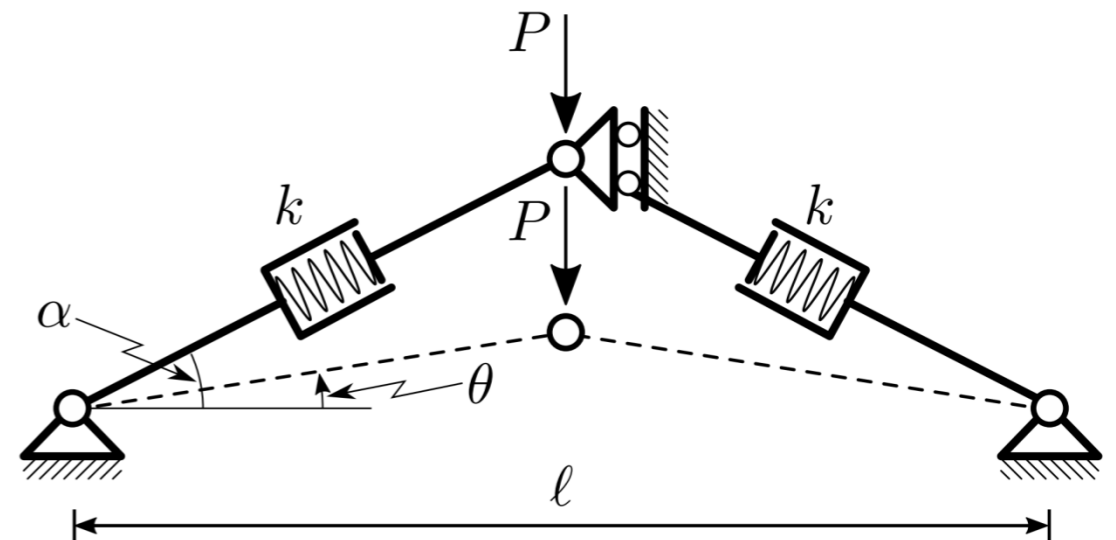
Per questo sistema semplice è possibile condurre un'analisi esatta. L'energia potenziale totale del sistema si scrive:

$$\Pi(\theta) = 2 \frac{1}{2} k \Delta \ell^2 - P \Delta$$

in cui  $\Delta \ell$  è l'allungamento di una singola asta, e  $\Delta$  lo spostamento della chiave dell'arco, positivo verso il basso:

$$\Delta \ell = \frac{\ell}{2} \left( \frac{1}{\cos \theta} - \frac{1}{\cos \alpha} \right)$$

$$\Delta = \frac{\ell}{2} (\tan \alpha - \tan \theta)$$



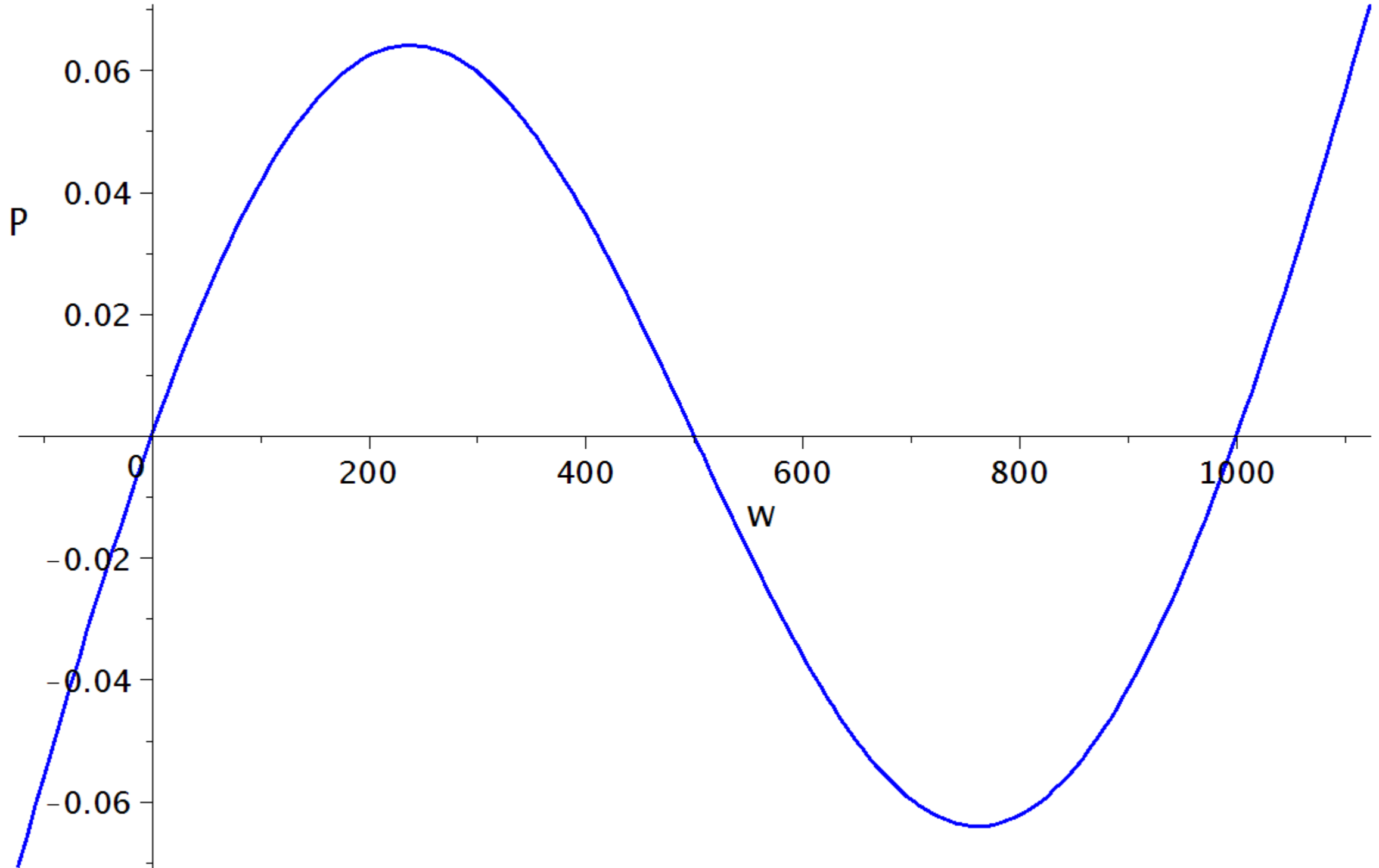


Dalla stazionarietà  $\frac{d\Pi}{d\theta} = 0$  si ricava l'equazione di equilibrio:

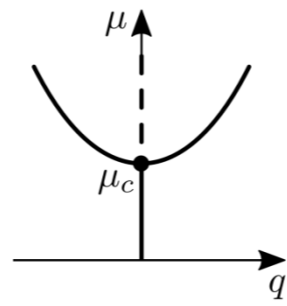
$$\frac{1}{2} \ell \frac{1}{\cos \theta} \left[ k \ell \tan \theta \left( \frac{1}{\cos \theta} - \frac{1}{\cos \alpha} \right) + P \frac{1}{\cos \theta} \right] = 0$$

la cui soluzione fornisce un unico percorso

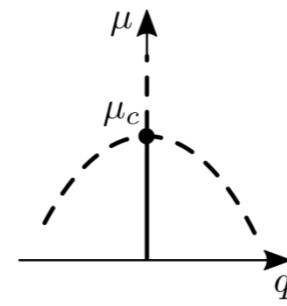
$$P = k \ell \sin \theta \left( \frac{1}{\cos \alpha} - \frac{1}{\cos \theta} \right)$$



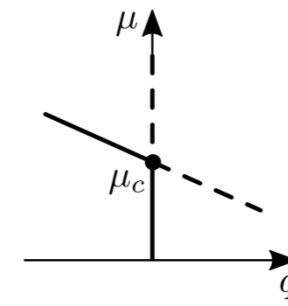
$\alpha = \pi/4, \quad \mathbf{w}=\Delta l, \quad P = P/k$



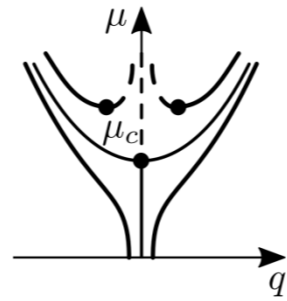
(a)



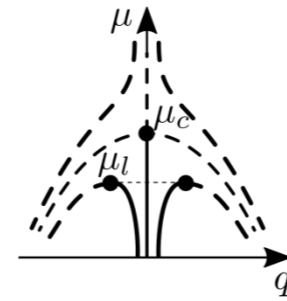
(b)



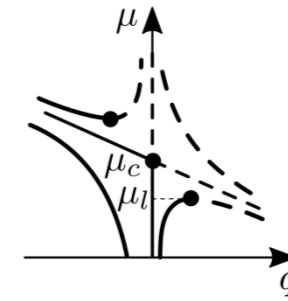
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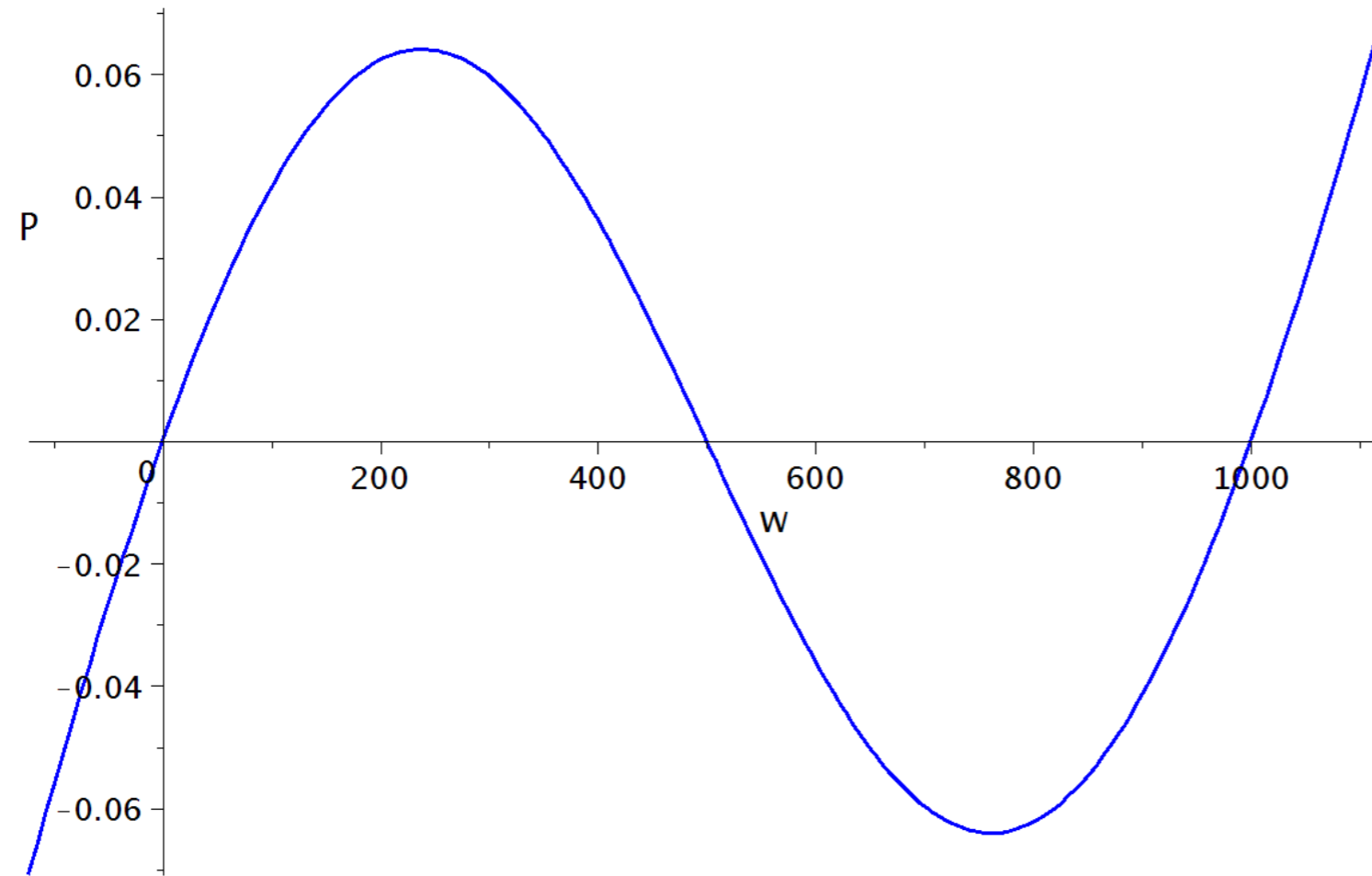
(d)



(e)



(f)





Per analizzare la stabilità dei diversi rami del percorso, si valuta la derivata seconda dell'energia potenziale:

$$\frac{d^2 \Pi}{d\theta^2} = \frac{1}{4} \ell \frac{1}{\cos^2 \alpha} \left[ k\ell \left( \frac{\cos(2\theta) - 3}{\cos \alpha \cos \theta} + 6 \frac{1}{\cos^2 \alpha} - 4 \right) + 4P \tan \theta \right]$$

che, espresso  $\cos \alpha$  dal percorso di equilibrio diventa:

$$\frac{d^2 \Pi}{d\theta^2} = \frac{k\ell^2}{\sin(2\theta)} \left( \tan^3 \theta - \frac{P}{k\ell} \right)$$

Per determinare i punti limite si impone  $\frac{dP}{d\theta} = 0$ , che implica:

$$\cos^3 \theta = \cos \alpha$$

Questa fornisce due radici uguali ed opposte,  $\theta = \theta_l^\pm$ , associate ai carichi limite:

$$P_l^\pm = \pm k\ell \tan^3 \theta_l$$

Da questa si evince che:

$$|\theta| > \theta_l \Rightarrow \frac{d^2 \Pi}{d\theta^2} > 0$$

$$|\theta| < \theta_l \Rightarrow \frac{d^2 \Pi}{d\theta^2} < 0$$

# L'analisi perturbativa

L'equazione di equilibrio, sviluppata intorno a  $\theta = 0$ , e fino ai termini del terzo grado, si scrive:

$$kl^2 \left[ \frac{1}{2} \left( 1 - \frac{1}{\cos \alpha} \right) \theta + \frac{1}{12} \left( 8 - 5 \frac{1}{\cos \alpha} \right) \theta^3 \right] + \frac{1}{2} P \ell (1 + \theta^2) = 0$$

Facendo uso degli sviluppi:

$$\theta(\epsilon) = \epsilon \theta_1 + \epsilon^2 \theta_2 + \epsilon^3 \theta_3 + \dots$$

$$P(\epsilon) = \epsilon P_1 + \epsilon^2 P_2 + \epsilon^3 P_3 + \dots$$

si ottengono le seguenti equazioni perturbative:

$$\epsilon^1 : kl^2 \left( 1 - \frac{1}{\cos \alpha} \right) \theta_1 + P_1 \ell = 0$$

$$\epsilon^2 : kl^2 \left( 1 - \frac{1}{\cos \alpha} \right) \theta_2 + P_2 \ell = 0$$

$$\epsilon^3 : kl^2 \left( 1 - \frac{1}{\cos \alpha} \right) \theta_3 + P_3 \ell = -P_1 \ell \theta_1^2 + \frac{1}{6} kl^2 \left( \frac{5}{\cos \alpha} - 8 \right) \theta_1^3$$

Queste, risolte in cascata, forniscono:

$$\epsilon^1 : P_1 = k\ell \left( \frac{1}{\cos \alpha} - 1 \right) \theta_1$$

$$\epsilon^2 : P_2 = k\ell \left( \frac{1}{\cos \alpha} - 1 \right) \theta_2$$

$$\epsilon^3 : P_3 = -P_1\theta_1^2 + \frac{1}{6}k\ell \left( \frac{5}{\cos \alpha} - 8 \right) \theta_1^3 + k\ell\theta_3 \left( \frac{1}{\cos \alpha} - 1 \right)$$

Scelta la normalizzazione  $\theta = \epsilon$ , deve essere  $\theta_1 = 1$ ,  $\theta_2 = \theta_3 = 0$  da cui, ricostituendo, si ottiene l'espressione asintotica del percorso:

$$P = k\ell \left[ \left( \frac{1}{\cos \alpha} - 1 \right) \theta - \frac{1}{6} \left( \frac{1}{\cos \alpha} + 2 \right) \theta^3 \right]$$



Il punto limite si ottiene risolvendo  $\frac{dP}{d\theta} = 0$ , cioè:

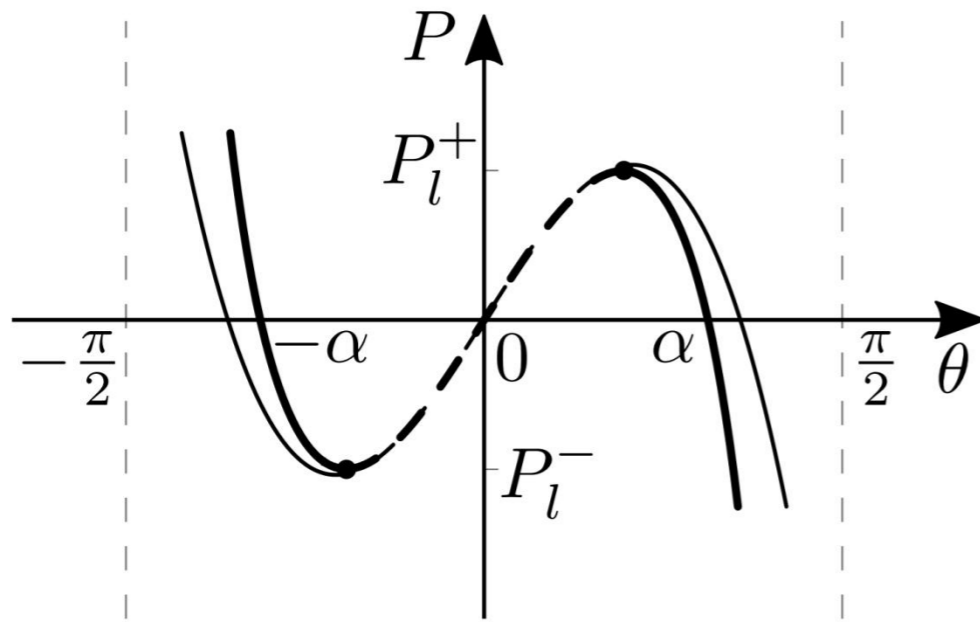
$$k\ell \left[ \left( \frac{1}{\cos \alpha} - 1 \right) - \frac{1}{2} \left( \frac{1}{\cos \alpha} + 2 \right) \theta^2 \right] = 0$$

che fornisce:

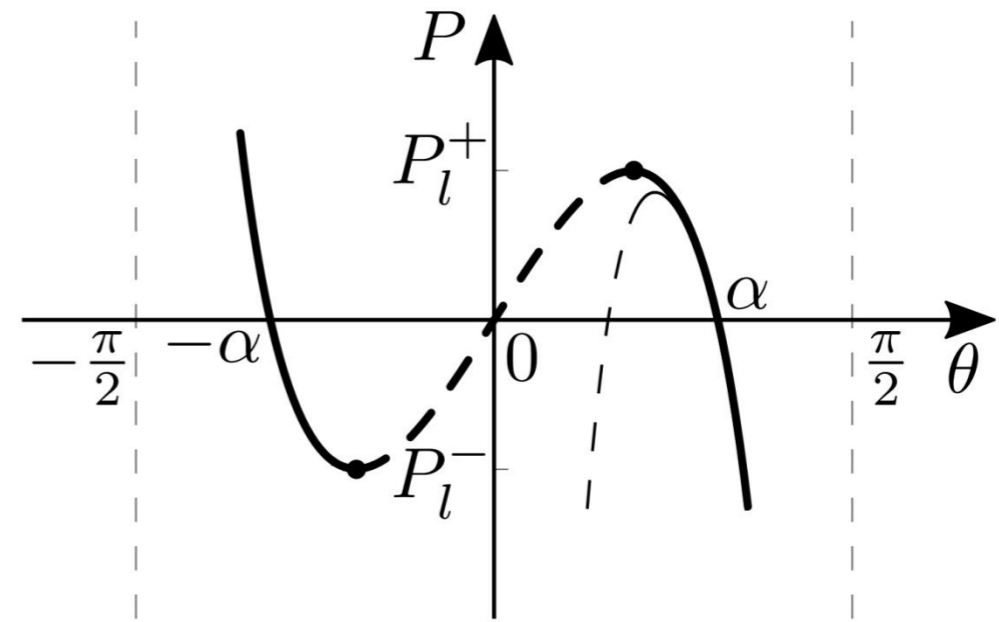
$$\theta_l = \pm \sqrt{2} \sqrt{\frac{1 - \cos \alpha}{1 + 2 \cos \alpha}}$$

a cui corrisponde il carico:

$$P_l = \pm k\ell \frac{2\sqrt{2}}{3} \frac{\left( \frac{1}{\cos \alpha} - 1 \right)^{3/2}}{\sqrt{\frac{1}{\cos \alpha} + 2}}$$



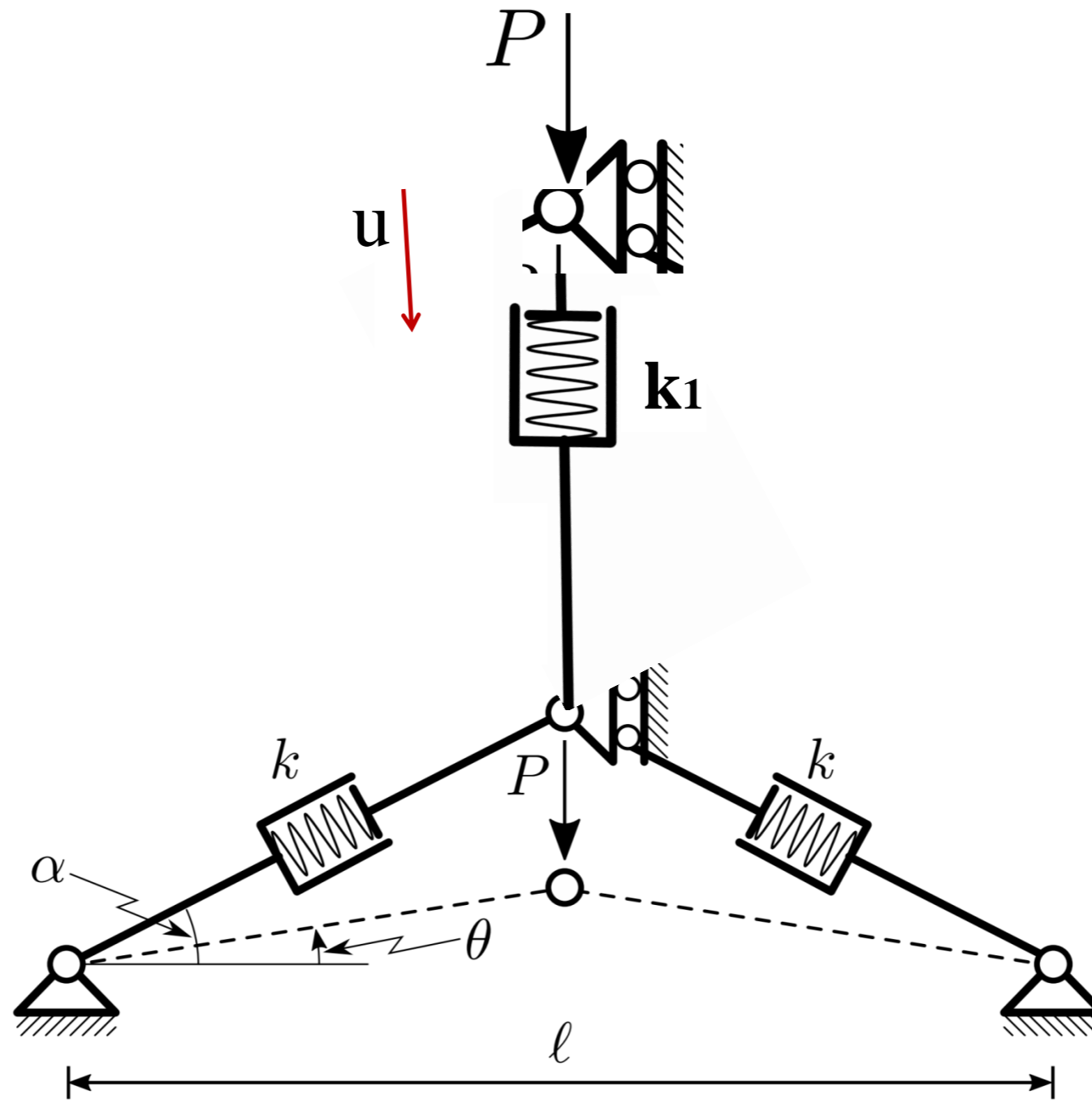
(a)



(b)

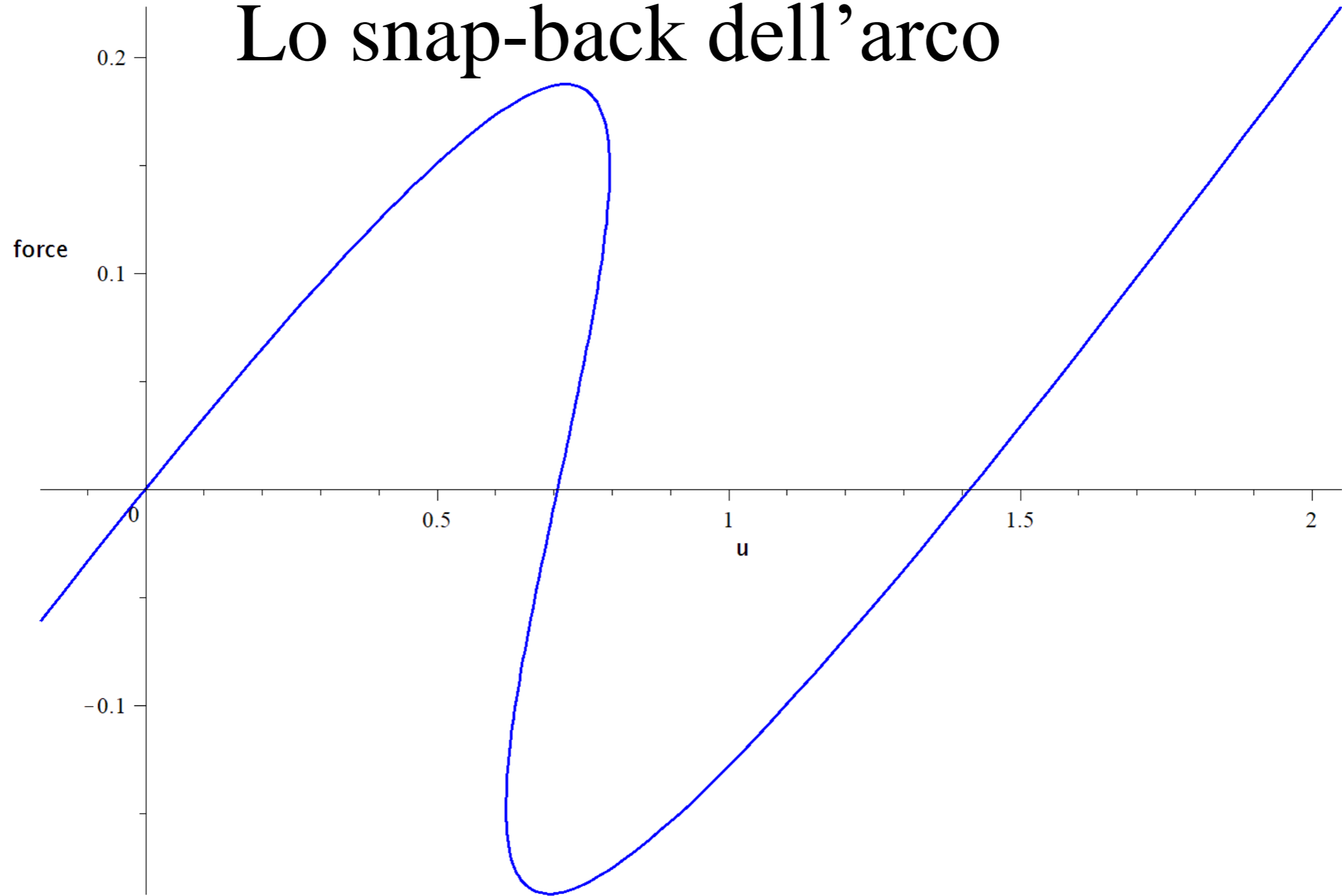
**Figura** Percorso di equilibrio asintotico dell'arco ribassato: (a) estrapolazione da  $\theta = 0$  (tre cerniere allineate), (b) estrapolazione da  $\theta = \alpha$  (arco nella posizione naturale superiore); tratto spesso: soluzione esatta; tratto sottile: soluzione asintotica.

# Lo snap-back dell'arco





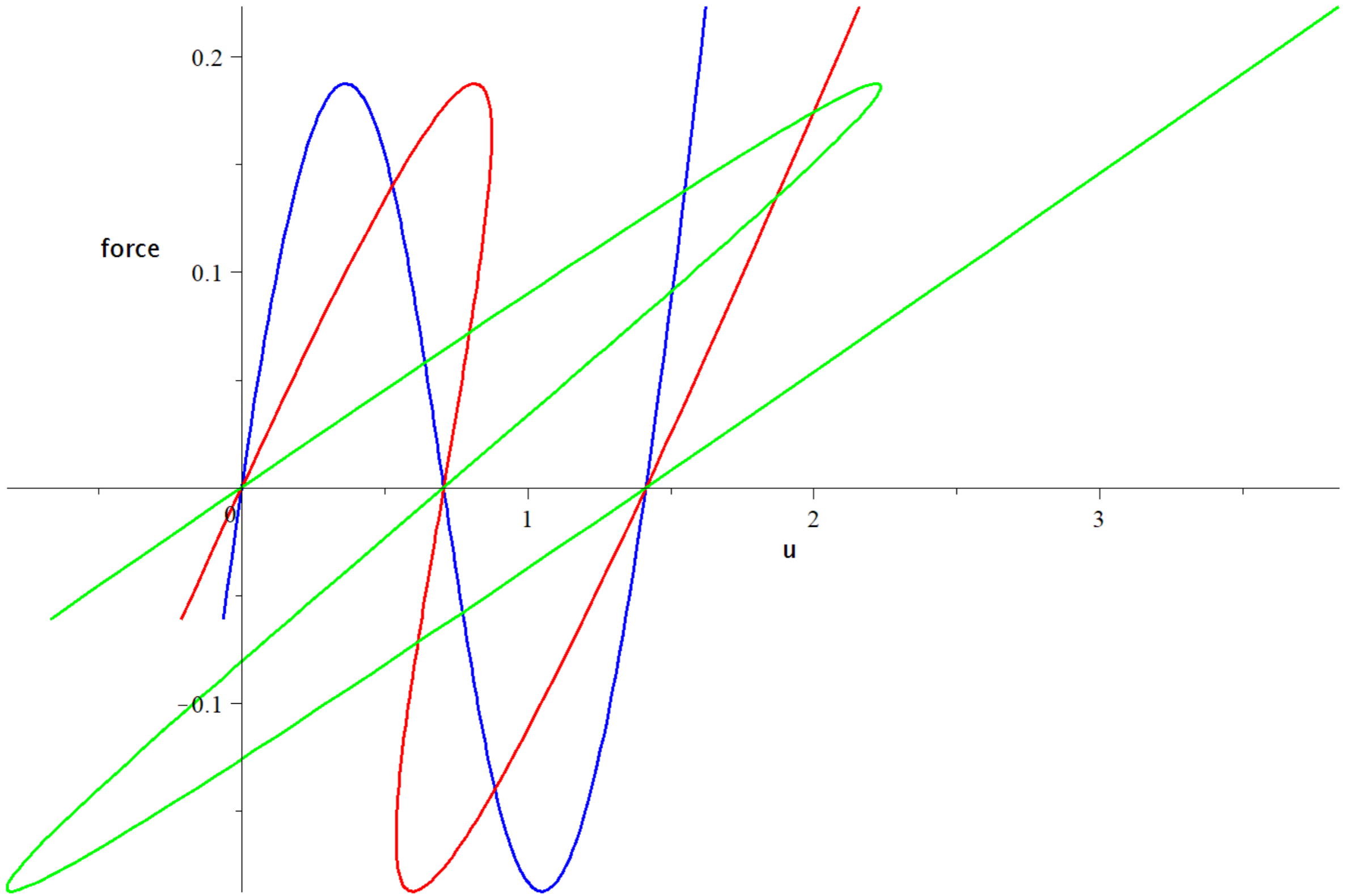
# Lo snap-back dell'arco



$$\alpha = \pi/4, \quad k_1 = 0.5k, \quad P = P/k$$



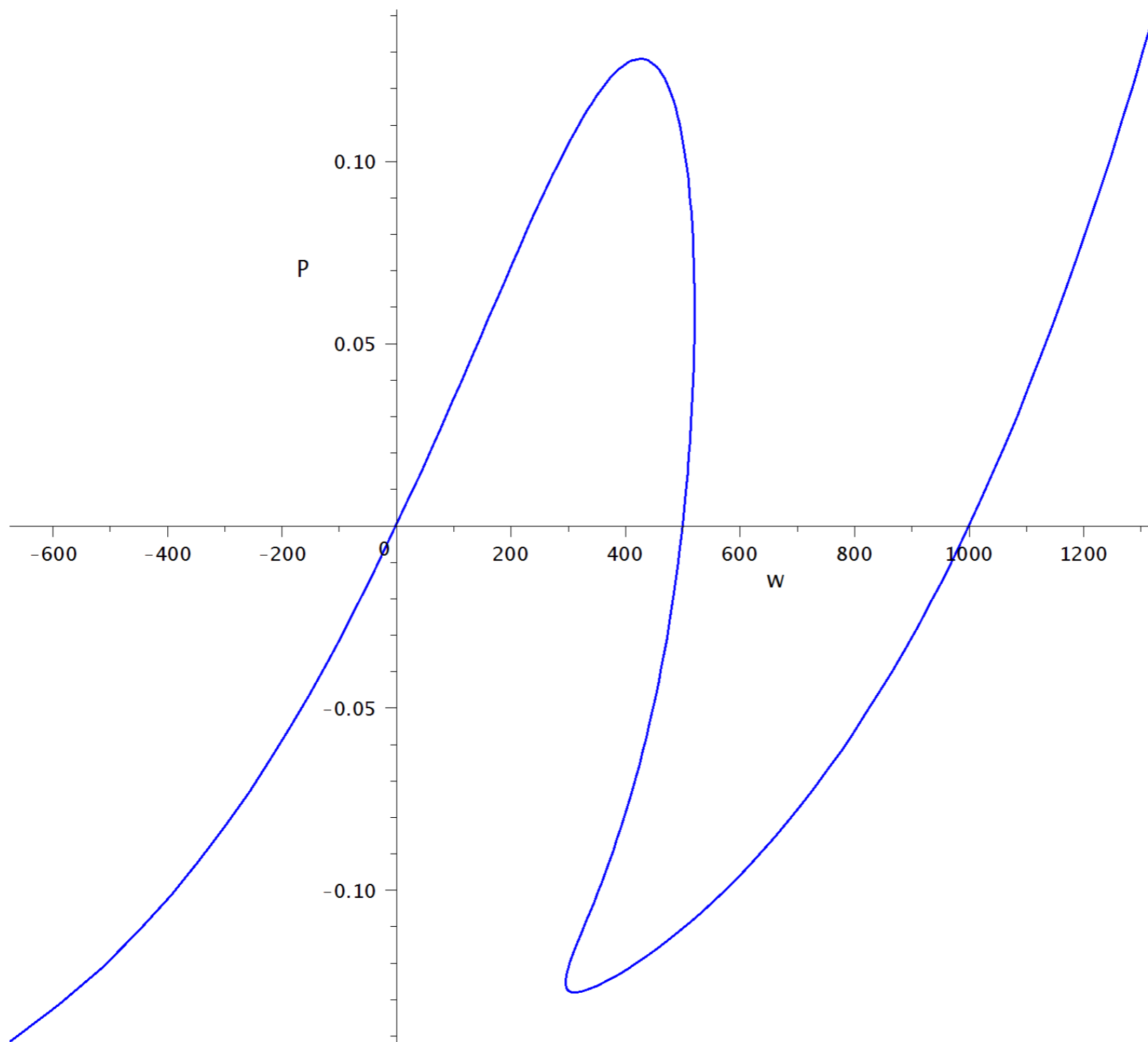
# Lo snap-back dell'arco





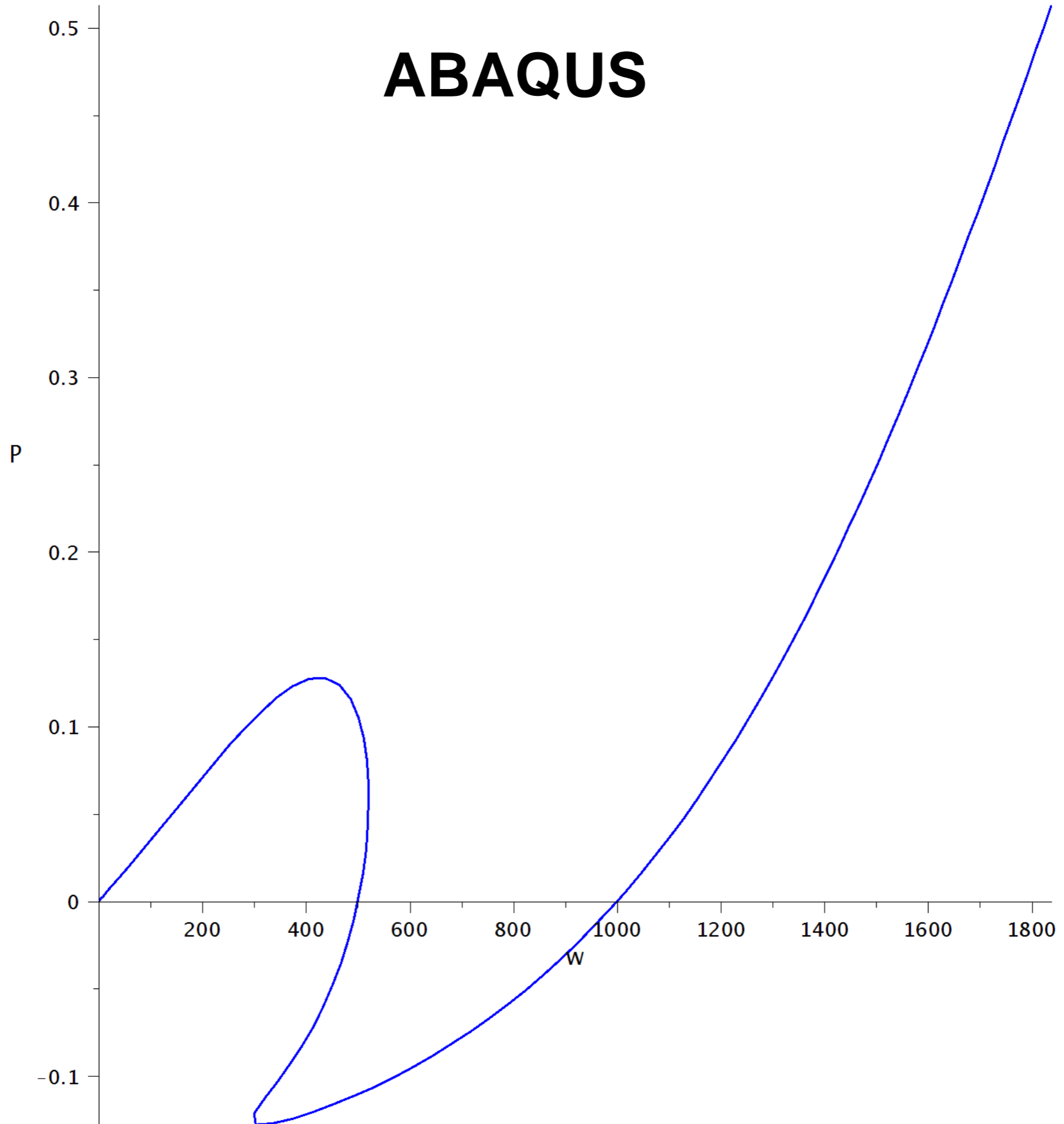
# Lo snap-back dell'arco.

## Deformazione di Hencky – $\epsilon = \ln(1/\lambda)$





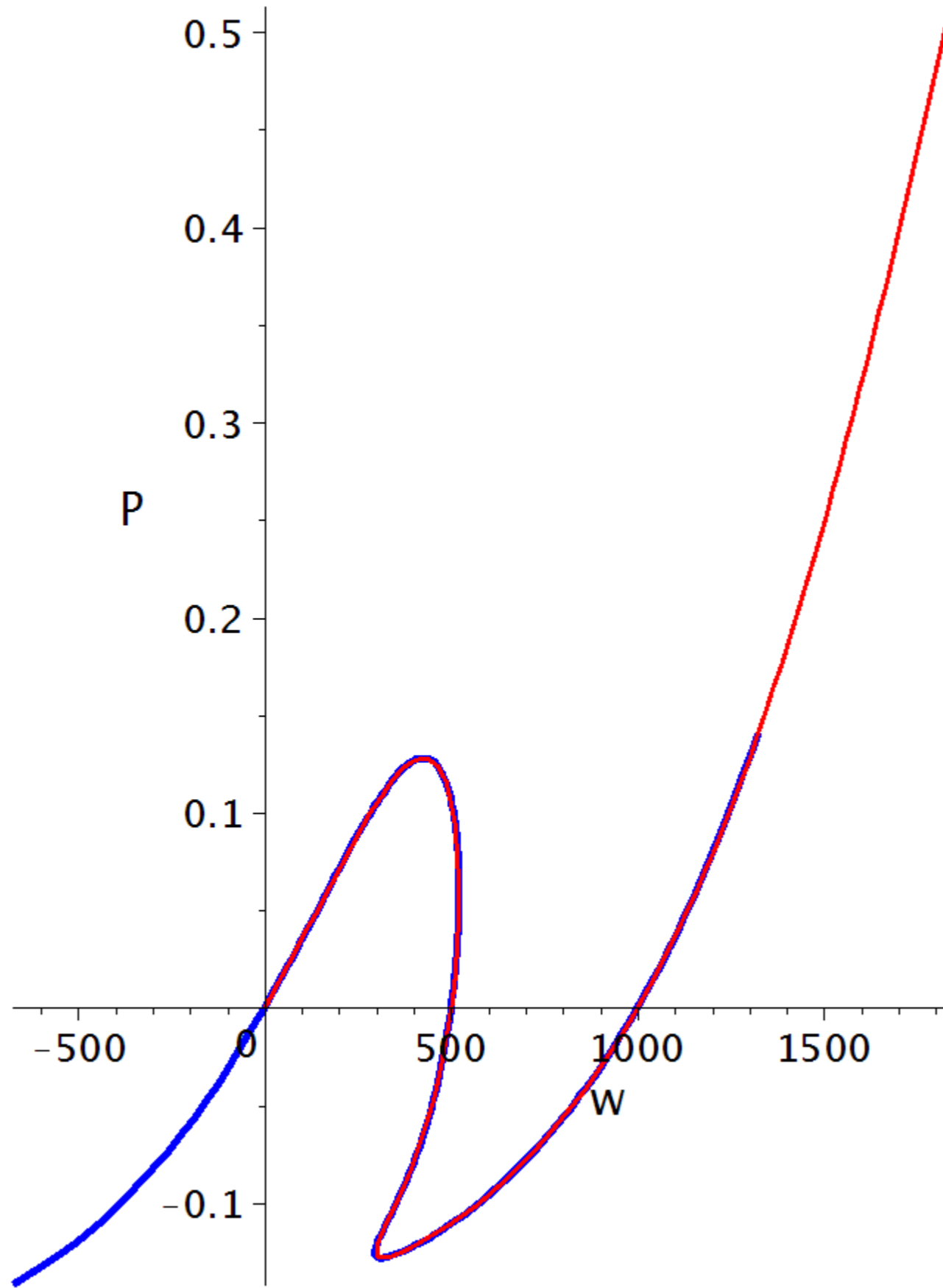
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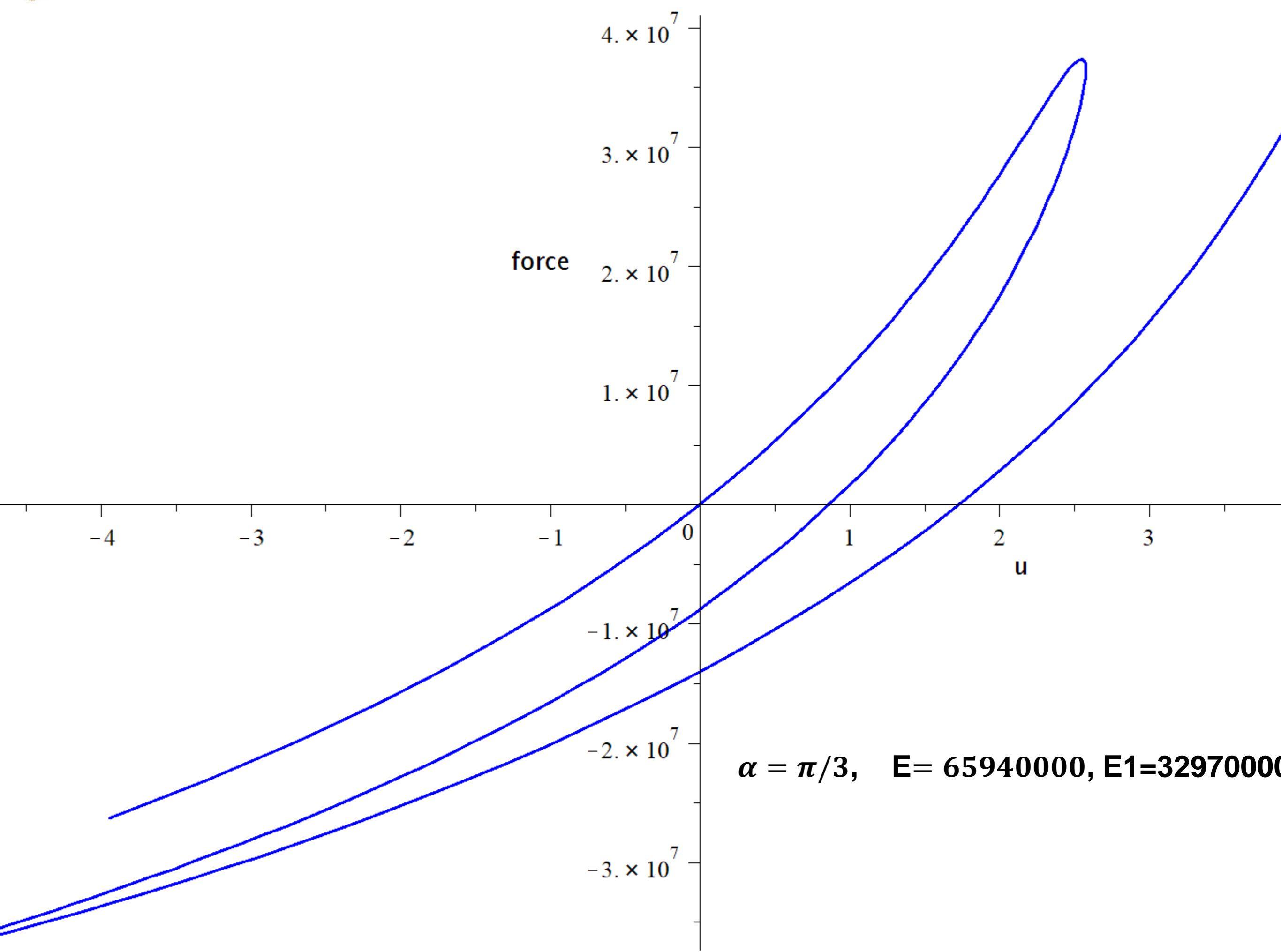


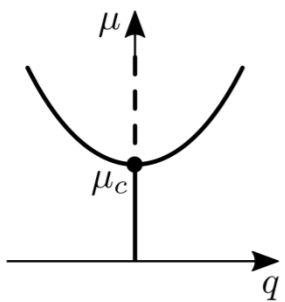
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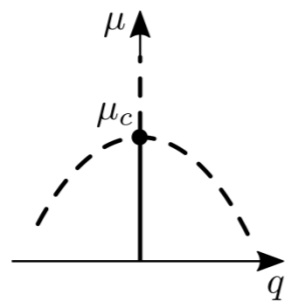
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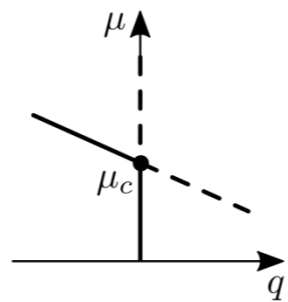




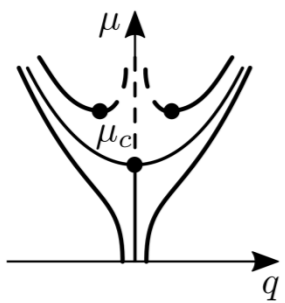
(a)



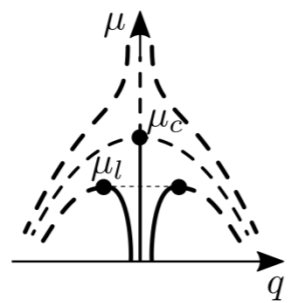
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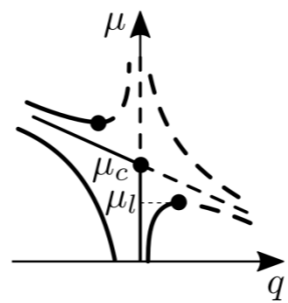
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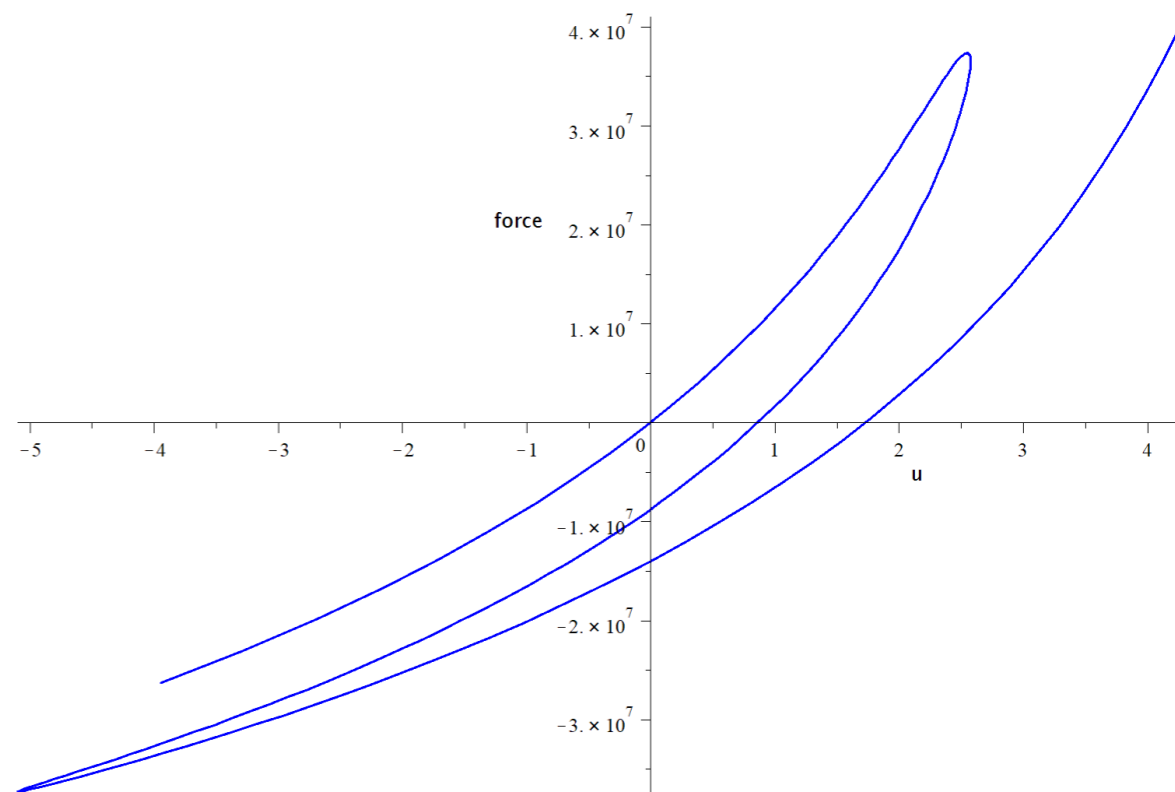
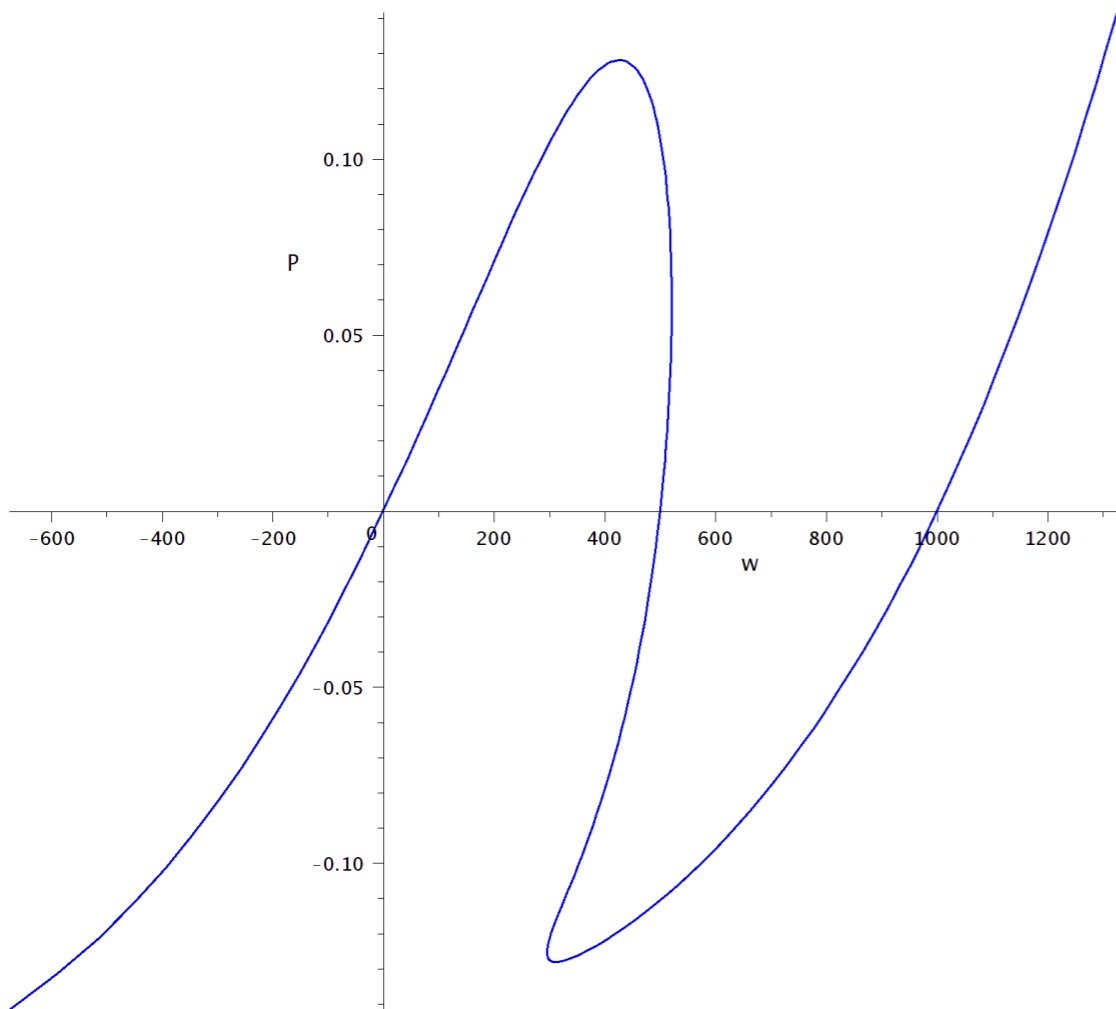
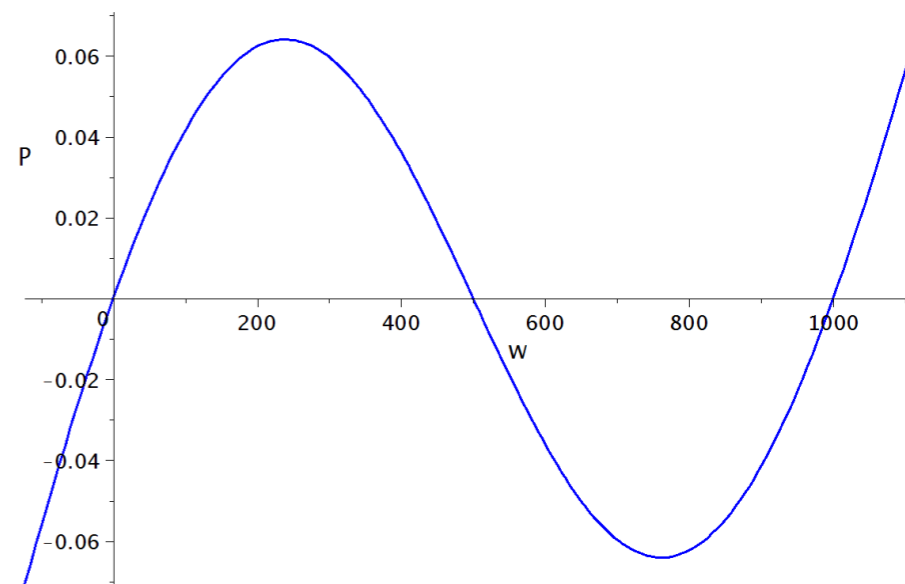
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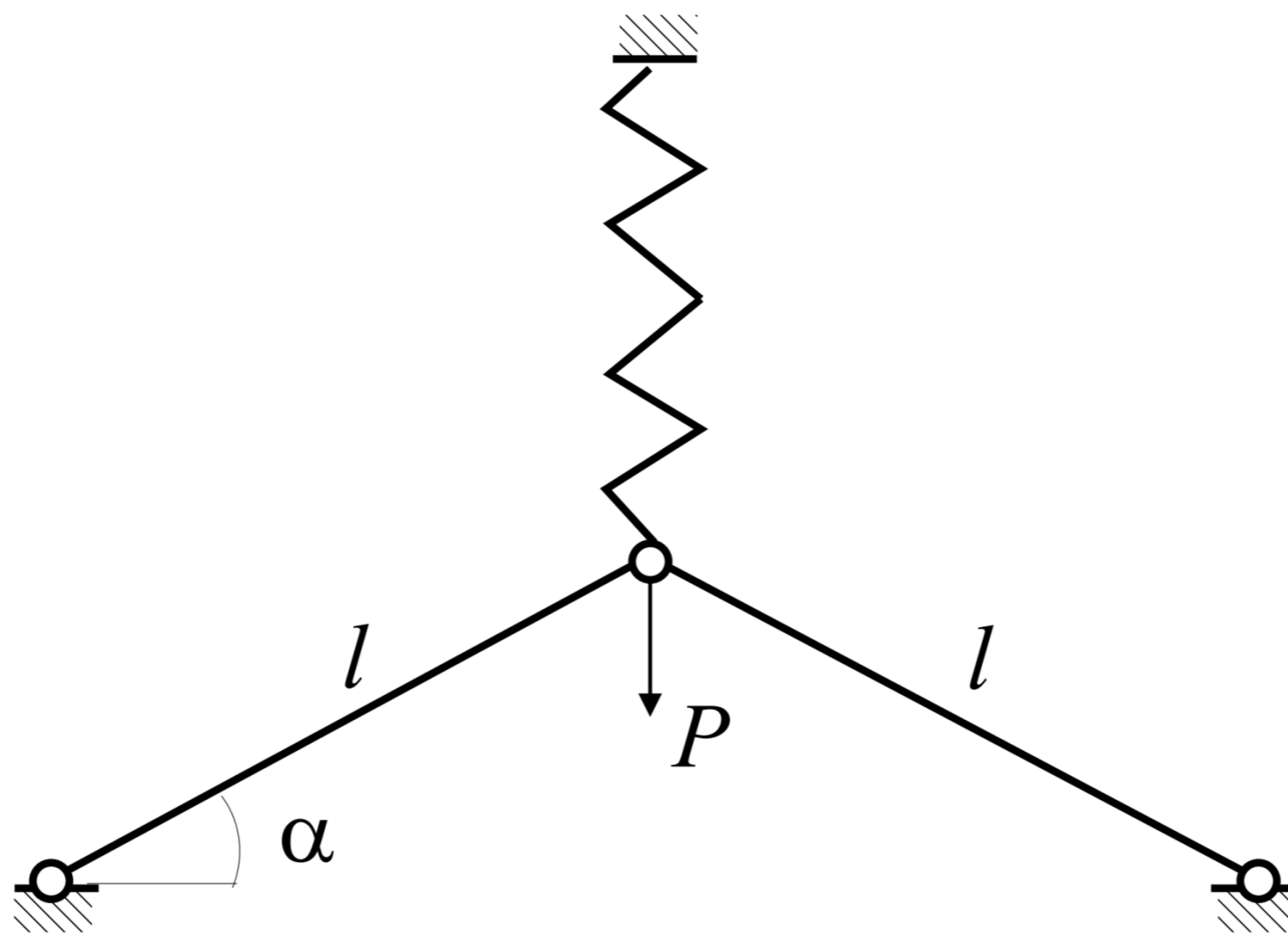
(e)



(f)



# Esempio





# Esempio

Classic von Mises truss with a spring

