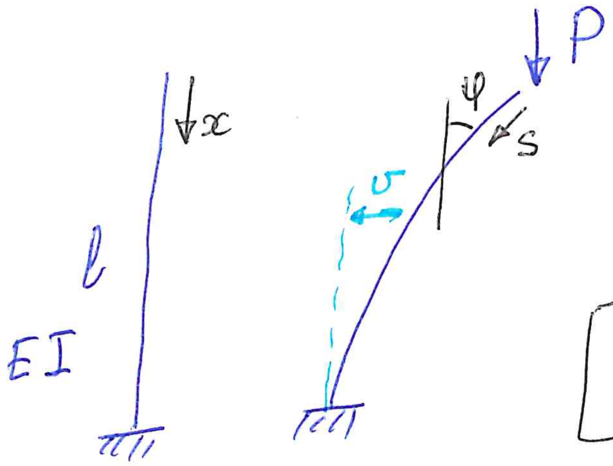
La trave reale

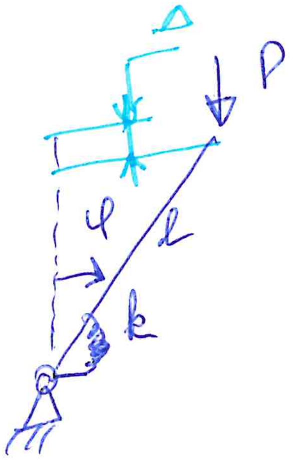
$v(x)$ è non piccolo.



$$\boxed{P_c = \frac{\pi^2}{4l^2} EI}$$

elastica di Eulero

1



$$\Pi = U - V$$

$$U = \frac{1}{2} k \varphi^2$$

$$V = P\Delta,$$

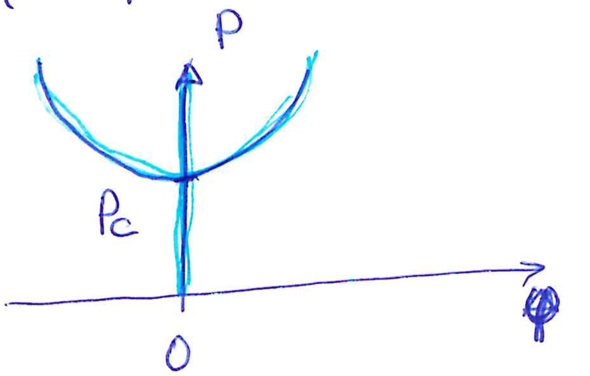
$$\Delta = l - l\cos\varphi$$

$$\Pi = \frac{1}{2} k\varphi^2 - Pl(1-\cos\varphi)$$

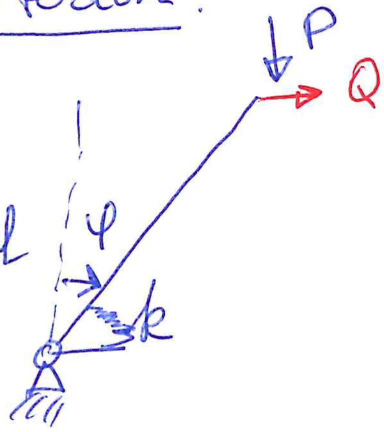
l'equilibrio $\quad \dfrac{d\Pi}{d\varphi} = 0$

$k\varphi - Pl\sin\varphi = 0 \Rightarrow$

1. $\varphi = 0$ (banale)
2. $P = \dfrac{k}{l}\dfrac{\varphi}{\sin\varphi}$

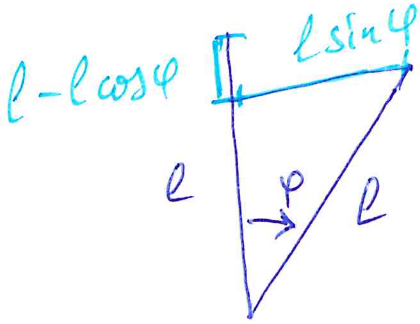


imperfezion:



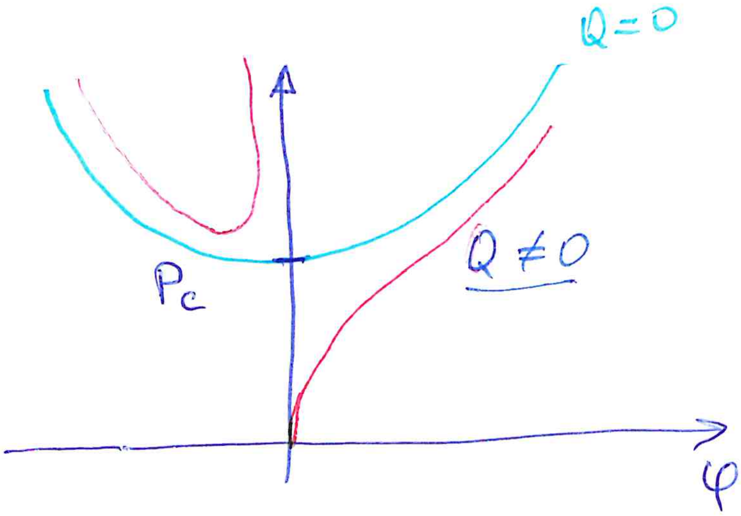
$Q \ll P$

$$\Pi = \frac{1}{2} k \varphi^2 - P l (1 - \cos\varphi) - Q l \sin\varphi$$

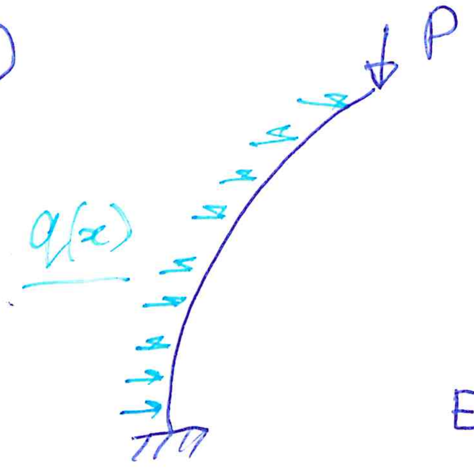


l'equilibrio $\quad \dfrac{d\Pi}{d\varphi} = 0$

$$k\varphi - P l \sin\varphi - Q l \cos\varphi = 0$$



2



$q\ell \ll P$

$$q = -Q \sin\frac{\pi}{2}\frac{x}{\ell}$$

$$EI\,v'' + Pv + q(x) = 0$$

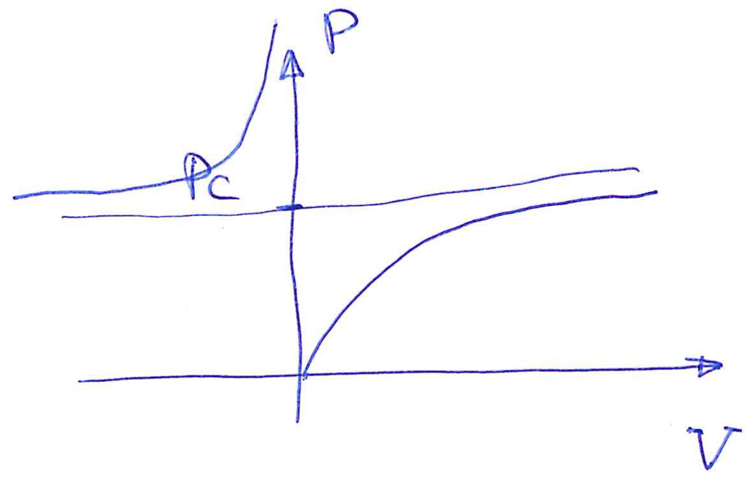
$$v \sim V \sin\frac{\pi}{2}\frac{x}{\ell}$$

$$\Rightarrow \; -EI\frac{\pi^2}{4\ell^2}V\cancel{\sin\frac{\pi}{2}\frac{x}{\ell}} + PV\cancel{\sin\frac{\pi}{2}\frac{x}{\ell}} - Q\cancel{\sin\frac{\pi}{2}\frac{x}{\ell}} = 0$$

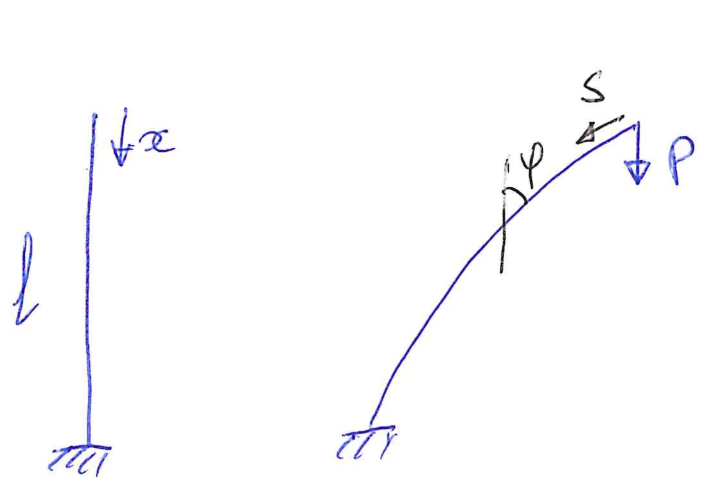
$$V\left[-EI\frac{\pi^2}{4\ell^2} + P\right] = Q$$

$$V = \frac{Q}{P - EI\,\pi/4\ell^2}$$

$V = \max|v|$



(3)



$$\Pi = U - V$$

$\varkappa$ - curvatura

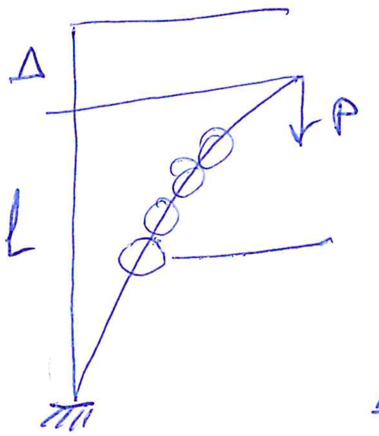
$$U = \frac{1}{2}\int_0^l EI\varkappa^2 ds$$

$$\varkappa = \varphi'(s)$$

$$(\cdot)' = \frac{d}{ds}(\cdot)$$

$$\boxed{U = \frac{1}{2}\int_0^l EI\varphi'^2 ds}$$

$$V = P\Delta$$



$ds\cos\varphi$

$$\Delta = l - \int_0^l \cos\varphi\, ds = \int_0^l [1 - \cos\varphi(s)]\, ds$$

$$\Rightarrow V = P\int_0^l [1 - \cos\varphi(s)]\, ds$$

$$\Pi = \frac{1}{2}\int_0^{\ell} EI\,\varphi'^2 ds - P\int_0^{\ell}[1-\cos\varphi]\,ds$$

$$= \int_0^{\ell}\left\{\frac{1}{2}EI\,\varphi'^2 - P[1-\cos\varphi]\right\}ds$$

l'equilibrio: invece $\dfrac{d\Pi}{d\varphi}=0$

$$\delta\Pi = 0$$

$$\delta\Pi = \delta\int_0^{\ell}\left\{\frac{1}{2}EI\,\varphi'^2 - P[1-\cos\varphi]\right\}ds =$$

$$= \int_0^{\ell}\left\{EI\,\varphi'\,\delta\varphi' + P\,\delta(\cos\varphi)\right\}ds$$

$$\delta\cos\varphi = -\sin\varphi\cdot\delta\varphi$$

$$= \int_0^{\ell}\left\{EI\,\varphi'\,\underline{\delta\varphi'} - P\sin\varphi\,\underline{\delta\varphi}\right\}ds$$

$$\int_0^{\ell} f g'\,ds = -\int_0^{\ell} f' g\,ds + \underbrace{fg\Big|_0^{\ell}}_{f(\ell)g(\ell)-f(0)g(0)}$$

$$\delta \Pi = \int_0^l \left[EI\varphi' \delta\varphi' - P\sin\varphi \, \delta\varphi \right] ds =$$

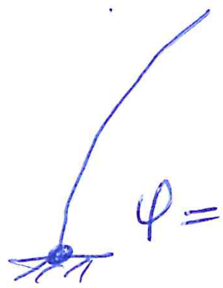
$$= \int_0^l \left[-(EI\varphi')' \delta\varphi - P\sin\varphi \, \delta\varphi \right] ds$$

$$+ EI\varphi' \delta\varphi \Big|_0^l$$

$$\varphi(0) = 0 \Rightarrow \delta\varphi(0) = 0$$

$$\delta\varphi : \quad \tilde{\varphi} = \varphi + \delta\varphi$$

$$\tilde{\varphi}(0) = 0 \Rightarrow \underline{\delta\varphi(0) = 0}$$



$\varphi = 0$

$$\delta\Pi = -\int_0^l \left[EI\varphi'' + P\sin\varphi \right] \delta\varphi \, ds$$

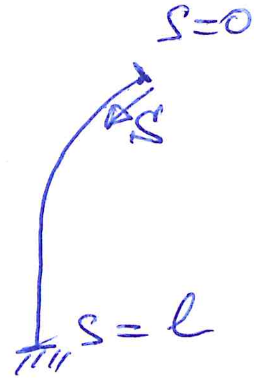
$$+ EI\varphi'(l)\,\delta\varphi(l)$$

l'equilibrio : $\delta\Pi = 0 \quad \forall \, \underline{\underline{\delta\varphi}}$

$\Rightarrow \quad EI\varphi'' + P\sin\varphi = 0$ – equazione di Eulero

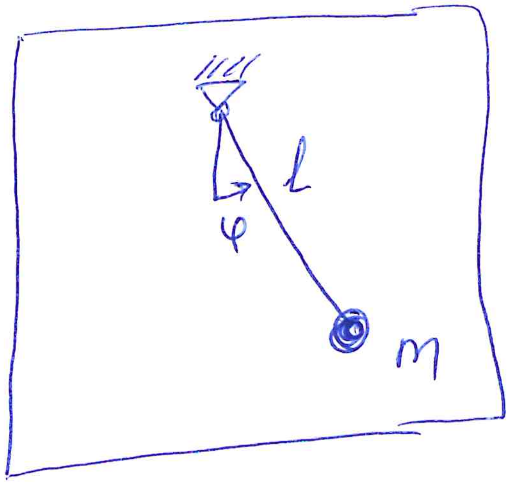
$EI\varphi'(l) = 0$ – la condizione naturale

$$\begin{cases} EI\varphi'' + P\sin\varphi = 0\,, & s\in(0,\ell) \\ \cancel{\varphi(0)=0} \\ \cancel{\varphi'(\ell)=0} \end{cases} \Rightarrow \begin{cases} \varphi'(0)=0 \\ \varphi(\ell)=0 \end{cases}$$



$$\varphi'' + \beta^2 \sin\varphi = 0$$

$$\beta^2 = \frac{P}{EI}$$



1. $\varphi = 0$ banale !

2. $\varphi \neq 0$ – ? non-banale ?

$$\varphi'' + \beta^2 \sin\varphi = 0 \quad \Big|\, \varphi'$$

$$\varphi''\varphi' + \beta^2 \sin\varphi\, \varphi' = 0$$

$$\left(\tfrac{1}{2}\varphi'^2\right)' + \beta^2(-\cos\varphi)' = 0$$

$$\left[\tfrac{1}{2}\varphi'^2 - \beta^2\cos\varphi\right]' = 0$$

$$\left(\tfrac{1}{2}\varphi'^2\right)' = \varphi''\varphi'$$

$$(\cos\varphi)' = -\sin\varphi\cdot\varphi'$$

$$\Rightarrow \quad \varphi'^2 - 2\beta^2\cos\varphi = C\,, \qquad C = \text{cost.}$$

$$\varphi'^2 - 2\beta^2 \cos\varphi = C$$

$$\varphi'(0) = 0 \Rightarrow \underbrace{\varphi'^2(0)}_{0} - 2\beta^2 \cos\varphi(0) = C$$

$$\Rightarrow \underline{C = -2\beta^2 \cos\varphi(0)}$$

Posto : $\varphi(0) = \Phi$

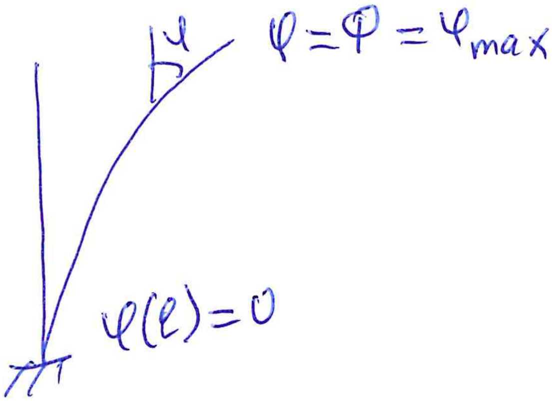
$$\varphi'^2 - 2\beta^2 \cos\varphi = -2\beta^2 \cos\Phi$$

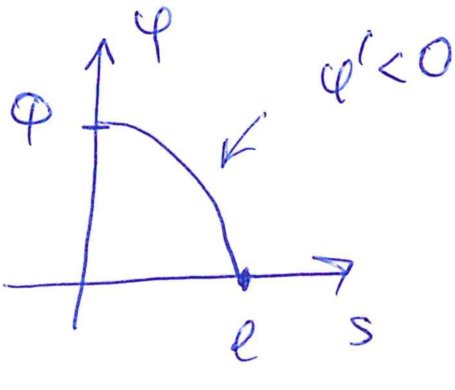
$$\varphi'^2 = 2\beta^2 \cos\varphi - 2\beta^2 \cos\Phi$$

$$\varphi' = \pm\sqrt{2\beta^2(\cos\varphi - \cos\Phi)}$$

$\pm$?

$$\varphi' = -\sqrt{2\beta^2(\cos\varphi - \cos\Phi)} \quad \text{perché } \varphi' < 0$$





$$\varphi' = -\beta\sqrt{2(\cos\varphi - \cos\phi)}$$

$$\frac{d\varphi}{ds} = -\beta\sqrt{2(\cos\varphi - \cos\phi)}$$

$$-\frac{d\varphi}{\beta\sqrt{2(\cos\varphi-\cos\phi)}} = ds \qquad \varphi(s) - ?$$

$$\boxed{-\int_{\phi}^{\varphi(s)} \frac{d\varphi}{\beta\sqrt{2(\cos\varphi-\cos\phi)}} = s}$$

$$\varphi(\ell) = 0 \qquad s = \ell$$

$$\ell = -\frac{1}{\beta}\int_{\phi}^{\varphi(\ell)} (\quad) = -\frac{1}{\beta}\int_{\phi}^{0} \frac{d\varphi}{\sqrt{2(\cos\varphi-\cos\phi)}}$$

$$\boxed{\ell = \frac{1}{\beta}\int_{0}^{\phi} \frac{d\varphi}{\sqrt{2(\cos\varphi-\cos\phi)}}} \qquad \beta, \phi$$

$$\cos\varphi = \cos^2\frac{\varphi}{2} - \sin^2\frac{\varphi}{2} = 1 - 2\sin^2\frac{\varphi}{2}$$

$$\cos\Phi = 1 - 2\sin^2\frac{\Phi}{2}$$

$$\cos\varphi - \cos\Phi = 1 - 2\sin^2\frac{\varphi}{2} - \left(1 - 2\sin^2\frac{\Phi}{2}\right) =$$

$$= 2\sin^2\frac{\Phi}{2} - 2\sin^2\frac{\varphi}{2}$$

$$\ell = \frac{1}{2\beta}\int_0^{\Phi}\frac{d\varphi}{\sqrt{\sin^2\frac{\Phi}{2} - \sin^2\frac{\varphi}{2}}}$$

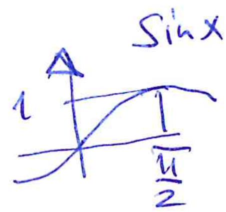
$$p = \sin\frac{\Phi}{2}\ , \qquad \sin^2\frac{\Phi}{2} = p^2$$

η: $\quad \sin\frac{\varphi}{2} = p\sin\eta \quad \left(= \sin\frac{\Phi}{2}\sin\eta\right)$

φ varia tra 0 e Φ

$\Rightarrow$ $\sin\eta$ varia tra 0 e 1 /

$\Rightarrow$ $\underline{\eta \text{ varia tra } 0 \text{ e } \pi/2}$



$\underline{d\varphi \to d\eta}$

$$d\sin\varphi/2 = \frac{1}{2}\cos\frac{\varphi}{2}\,d\varphi$$

$$d\left(\sin\frac{\varphi}{2}\right) = d\left(p\sin\eta\right) = p\cos\eta\,d\eta$$

$$\Rightarrow \underline{\frac{1}{2}\cos\frac{\varphi}{2}\,d\varphi = p\cos\eta\,d\eta}$$

$$d\varphi = \frac{2p\cos\eta\, d\eta}{\cos\frac{\varphi}{2}} = \frac{2p\cos\eta\, d\eta}{\sqrt{1-\sin^2\frac{\varphi}{2}}} =$$

$$= \frac{2p\cos\eta\, d\eta}{\sqrt{1-p^2\sin^2\eta}}$$

$$l = \frac{1}{\cancel{2}\beta}\int_0^{\pi/2} \frac{\cancel{2}\cancel{p}\cancel{\cos\eta}\, \cancel{d\eta}}{\sqrt{1-p^2\sin^2\eta}} \; \frac{1}{\cancel{p}} \; \frac{1}{\underbrace{\sqrt{1-\sin^2\eta}}_{\cancel{\cos\eta}}} \, d\eta$$

$$l = \frac{1}{\beta}\int_0^{\pi/2} \frac{d\eta}{\sqrt{1-p^2\sin^2\eta}}$$

$$\beta l = \int_0^{\frac{\pi}{2}} \frac{d\eta}{\sqrt{1-p^2\sin^2\eta}} \equiv K(p)$$

$K(p)$ è detto come integrale ellittico completo di prima specie.

$$\boxed{\beta l = K(p)}$$

$$\beta l = K(p) \quad , \quad \beta^2 = \frac{P}{EI}$$

$$\varphi \rightarrow p \equiv \sin\frac{\varphi}{2} \rightarrow \beta l \rightarrow P$$

Quando $\varphi \ll 1$, $\varphi \ll 1$, $\eta \ll 1$

$$\sin\frac{\varphi}{2} = \sin\frac{\varphi}{2}\sin\eta$$

$$\int_0^{\pi/2} \frac{d\eta}{\sqrt{1-\underbrace{p^2\sin^2\eta}_{\ll 1}}} \approx \int_0^{\pi/2} d\eta = \frac{\pi}{2}$$

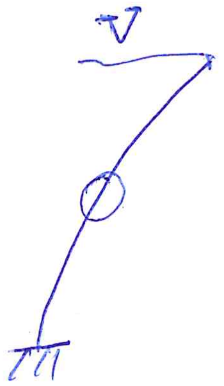
$$\Rightarrow \beta l = \frac{\pi}{2} \Rightarrow \boxed{P_c = \frac{\pi^2}{4l^2} EI}$$

$$\beta l = K(p)$$

$$\frac{P l^2}{EI} = K^2(p)$$

$$P = K^2(p)\frac{EI}{l^2} = K^2(p)\frac{P_c \cdot 4}{\pi^2}$$

$$\boxed{P = P_c \cdot \frac{4}{\pi^2} K^2(p)}$$



$$P \sim V$$

$$dy = ds \sin\varphi$$

$$V = \int_0^{l} dy = \int_0^{l} \sin\varphi \, ds$$

$$V = \frac{2p}{\beta} = \frac{2pl}{K(p)}$$