Chapter 2

Rectangular Plates

§2.1 Introduction

This chapter focuses on rectangular plates, which are common structural elements in the floors, roofs, and walls of buildings and vehicles. For the analysis of rectangular plates, a natural reference coordinate system is a Cartesian coordinate system, as depicted in Fig. 2-1.

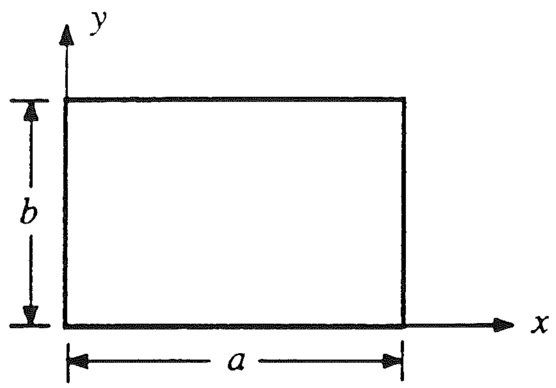

Figure 2-1. Rectangular plate and Cartesian coordinate system.

The analyses presented in this chapter utilize the governing equations for plate behavior which were derived in Chapter 1. For notational simplicity, subscripts 0 and superscripts 0 which denote that a quantity is evaluated at the plate midplane are henceforth omitted. Thus w_0 and ε_{xx}^0 are represented by w and ε_{xx}, with the understanding that the *midplane* displacement or strain is referred to, unless stated otherwise.

§2.2 Navier's Soluticn for Simply-Supported Rectangular Plates under Transverse Load

Consider a rectangular plate which is simply supported along all four edges, and is subjected to some distribution of transverse load $p_z = p(x, y)$. For a plate with length a and width b as shown in Fig. 2-1, the load and displacement can be represented in terms of the Fourier series expansions

$$p(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} p_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{2.1a}$$

$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{2.1b}$$

The form (2.1b) for w satisfies the simply-supported boundary conditions for plate bending, which are

$$\text{Along } x = 0 \text{ and } x = a : \begin{cases} w = 0 \\ M_x = 0 \end{cases} \tag{2.2a}$$

$$\text{Along } y = 0 \text{ and } y = b : \begin{cases} w = 0 \\ M_y = 0 \end{cases} \tag{2.2b}$$

Because the plate is loaded just by transverse external forces, only the plate bending response, governed by Eq. (1.46), need be considered. In this case, Eq. (1.46) can be written

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} - \frac{p}{D} = 0 \tag{2.3}$$

Substitution of the expansions (2.1) into Eq. (2.3) yields

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{w_{mn}\left[\left(\frac{m\pi}{a}\right)^4 + 2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + \left(\frac{n\pi}{b}\right)^4\right] - \frac{p_{mn}}{D}\right\} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} = 0 \tag{2.4}$$

Each term in the series (2.4) must vanish independently; hence it follows that

$$w_{mn} = \frac{p_{mn}}{D[(m\pi/a)^2 + (n\pi/b)^2]^2} \tag{2.5}$$

If the Fourier coefficients p_{mn} of the load are known, then Eq. (2.5) can be used to obtain the Fourier coefficients w_{mn} in the expansion (2.1b), and the normal displacement $w(x, y)$ is determined.†

† The double Fourier series solution for the simply-supported rectangular plate under transverse load was presented to the French Academy of Sciences by Navier in 1820.

For a given transverse load distribution $p(x,y)$, the Fourier coefficients p_{mn} in Eq. (2.1a) are obtained by taking advantage of the *orthogonality relation*

$$\int_0^\ell \sin\frac{n\pi x}{\ell}\sin\frac{n'\pi x}{\ell} = \frac{\ell}{2}\delta_{nn'} \tag{2.6}$$

where

$$\delta_{nn'} = \begin{cases} 1 & \text{if } n = n' \\ 0 & \text{if } n \neq n' \end{cases} \tag{2.7}$$

is the *Kronecker delta.*

Multiplying Eq. (2.1a) by $\sin\frac{m'\pi x}{a}\sin\frac{n'\pi y}{b}$ and integrating over the area of the plate yields

$$\begin{aligned}
&\int_0^b\int_0^a \sin\frac{m'\pi x}{a}\sin\frac{n'\pi y}{b}\,p(x,y)\,dx\,dy \\
&= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} p_{mn}\int_0^b\int_0^a \sin\frac{m'\pi x}{a}\sin\frac{n'\pi y}{b}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy \\
&= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} p_{mn}\left(\frac{a}{2}\delta_{mm'}\right)\left(\frac{b}{2}\delta_{nn'}\right) \\
&= p_{m'n'}\frac{ab}{4}
\end{aligned} \tag{2.8}$$

from which it follows that

$$p_{mn} = \frac{4}{ab}\int_0^b\int_0^a \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,p(x,y)\,dx\,dy \tag{2.9}$$

where the relation (2.6) is employed in order to evaluate the integrals on the right hand side of Eq. (2.8).

Example 2.1: Simply-Supported Rectangular Plate under Uniform Transverse Load

Consider a rectangular plate with four simply-supported edges, subjected to a uniform transverse load $p(x,y) = p_0$ (see Fig. 2-1). Equation (2.9) provides the Fourier coefficients of the load:

$$\begin{aligned}
p_{mn} &= \frac{4p_0}{ab}\int_0^b\int_0^a \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy \\
&= \frac{4p_0}{\pi^2 mn}(1-\cos m\pi)(1-\cos n\pi) \\
&= \frac{4p_0}{\pi^2 mn}[1-(-1)^m][1-(-1)^n]
\end{aligned} \tag{a}$$

The coefficients p_{mn} given by (a) vanish for even values of m or n; thus the series expansion (2.1a) for the load can be written

$$p(x,y) = \sum_{m=1}^{m_{max}} \sum_{n=1}^{n_{max}} \frac{16p_0}{\pi^2 mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \quad \begin{cases} m = 1,3,\ldots \\ n = 1,3,\ldots \end{cases} \qquad (b)$$

which is a truncated version of the infinite series (2.1a), with m_{max} and n_{max} as the upper limits of m and n.

Equation (b) is the Fourier series representation of a uniform load p_0. The nature of this representation is illustrated by Fig. 2-2, which depicts the load distribution $p(x, b/2)$ given by the series (b). As the number of terms in the series is increased, the sine terms add in such a manner as to approximate the uniform load distribution $p/p_0 = 1$ (see Fig. 2-2). The series (b) must always give $p = 0$ at the plate boundary, however, because the sine functions vanish on the boundary.

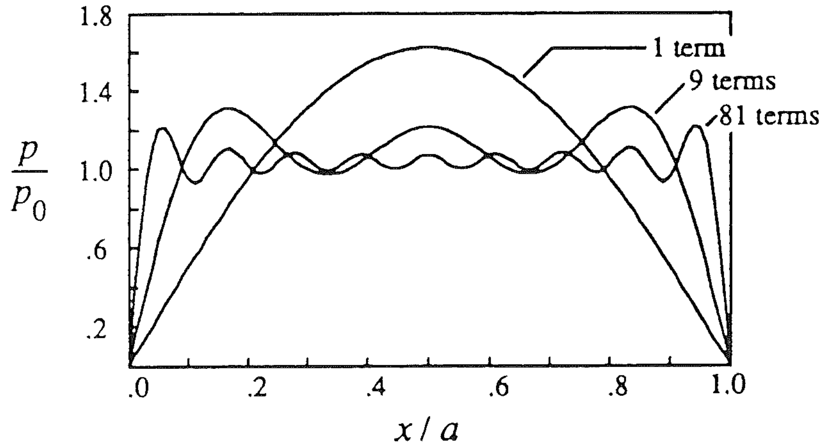

Figure 2-2. Fourier series representation of the uniform load along the line $y = b/2$. Values obtained from Eq. (b), with 1, 9, and 81 terms in the series, corresponding to $m_{max} = n_{max} = 1$, 5, and 17.

From (a) and Eq. (2.5), the coefficients w_{mn} are readily obtained, and the normal displacement w, given by Eq. (2.1b), is

$$w(x,y) = \frac{16p_0}{\pi^6 D} \sum_{m=1}^{m_{max}} \sum_{n=1}^{n_{max}} \frac{\sin(m\pi x/a)\sin(n\pi y/b)}{mn[(m/a)^2 + (n/b)^2]^2} \quad \begin{cases} m = 1,3,\ldots \\ n = 1,3,\ldots \end{cases} \qquad (c)$$

For the case of a square plate ($a = b$ in Fig. 2-1), the displacement w at the center of the plate ($x = a/2$, $y = a/2$) is given in Table 2-1. The series (c) converges quite rapidly, and with four terms ($m_{max} = n_{max} = 3$), yields

$$w(a/2, a/2) = 0.00406 \frac{p_0 a^4}{D} \qquad (d)$$

Table 2-1. Convergence of Navier's solution for the maximum displacement w, maximum bending moment M_x, corner force F_c, and maximum transverse edge shear V_x for a simply-supported square plate under uniform load p_0 ($\nu = 0.3$).

No. of Terms	$wD/(p_0 a^4)$	$M_x/(p_0 a^2)$	$F_c/(p_0 a^2)$	$V_x/(p_0 a)$
1	0.00416	0.0534	0.0575	0.348
4	0.00406	0.0469	0.0628	0.363
9	0.00406	0.0482	0.0640	0.391
∞	0.00406	0.0479	0.0650	0.420

Table 2-1 also lists the values for the maximum bending moment $M_x(a/2, a/2)$, the corner force $F_c(0,0)$, and the maximum transverse edge shear $V_x(0, a/2)$. Series expansions for these quantities are obtained by substituting the expansion (c) into the relations

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \tag{e}$$

$$F_c = -2M_{xy} = 2D(1-\nu)\frac{\partial^2 w}{\partial x \partial y} \tag{f}$$

$$V_x = -D\left[\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x \partial y^2}\right] \tag{g}$$

which follow respectively from Eqs. (1.80), (1.62), and (1.82).

Note that the values for M_x, F_c, and V_x in Table 2-1 converge more slowly than the displacement w. This is due to the fact that *derivatives* of the trigonometric series (c) converge more slowly than (c), because each differentiation brings an extra factor of m or n into the numerator of each term. The transverse shear given by Eq. (g), which involves three derivatives of the series (c), exhibits the slowest convergence in Table 2-1.

The maximum bending stresses in the plate occur at the center, on the surfaces $z = \pm t/2$. From Eq. (1.25) and Table 2-1,

$$\sigma_{xx}^{\max} = \pm\frac{6M_x^{\max}}{t^2} = \pm 0.29\,\frac{p_0 a^2}{t^2} \tag{h}$$

Note, however, that due to the corner forces the corners will be regions of significant stress concentration. Interestingly, the four corner forces act in the *same direction* as the applied load, and contribute a combined reaction force which is approximately equal to one-quarter of the total applied load $p_0 a^2$.

§2.3 Lévy's Solution for Rectangular Plates under Transverse Load

The example of the previous section showed that a double Fourier series representation for the normal displacement w leads to series expansions for the moment and shear resultants which converge relatively slowly. An alternative approach, commonly referred to as "Lévy's solution," is the single Fourier series

$$w(x,y) = \begin{cases} \sum_{m=1}^{\infty} f_m(y) \sin \frac{m\pi x}{a} \\ \text{or} \\ \sum_{m=0}^{\infty} g_m(y) \cos \frac{m\pi x}{a} \end{cases} \qquad (2.10a,b)$$

where $f_m(y)$ and $g_m(y)$ are unknown functions.†

Whereas Navier's solution (2.1b) is applicable only to problems in which all four edges are simply supported, Lévy's solution (2.10) may be applied to problems in which arbitrary boundary conditions are specified along the two edges $y = 0$ and $y = b$ (see Fig. 2-1). Along the two edges $x = 0$ and $x = a$, however, the boundary conditions which can be satisfied by (2.10) depend upon the form which is assumed for the series in x.‡

We will consider the sine series (2.10a), which corresponds to simply-supported edges at $x = 0, a$. The transverse load $p(x, y)$ can also be written in terms of a Fourier sine series in x:

$$p(x,y) = \sum_{m=1}^{\infty} p_m(y) \sin \frac{m\pi x}{a} \qquad (2.11)$$

where, by the orthogonality relation (2.6), the coefficients p_m are

$$p_m = \frac{2}{a} \int_0^a \sin \frac{m\pi x}{a}\, p(x,y)\, dx \qquad (2.12)$$

Substitution of the expansions (2.10a) and (2.11) into Eq. (2.3) yields

$$\sum_{m=1}^{\infty} \left[\frac{d^4 f_m}{dy^4} - 2\left(\frac{m\pi}{a}\right)^2 \frac{d^2 f_m}{dy^2} + \left(\frac{m\pi}{a}\right)^4 f_m - \frac{p_m}{D} \right] \sin \frac{m\pi x}{a} = 0 \qquad (2.13)$$

† Although the form (2.10) is frequently attributed to M. Lévy, Timoshenko and Goodier (1970) point out (p. 53) that the earliest thorough discussion of single Fourier series solutions for rectangular plates appears to be the one by E. Mathieu (1890).

‡ More general boundary conditions can be handled by superposing several Lévy solutions of the form (2.10). The case of completely free edges is discussed by Hagedorn, Kelkel, and Wallaschek (1986).

from which is obtained the set of independent ordinary differential equations

$$\frac{d^4 f_m}{dy^4} - 2\left(\frac{m\pi}{a}\right)^2 \frac{d^2 f_m}{dy^2} + \left(\frac{m\pi}{a}\right)^4 f_m = \frac{p_m}{D} \tag{2.14}$$

The solution to (2.14) can be expressed in terms of particular and complementary parts

$$f_m = f_m^{(p)} + f_m^{(c)} \tag{2.15}$$

where $f_m^{(c)}$ is the solution to the homogeneous equation

$$\frac{d^4 f_m}{dy^4} - 2\left(\frac{m\pi}{a}\right)^2 \frac{d^2 f_m}{dy^2} + \left(\frac{m\pi}{a}\right)^4 f_m = 0 \tag{2.16}$$

and is given by

$$f_m^{(c)} = A'_m e^{m\pi y/a} + B'_m e^{-m\pi y/a} + y\,(C'_m e^{m\pi y/a} + D'_m e^{-m\pi y/a}) \tag{2.17}$$

or equivalently,

$$\begin{aligned} f_m^{(c)} = A_m \cosh\frac{m\pi y}{a} + B_m\, y \sinh\frac{m\pi y}{a} \\ + C_m \sinh\frac{m\pi y}{a} + D_m\, y \cosh\frac{m\pi y}{a} \end{aligned} \tag{2.18}$$

in which A_m, B_m, C_m, and D_m (and primed counterparts) are undetermined constants.

The form of the particular solution $f_m^{(p)}$ depends on the function $p_m(y)$ in Eq. (2.14). For the cases of constant, linear, or exponential variation of p_m with y, the particular solutions are

$$p_m = \begin{cases} A \\ Ay \\ Ae^{\alpha y} \end{cases} ; \qquad f_m^{(p)} = \begin{cases} \frac{A}{D}\left(\frac{a}{m\pi}\right)^4 \\ \frac{Ay}{D}\left(\frac{a}{m\pi}\right)^4 \\ \frac{Ae^{\alpha y}}{D[\alpha^4 - 2\alpha^2(m\pi/a)^2 + (m\pi/a)^4]} \end{cases} \tag{2.19}$$

in which A and α are constant for a given value of m.

Example 2.2: Simply-Supported Rectangular Plate under Uniform Transverse Load

Consider the problem of the previous example: a simply-supported rectangular plate subjected to a uniform transverse load $p(x, y) = p_0$. Due to the problem's symmetry, the Lévy solution is more conveniently obtained in terms of the coordinate system shown in Fig. 2-3. The series representations for the displacement w and the load p are

$$w = \sum_{m=1,3,\ldots}^{m_{max}} f_m \sin\frac{m\pi x}{a}; \qquad p = \sum_{m=1,3,\ldots}^{m_{max}} p_m \sin\frac{m\pi x}{a} \tag{a}$$

where, from Eq. (2.12),

$$p_m = \frac{2p_0}{a}\int_0^a \sin\frac{m\pi x}{a}\,dx = \frac{4p_0}{m\pi} \tag{b}$$

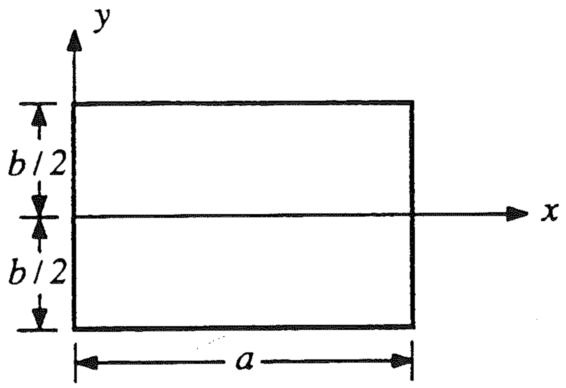

Figure 2-3. Rectangular plate and coordinate system for Example 2.2.

From Eqs. (2.18) and (2.19),

$$\begin{aligned} f_m^{(c)} &= A_m \cosh \frac{m\pi y}{a} + B_m y \sinh \frac{m\pi y}{a} \\ f_m^{(p)} &= \frac{4p_0 a^4}{Dm^5\pi^5} \end{aligned} \qquad (c)$$

In the preceding expression for $f_m^{(c)}$, only the terms of Eq. (2.18) which are symmetric about the x axis are retained. The displacement w, from Eqs. (a), (c), and (2.15), is given by

$$w = \sum_{m=1,3,\ldots}^{m_{max}} \left(A_m \cosh \frac{m\pi y}{a} + B_m y \sinh \frac{m\pi y}{a} + \frac{4p_0 a^4}{Dm^5\pi^5} \right) \sin \frac{m\pi x}{a} \qquad (d)$$

The constants A_m and B_m in Eq. (d) can be determined by enforcing the boundary conditions, which are

$$\text{Along } x = 0, a : \begin{cases} w = 0 \\ M_x = 0 \end{cases}; \qquad \text{Along } y = \pm b/2 : \begin{cases} w = 0 \\ M_y = 0 \end{cases} \qquad (e)$$

Note that the conditions along the edges $x = 0, a$ are automatically satisfied by Eq. (d). The conditions along the edges $y = \pm b/2$ can be expressed in terms of w as

$$w \Big|_{y=\pm\frac{b}{2}} = 0; \qquad \frac{\partial^2 w}{\partial y^2}\Big|_{y=\pm\frac{b}{2}} = 0 \qquad (f)$$

Substitution of Eq. (d) into Eqs. (f) yields the two equations

$$\begin{aligned} A_m \cosh \alpha_m + B_m \frac{b}{2} \sinh \alpha_m &= -\frac{4p_0 a^4}{Dm^5\pi^5} \\ A_m \frac{m\pi}{a} \cosh \alpha_m + B_m(\alpha_m \sinh \alpha_m + 2\cosh \alpha_m) &= 0 \end{aligned} \qquad (g)$$

in which the quantity α_m is defined by

$$\alpha_m = \frac{m\pi b}{2a} \qquad (h)$$

The equations (g) in the two unknowns A_m and B_m are readily solved to obtain

$$\begin{aligned} A_m &= -\frac{4p_0 a^4 + p_0 a^3 b\, m\pi \tanh\alpha_m}{Dm^5\pi^5 \cosh\alpha_m} \\ B_m &= \frac{2p_0 a^3}{Dm^4\pi^4 \cosh\alpha_m} \end{aligned} \qquad (i)$$

The final solution for the displacement w is therefore given by Eq. (d), where A_m and B_m are evaluated from Eqs. (i).

Series expansions for the bending moment, corner force, and transverse shear are obtained by substituting the series (d) into the equations (e, f, g) of Example 2.1. For the case of a square plate ($a = b$ in Fig. 2-3), Table 2-2 lists the maximum displacement $w(a/2, 0)$, the maximum bending moment $M_x(a/2, 0)$, the corner force $F_c(0, -b/2)$, and the maximum transverse edge shear $V_x(0, 0)$, computed from the Lévy solution (d).

Table 2-2. Convergence of Lévy's solution for the maximum displacement w, maximum bending moment M_x, corner force F_c, and maximum transverse edge shear V_x for a simply-supported square plate under uniform load p_0 ($\nu = 0.3$).

No. of Terms	$wD/(p_0a^4)$	$M_x/(p_0a^2)$	$F_c/(p_0a^2)$	$V_x/(p_0a)$
1	0.00411	0.0517	0.0603	0.325
4	0.00406	0.0478	0.0646	0.395
9	0.00406	0.0479	0.0649	0.409
∞	0.00406	0.0479	0.0650	0.420

A comparison of Table 2-2 with Table 2-1 reveals that as the numbers of terms in the series are increased, the Lévy solution converges considerably more rapidly than the Navier solution, especially for the force quantities, which involve derivatives of w. This demonstrates an important computational advantage of the single Fourier series (2.10) over the double Fourier series (2.1).

§2.4 Lévy's Solution for Rectangular Plates under In-plane Edge Loads

Because the governing equation (1.40) for plane stress is of the same form as the biharmonic equation (2.3) for plate bending, the Navier and Lévy solutions can also be applied to plane stress problems of rectangular plates. In this section, we discuss the Lévy-type solution for the rectangular plate under in-plane edge loads. The governing equation in terms of the Airy stress function ϕ is given by Eq. (1.40) with $p_x = p_y = 0$:

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \tag{2.20}$$

The stress function ϕ is taken to have the form

$$\phi(x, y) = \sum_{m=1}^{\infty} f_m(y) \sin \frac{m\pi x}{a} \tag{2.21}$$

which allows the satisfaction of arbitrary boundary conditions along the edges $y = \pm b/2$ (see Fig. 2-3). Along the edges $x = 0$ and $x = a$, however, the form (2.21) satisfies the particular boundary conditions

$$\text{Along } x = 0, a : \begin{cases} \varepsilon_{yy} = 0 \\ N_x = 0 \end{cases} \tag{2.22}$$

which can be verified by subtituting Eq. (2.21) into the definitions (1.39).

Substitution of Eq. (2.21) into Eq. (2.20) results in the same homogeneous equation (2.16) obtained for the plate bending problem, and the functions $f_m(y)$ in Eq. (2.21) are therefore given by Eq. (2.17) or (2.18).

Example 2.3: Shear Lag in a Rectangular Plate with an Inextensional Border

Figure 2-4 depicts a rectangular plate with a uniform tensile load $N_y = N_0$ applied along the edges $y = \pm b/2$. The plate is bordered by an inextensional frame, which prevents the edges from stretching or contracting. The boundary conditions can therefore be written

$$\text{Along } x = 0, a : \begin{cases} \varepsilon_{yy} = 0 \\ N_x = 0 \end{cases}; \qquad \text{Along } y = \pm b/2 : \begin{cases} \varepsilon_{xx} = 0 \\ N_y = N_0 \end{cases} \tag{a}$$

Although N_y is uniform along the edges where the load is applied, it is *not* uniform in the plate's interior. We are interested in determining the distribution of stress within the plate.

The stress function $\phi(x, y)$, from Eq. (2.21), is given by

$$\phi = \sum_{m=1,3,\ldots}^{m_{max}} \left(A_m \cosh \frac{m\pi y}{a} + B_m y \sinh \frac{m\pi y}{a} \right) \sin \frac{m\pi x}{a} \tag{b}$$

in which only the symmetric terms of $f_m(y)$, given by Eq. (2.18), and the symmetric terms in x, which correspond to $m = 1, 3, \ldots$, are retained. Equation (b) automatically satisfies the boundary conditions along $x = 0, a$.

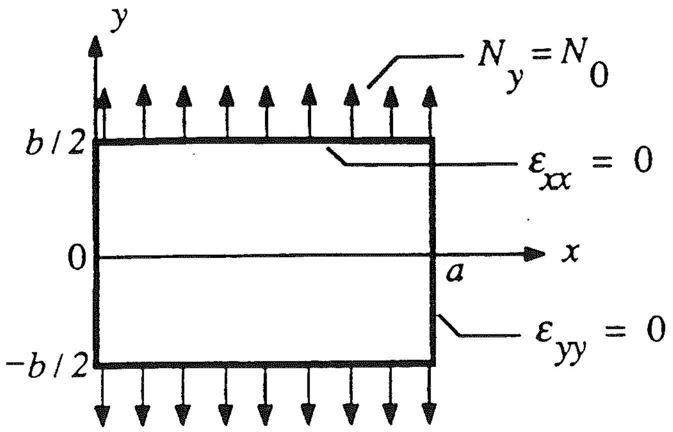

Figure 2-4. Plane stress problem of a rectangular plate bordered by an inextensional frame. A uniform tensile load $N_y = N_0$ is applied along the edges $y = \pm b/2$.

By means of the equations (1.39) and (1.23a), the boundary conditions (a) along $y = \pm b/2$ can be expressed in terms of ϕ as

$$\left(\frac{\partial^2\phi}{\partial y^2} - \nu\frac{\partial^2\phi}{\partial x^2}\right)\Bigg|_{y=\pm\frac{b}{2}} = 0 \qquad\qquad (c)$$
$$\frac{\partial^2\phi}{\partial x^2}\Bigg|_{y=\pm\frac{b}{2}} = N_0$$

Substituting the series (b) into the first of Eqs. (c) and setting the coefficients of each term in $\sin m\pi x/a$ equal to zero yields

$$A_m(1+\nu)\frac{m\pi}{a}\cosh\alpha_m + B_m[(1+\nu)\alpha_m\sinh\alpha_m + 2\cosh\alpha_m] = 0 \qquad (d)$$

in which α_m is the same parameter defined by Eq. (h) of Example 2.2.

Substituting the series (b) into the second of Eqs. (c), and employing the orthogonality condition (2.6) yields

$$-(A_m\cosh\alpha_m + B_m\frac{b}{2}\sinh\alpha_m)\frac{m^2\pi^2}{2a} = \int_0^a N_0\sin\frac{m\pi x}{a}dx \qquad (e)$$

which reduces to

$$A_m\frac{m\pi}{a}\cosh\alpha_m + B_m\alpha_m\sinh\alpha_m = -\frac{4aN_0}{m^2\pi^2} \qquad (f)$$

for odd values of m.

The two equations (d) and (f) in the two unknowns A_m and B_m are solved to obtain

$$A_m = -\frac{2a^2N_0[(1+\nu)\alpha_m\sinh\alpha_m + 2\cosh\alpha_m]}{(m\pi)^3\cosh^2\alpha_m} \qquad\qquad (g)$$
$$B_m = \frac{2(1+\nu)aN_0}{(m\pi)^2\cosh\alpha_m}$$

The solution for the stress function ϕ is therefore given by Eq. (b), where A_m and B_m are evaluated from Eqs. (g).

Series expansions for the stress resultants are obtained by substituting the series (b) into the definitions (1.39). Figure 2-5a shows the stress resultant N_y along the line $y = 0$, for four different values of the plate aspect ratio b/a. As the aspect ratio increases, the relative magnitude of the stress N_y at the center of the plate decreases. This is called "shear lag," because the load is transferred by shear stresses along the edges $x = 0, a$. For small values of b/a, the distribution of N_y is more uniform, with N_y/N_0 closer to unity (see Fig. 2-5a).

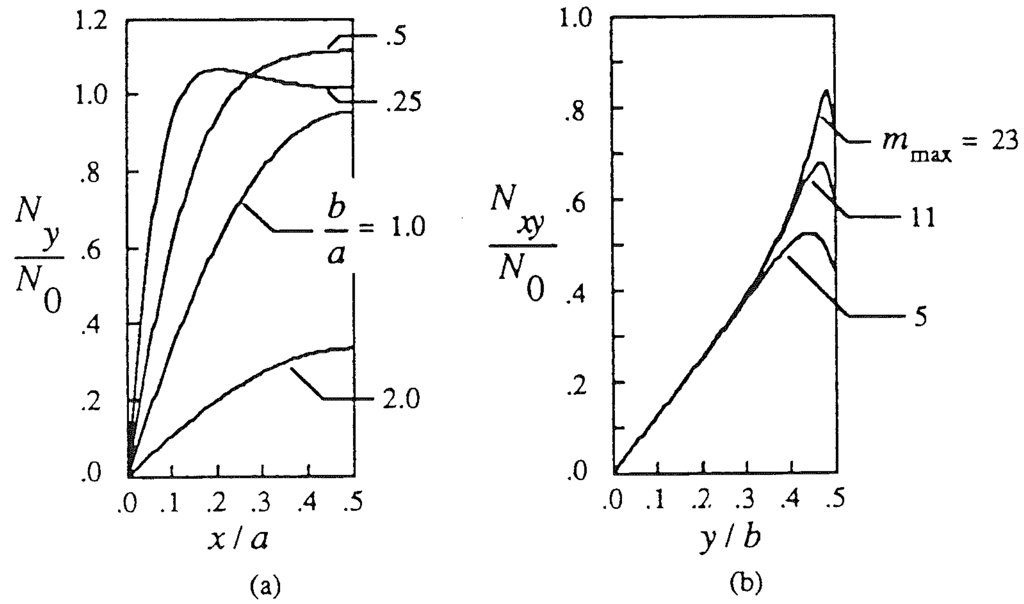

Figure 2-5. Stress resultants in a plate with an inextensional border and applied edge loads as shown in Fig. 2-4. (a) Converged values of the stress resultant $N_y(x, 0)$ for $0 \leq x/a \leq 0.5$ and aspect ratios $b/a = 0.25$, 0.5, 1.0, and 2.0. (b) Shear stress resultant $N_{xy}(0, y)$ for $0 \leq y/b \leq 0.5$, $b/a = 1.0$. The Levy solutions for $m_{max} = 5$, 11, and 23 exhibit nonuniform convergence. The poor convergence in the neighborhood of $y = b/2$ is indicative of a stress singularity at the corner of the plate. (Results shown are for $\nu = 0.3$.)

Figure 2-5b shows the shear resultant N_{xy} along the edge $x = 0$, for the case of a square plate ($b/a = 1$). The shear stress is zero at $y = 0$, and grows to a peak value at the corner. Note that the shear resultants given by 3 terms, 6 terms, and 12 terms exhibit nonuniform convergence, with very slow convergence near the corner $y = b/2$. This is indicative of a stress singularity at the corner. Because the plane stress theory used here is valid only for small strains, it is not accurate at the corners of the plate, where large strains and stresses occur.

Chapter 3

Circular Plates

§3.1 Introduction

Circular plates are common elements of axially symmetric structures such as containment vessels and rotating machinery. Some examples of circular plates are computer disks, clutch plates, disk brakes, manhole covers, and the floors of cylindrical storage tanks. For the analysis of a circular or annular plate, it is convenient to exploit the symmetry by employing polar coordinates as shown in Fig. 3-1. The polar coordinates (r, θ) are related to the Cartesian coordinates (x, y) by the equations

$$\begin{aligned} x &= r\cos\theta \qquad & r &= \sqrt{x^2+y^2} \\ y &= r\sin\theta \qquad & \theta &= \tan^{-1}\frac{y}{x}. \end{aligned} \tag{3.1}$$

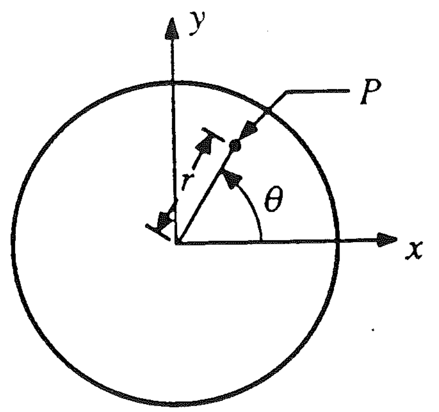

Figure 3-1. Circular plate and polar coordinate system.

In this chapter, the plate equations are transformed from the Cartesian form of Chapter 1 to polar form. It is important to keep in mind, however, that the states of deformation and stress in a plate are physical phenomena which are independent of the coordinate system employed. Some vector notation will be introduced in order to illustrate and exploit this coordinate system invariance. To begin, we consider the harmonic, or Laplacian, differential operator which appears in Eqs. (1.40) and (1.46).

§3.2 The Gradient and the Laplacian of a Scalar Function

Consider the two points P_1 and P_2 in Fig. 3-2. The position vector ds of P_2 relative to P_1 is given by

$$ds = \mathbf{r}_2 - \mathbf{r}_1 \tag{3.2}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ represent the vectors which connect the origin to the points P_1 and P_2, respectively. The vector ds can also be expressed

$$ds = dx\,\mathbf{e}_x + dy\,\mathbf{e}_y \tag{3.3}$$

where (x, y) and $(x + dx, y + dy)$ are the Cartesian coordinates of P_1 and P_2, and the vectors $\mathbf{e}_x$ and $\mathbf{e}_y$, which satisfy

$$\mathbf{e}_x \cdot \mathbf{e}_x = \mathbf{e}_y \cdot \mathbf{e}_y = 1; \qquad \mathbf{e}_x \cdot \mathbf{e}_y = 0 \tag{3.4}$$

are the orthonormal basis vectors for the Cartesian coordinate system.

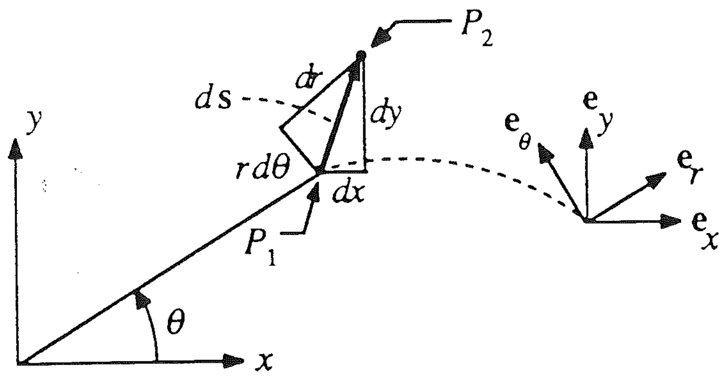

Figure 3-2. The position vector ds of P_2 relative to P_1 has the components (dx, dy) in the local Cartesian coordinate system, but has the components $(dr, rd\theta)$ in the local polar coordinate system.

In the polar coordinate system, the coordinates of P_1 and P_2 are (r, θ) and $(r + dr, \theta + d\theta)$, and the orthonormal basis vectors in the r and θ directions are denoted by $\mathbf{e}_r$ and $\mathbf{e}_\theta$. Note that $\mathbf{e}_r$ and $\mathbf{e}_\theta$, unlike $\mathbf{e}_x$ and $\mathbf{e}_y$, are functions of position (see Fig. 3-2). In terms of the basis vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ at the point P_1, the vector ds can be expressed

$$ds = dr\,\mathbf{e}_r + rd\theta\,\mathbf{e}_\theta \tag{3.5}$$

where quadratic terms in dr and $d\theta$ are neglected because P_2 is assumed to be infinitesimally close to P_1.

Now consider a scalar function $F(x,y)$. The change in F in moving from P_1 to P_2 is

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy \tag{3.6}$$

which, in terms of the Cartesian basis vectors, can be written

$$dF = d\mathbf{s} \cdot \left(\mathbf{e}_x \frac{\partial F}{\partial x} + \mathbf{e}_y \frac{\partial F}{\partial y} \right) \tag{3.7}$$

where $d\mathbf{s}$ is given by Eq. (3.3).

In terms of the polar coordinates, the change in F in moving from P_1 to P_2 is

$$\begin{aligned} dF &= \frac{\partial F}{\partial r}dr + \frac{\partial F}{\partial \theta}d\theta \\ &= d\mathbf{s} \cdot \left(\mathbf{e}_r \frac{\partial F}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial F}{\partial \theta} \right) \end{aligned} \tag{3.8}$$

where $d\mathbf{s}$ is given by Eq. (3.5)

The vector $d\mathbf{s}$ and the change dF in the function F depend only on the locations of P_1 and P_2 — not on the coordinate system. The equations (3.7) and (3.8) are therefore equivalent to one another, and can be rewritten

$$dF = d\mathbf{s} \cdot \nabla F \tag{3.9}$$

in which ∇F, the *gradient* of F, is the "rate of change vector." Equation (3.9) is a vector equation, and is completely independent of any particular coordinate system.

In terms of Cartesian or polar basis vectors, the vector ∇F takes the explicit forms

$$\nabla F = \mathbf{e}_x \frac{\partial F}{\partial x} + \mathbf{e}_y \frac{\partial F}{\partial y} = \left(\mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y} \right) F \tag{3.10a}$$

$$= \mathbf{e}_r \frac{\partial F}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial F}{\partial \theta} = \left(\mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} \right) F \tag{3.10b}$$

The coefficients of the vectors $\mathbf{e}_x$ and $\mathbf{e}_y$ in Eq. (3.10a) are the *Cartesian components* of the gradient, while the coefficients of the vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ in Eq. (3.10b) are the *polar components*.

In Cartesian coordinates, the harmonic, or Laplacian, differential operator Δ is given by

$$\Delta F = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) F \tag{3.11}$$

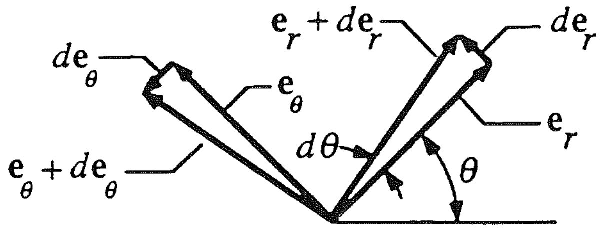

Figure 3-3. Change in orientation of the unit basis vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ in moving from a point with coordinates (r, θ) to a point with coordinates $(r + dr, \theta + d\theta)$.

where F is a scalar function of x and y. By means of the representation (3.10a) for the gradient operator, Eq. (3.11) can be written in vector form as follows:

$$\Delta F = \left(\mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y}\right) \cdot \left(\mathbf{e}_x \frac{\partial F}{\partial x} + \mathbf{e}_y \frac{\partial F}{\partial y}\right) \tag{3.12a}$$

$$= \nabla \cdot \nabla F \tag{3.12b}$$

The Cartesian basis vectors $\mathbf{e}_x$ and $\mathbf{e}_y$ are the same at any point (x, y); these vectors can therefore be treated as constants under the differentiations in Eq. (3.12a).

The vector representation (3.12b) for the Laplacian operator can be expressed in terms of polar coordinates by means of Eq. (3.10b):

$$\Delta F = \left(\mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}\right) \cdot \left(\mathbf{e}_r \frac{\partial F}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial F}{\partial \theta}\right) \tag{3.13}$$

Note, however, that in carrying out the differentiations in Eq. (3.13), the dependencies of the basis vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ on r and θ must be accounted for.

In moving from a point (r, θ) to a point $(r+dr, \theta+d\theta)$, the basis vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ undergo a rotation $d\theta$ (see Fig. 3-3). Figure 3-3 reveals that for $d\theta$ infinitesmally small, the changes $d\mathbf{e}_r$ and $d\mathbf{e}_\theta$ in the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ have magnitudes equal to $d\theta$, and are given by

$$\begin{aligned} d\mathbf{e}_r &= \mathbf{e}_\theta d\theta \\ d\mathbf{e}_\theta &= -\mathbf{e}_r d\theta \end{aligned} \tag{3.14}$$

which yields

$$\begin{aligned} \frac{\partial \mathbf{e}_r}{\partial \theta} &= \mathbf{e}_\theta \\ \frac{\partial \mathbf{e}_\theta}{\partial \theta} &= -\mathbf{e}_r \\ \frac{\partial \mathbf{e}_r}{\partial r} = \frac{\partial \mathbf{e}_\theta}{\partial r} &= 0 \end{aligned} \tag{3.15}$$

Equation (3.13) can now be expanded and evaluated:

$$\begin{aligned}\Delta F &= \mathbf{e}_r \cdot \frac{\partial}{\partial r}\left(\mathbf{e}_r \frac{\partial F}{\partial r}\right) + \mathbf{e}_r \cdot \frac{\partial}{\partial r}\left(\mathbf{e}_\theta \frac{1}{r}\frac{\partial F}{\partial \theta}\right) \\ &\quad + \mathbf{e}_\theta \cdot \frac{1}{r}\frac{\partial}{\partial \theta}\left(\mathbf{e}_r \frac{\partial F}{\partial r}\right) + \mathbf{e}_\theta \cdot \frac{1}{r}\frac{\partial}{\partial \theta}\left(\mathbf{e}_\theta \frac{1}{r}\frac{\partial F}{\partial \theta}\right) \qquad (3.16a) \\ &= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) F \qquad (3.16b)\end{aligned}$$

where Eqs. (3.15) and the orthogonality relations

$$\mathbf{e}_r \cdot \mathbf{e}_r = \mathbf{e}_\theta \cdot \mathbf{e}_\theta = 1; \qquad \mathbf{e}_r \cdot \mathbf{e}_\theta = 0 \qquad (3.17)$$

have been employed in arriving at Eq. (3.16b), which is the representation of the Laplacian differential operator (3.12b) in terms of polar coordinates.

§3.3 Plate Bending in Polar Coordinates

The governing equation (1.46) for plate bending, in the absence of distributed in-plane moment loads ($m_x = m_y = 0$), is the biharmonic equation

$$\Delta\Delta w = \frac{p}{D} \qquad (3.18)$$

where w is the transverse displacement of the plate midplane, and p is the transverse load. Substitution of the expression (3.16b) for Δ yields

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) = \frac{p}{D} \qquad (3.19)$$

which is the governing equation for plate bending in terms of the polar coordinates (r, θ).

The midplane curvatures can be expressed in terms of r and θ by transforming the kinematic relations (1.21). At the point P, located at (r_P, θ_P) as shown in Fig. 3-4, the basis $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ is aligned with the rotated Cartesian coordinate system $(\bar{x}, \bar{y})$. The coordinates $(\bar{x}, \bar{y})$ are related to the polar coordinates (r, θ) by the equations

$$\begin{aligned}\bar{x} &= r\cos(\theta - \theta_P) \qquad & r &= \sqrt{\bar{x}^2 + \bar{y}^2} \\ \bar{y} &= r\sin(\theta - \theta_P) \qquad & \tan(\theta - \theta_P) &= \tfrac{\bar{y}}{\bar{x}}\end{aligned} \qquad (3.20)$$

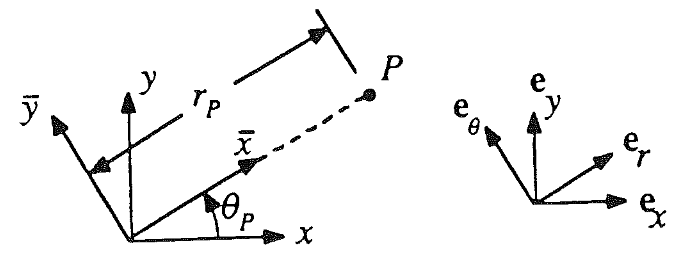

Figure 3-4. Rotated Cartesian coordinates $\bar{x}$ and $\bar{y}$, and point P, located at (r_P, θ_P).

The curvatures and twist of the plate midplane at P are

$$\kappa_r = \kappa_{\bar{x}} = -\left.\frac{\partial^2 w}{\partial \bar{x}^2}\right|_P \tag{3.21a}$$

$$\kappa_\theta = \kappa_{\bar{y}} = -\left.\frac{\partial^2 w}{\partial \bar{y}^2}\right|_P \tag{3.21b}$$

$$\kappa_{r\theta} = \kappa_{\bar{x}\bar{y}} = -\left.\frac{\partial^2 w}{\partial \bar{x}\partial \bar{y}}\right|_P \tag{3.21c}$$

which can be written in terms of r and θ by means of the transformation equations

$$\frac{\partial}{\partial \bar{x}} = \frac{\partial r}{\partial \bar{x}}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial \bar{x}}\frac{\partial}{\partial \theta} \quad ; \quad \frac{\partial}{\partial \bar{y}} = \frac{\partial r}{\partial \bar{y}}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial \bar{y}}\frac{\partial}{\partial \theta} \tag{3.22a, b}$$

From the relations (3.20), the derivatives in Eq. (3.22) are

$$\begin{aligned} \frac{\partial r}{\partial \bar{x}} &= \cos(\theta - \theta_P) & \frac{\partial \theta}{\partial \bar{x}} &= -\frac{\sin(\theta - \theta_P)}{r} \\ \frac{\partial r}{\partial \bar{y}} &= \sin(\theta - \theta_P) & \frac{\partial \theta}{\partial \bar{y}} &= \frac{\cos(\theta - \theta_P)}{r} \end{aligned} \tag{3.23}$$

The equations (3.21) can now be expressed in terms of r and θ by substituting Eqs. (3.22) and (3.23), carrying out the derivatives, and evaluating at (r_P, θ_P). The result, valid at any point (r, θ), is

$$\kappa_r = -\frac{\partial^2 w}{\partial r^2} \tag{3.24a}$$

$$\kappa_\theta = -\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} - \frac{1}{r}\frac{\partial w}{\partial r} \tag{3.24b}$$

$$\kappa_{r\theta} = \kappa_{\theta r} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right) \tag{3.24c}$$

The directions of positive action for the bending and twisting moment resultants M_r, M_θ, $M_{r\theta}$, and $M_{\theta r}$ are shown in Fig. 3-5. For isotropic material behavior, the relationship between the moment resultants M_r, M_θ, $M_{r\theta}$ and the midplane curvatures κ_r, κ_θ, $\kappa_{r\theta}$ is

$$\begin{Bmatrix} M_r \\ M_\theta \\ M_{r\theta} \end{Bmatrix} = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} \kappa_r \\ \kappa_\theta \\ \kappa_{r\theta} \end{Bmatrix} \tag{3.25}$$

Equation (3.25) follows directly from Eq. (1.43) for Cartesian coordinates, because the constitutive relations for an isotropic material do not depend on the material's orientation.

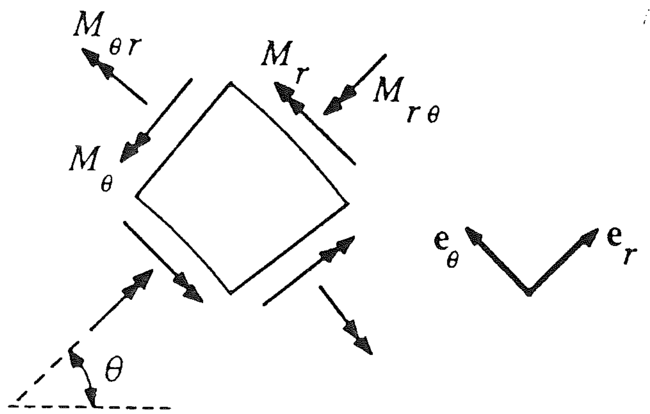

Figure 3-5. Moment resultants M_r, M_θ, $M_{r\theta}$, and $M_{\theta r}$.

In order to obtain expressions for the transverse shear resultants in terms of w, we note that Eqs. (1.7d), (1.21) and (1.43) provide

$$\begin{aligned} Q_x &= \frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} + m_x \\ &= D\left[\frac{\partial \kappa_x}{\partial x} + \nu\frac{\partial \kappa_y}{\partial x} + (1-\nu)\frac{\partial \kappa_{xy}}{\partial y}\right] + m_x \\ &= -D\frac{\partial}{\partial x}(\Delta w) + m_x \end{aligned} \tag{3.26a}$$

and similarly

$$Q_y = -D\frac{\partial}{\partial y}(\Delta w) + m_y \tag{3.26b}$$

Relative to the rotated Cartesian coordinates $(\bar{x}, \bar{y})$ shown in Fig. 3-5, the transverse shear resultants at P are therefore

$$Q_r = Q_{\bar{x}} = -D\frac{\partial}{\partial \bar{x}}(\Delta w) + m_r \tag{3.27a}$$

$$Q_\theta = Q_{\bar{y}} = -D\frac{\partial}{\partial \bar{y}}(\Delta w) + m_\theta \tag{3.27b}$$

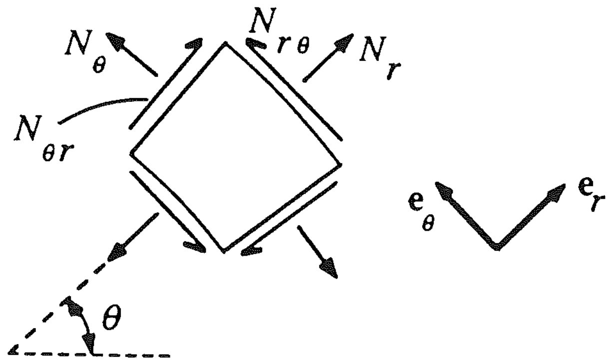

Figure 3-6. Stress resultants N_r, N_θ, $N_{r\theta}$, and $N_{\theta r}$.

where m_r and m_θ are the distributed moment loads which act in the $\mathbf{e}_\theta$ and $-\mathbf{e}_r$ directions. The transformation relations (3.22) then yield

$$Q_r = -D\frac{\partial}{\partial r}(\Delta w) + m_r \tag{3.28a}$$

$$Q_\theta = -D\frac{1}{r}\frac{\partial}{\partial \theta}(\Delta w) + m_\theta \tag{3.28b}$$

which are the transverse shear resultants which act on edges with normals $\mathbf{e}_r$ and $\mathbf{e}_\theta$.

In addition, it follows from Eq. (1.59) that

$$V_r = Q_r + \frac{1}{r}\frac{\partial M_{r\theta}}{\partial \theta} \tag{3.29a}$$

$$V_\theta = Q_\theta + \frac{\partial M_{r\theta}}{\partial r} \tag{3.29b}$$

where V_r and V_θ are the Kirchhoff shear resultants, which include the effective transverse shear due to the twisting moment $M_{r\theta}$.

§3.4 Stresses Relative to Polar Coordinates

The directions of positive action for the in-plane stress resultants N_r, N_θ, $N_{r\theta}$, and $N_{\theta r}$ are shown in Fig. 3-6. Further discussion of the plane stress problem is deferred to §3.9. As discussed in Chapter 1, the in-plane resultants and moment resultants are associated respectively with direct stresses and bending stresses. In terms of polar components, the stresses which act parallel to the plate midplane are

$$\begin{Bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{r\theta} \end{Bmatrix} = \frac{1}{t}\begin{Bmatrix} N_r \\ N_\theta \\ N_{r\theta} \end{Bmatrix} + \frac{12z}{t^3}\begin{Bmatrix} M_r \\ M_\theta \\ M_{r\theta} \end{Bmatrix} \tag{3.30}$$

where σ_{rr} and $\sigma_{\theta\theta}$ are the stresses which act in the r and θ directions, on surfaces with normals $\mathbf{e}_r$ and $\mathbf{e}_\theta$, and $\sigma_{r\theta}$ is the shear stress which acts in the θ direction, on a surface whose normal is $\mathbf{e}_r$. Note that Eq. (3.30) is identical in form to Eq. (1.25).

If there are no tangential shear tractions acting on the plate's surfaces, the transverse shear stresses vary parabolically through the thickness, and are related to the stress resultants Q_r and Q_θ by

$$\left\{ \begin{matrix} \sigma_{rz} \\ \sigma_{\theta z} \end{matrix} \right\} = \left\{ \begin{matrix} Q_r \\ Q_\theta \end{matrix} \right\} \frac{3}{2t} \left[1 - \left(\frac{2z}{t} \right)^2 \right] \tag{3.31}$$

which is identical in form to Eqs. (1.32).

As in the case of Cartesian coordinates, equilibrium requires that

$$\sigma_{r\theta} = \sigma_{\theta r} \quad ; \quad \sigma_{rz} = \sigma_{zr} \quad ; \quad \sigma_{\theta z} = \sigma_{z\theta} \tag{3.32a, b, c}$$

and

$$N_{r\theta} = N_{\theta r} \quad ; \quad M_{r\theta} = M_{\theta r} \tag{3.33a, b}$$

§3.5 Circular Plates under Transverse Load

For the plate bending problem of a circular or annular plate subjected to a transverse load $p_z = p(r, \theta)$, the solution to Eq. (3.19) can be expressed

$$w = w^{(p)} + w^{(c)} \tag{3.34}$$

where $w^{(p)}$ is a *particular solution* to Eq. (3.19) and $w^{(c)}$ is a *complementary solution*, which satisfies the homogeneous equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right) \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} \right) = 0 \tag{3.35}$$

The particular solution $w^{(p)}$ satisfies Eq. (3.19), but does not in general satisfy the boundary conditions at the edges of the plate. A complementary solution $w^{(c)}$ must be found such that the total solution w satisfies the boundary conditions.

Because Eq. (3.35) is homogeneous, the complementary solution corresponds to a distribution of moment and shear loads *applied to the plate's edges*. From a physical standpoint, the plate can be pictured initially as carrying the transverse load in the state described by the particular solution. The edge loading of the complementary solution is then superposed in order to bend the plate's edges into the configuration required to satisfy the boundary conditions.

3.5.1 Axisymmetric Problems

If the ~~the~~ applied load and support conditions are independent of θ, it can be assumed that the displacement w of a circular or annular plate will also be independent of θ. The plate bending equation (3.19) thus reduces to the ordinary differential equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right) = \frac{p}{D} \tag{3.36}$$

which can be expressed

$$\frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]\right\} = \frac{p}{D} \tag{3.37}$$

Equation (3.37) can be integrated directly to obtain

$$\begin{aligned} w(r) &= w^{(p)}(r) + w^{(c)}(r) \\ &= \int \frac{1}{r} \int r \int \frac{1}{r} \int \frac{r\, p(r)}{D}\, dr\, dr\, dr\, dr \\ &\quad + A_0 + B_0 \ln r + C_0 r^2 + D_0 r^2 \ln r \end{aligned} \tag{3.38}$$

where A_0, B_0, C_0, and D_0 are arbitrary constants of integration.

The first term on the right-hand side of Eq. (3.38) is the particular solution, which depends on the load $p(r)$, and the last four terms are the complementary solution for the axisymmetric problem. The solution (3.38) can be evaluated for any given axisymmetric load distribution $p(r)$. The four constants A_0, B_0, C_0, and D_0 are evaluated by satisfying four boundary conditions. In the case of an annular plate, two conditions are applied at the inner edge and two conditions are applied at the outer edge.

Example 3.1: Simply-Supported Circular Plate under Uniform Transverse Load

Consider a simply-supported circular plate, subjected to a uniform load $p(r) = p_0$, as shown in Fig. 3-7a. The boundary conditions are

$$w = M_r = 0 \qquad (\text{At } r = a) \tag{a}$$

The moment resultant M_r can be expressed in terms of w by means of the constitutive relations (3.25):

$$\begin{aligned} M_r &= D(\kappa_r + \nu\kappa_\theta) \\ &= -D\left[\frac{\partial^2 w}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right] \\ &= -D\left(\frac{d^2 w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right) \end{aligned} \tag{b}$$

Thus the boundary conditions (a) become

$$\text{At } r = a: \begin{cases} w = 0 \\ \dfrac{d^2 w}{dr^2} + \dfrac{\nu}{r}\dfrac{dw}{dr} = 0 \end{cases} \tag{c}$$

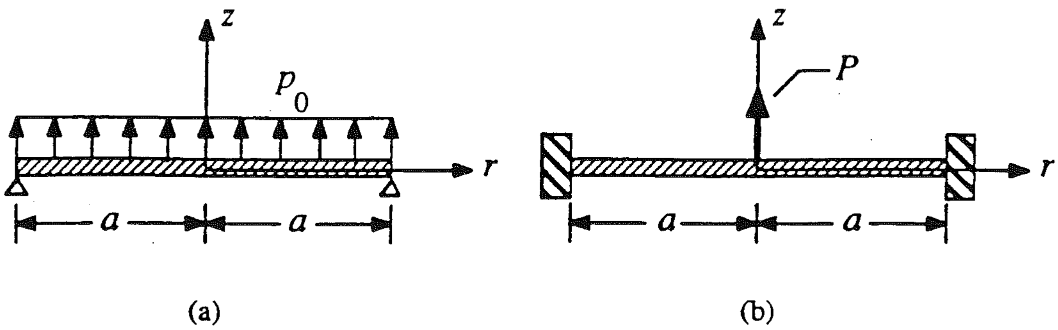

Figure 3-7. (a) Simply-supported circular plate subjected to a uniform load $p(r) = p_0$. (b) Clamped circular plate subjected to a concentrated load P at the center $r = 0$.

The displacement w can be calculated by substituting $p(r) = p_0$ into Eq. (3.38) and integrating, which yields

$$w = \frac{p_0 r^4}{64D} + A_0 + B_0 \ln r + C_0 r^2 + D_0 r^2 \ln r \qquad (d)$$

It can be seen that the term in $\ln r$ in Eq. (d) leads to an infinite displacement at $r = 0$, while the term in $r^2 \ln r$ leads to infinite stresses. For the uniform load under consideration, however, it is clear that the displacement and stresses at $r = 0$ must be finite. We therefore deduce the *regularity conditions*

$$B_0 = D_0 = 0 \qquad (e)$$

which can be thought of as "boundary conditions" at $r = 0$.

By substituting Eq. (d) into the boundary conditions (c), the two equations

$$\begin{aligned} \frac{p_0 a^4}{64D} + A_0 + C_0 a^2 &= 0 \\ (3+\nu)\frac{p_0 a^2}{16D} + 2C_0(1+\nu) &= 0 \end{aligned} \qquad (f)$$

in the two unknowns A_0 and C_0 are obtained; thus

$$A_0 = \frac{p_0 a^4}{64D}\left(\frac{5+\nu}{1+\nu}\right) \quad ; \quad C_0 = -\frac{p_0 a^2}{32D}\left(\frac{3+\nu}{1+\nu}\right) \qquad (g)$$

and from Eq. (d) and the results (e, g):

$$w(r) = \frac{p_0 a^4}{64D}\left[\left(\frac{r}{a}\right)^4 - 2\left(\frac{3+\nu}{1+\nu}\right)\left(\frac{r}{a}\right)^2 + \frac{5+\nu}{1+\nu}\right] \qquad (h)$$

The bending stresses, calculated from Eq. (3.30), are:

$$\sigma_{rr}(r) = \frac{12zM_r}{t^3} = -\frac{12zD}{t^3}\left(\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right)$$

$$= \frac{3p_0}{8}\frac{2z}{t}\left(\frac{a}{t}\right)^2 (3+\nu)\left[1-\left(\frac{r}{a}\right)^2\right] \qquad (i)$$

$$\sigma_{\theta\theta}(r) = \frac{12zM_\theta}{t^3} = -\frac{12zD}{t^3}\left(\frac{1}{r}\frac{dw}{dr}+\nu\frac{d^2w}{dr^2}\right)$$

$$= \frac{3p_0}{8}\frac{2z}{t}\left(\frac{a}{t}\right)^2\left[(3+\nu)-(1+3\nu)\left(\frac{r}{a}\right)^2\right] \qquad (j)$$

The maximum displacement and stresses, which occur at the center of the plate ($r=0$) are therefore:

$$w(0) = \frac{p_0 a^4}{64D}\left(\frac{5+\nu}{1+\nu}\right) \quad ; \quad \sigma_{rr}(0) = \sigma_{\theta\theta}(0) = \frac{3(3+\nu)p_0}{8}\left(\frac{a}{t}\right)^2 \qquad (k)$$

Example 3.2: Circular Plate with a Concentrated Load at the Center

A circular plate is subjected to a concentrated load P at the center $r=0$, as shown in Fig. 3-7b. The edges of the plate are built in to a rigid support, which implies

$$w = \frac{dw}{dr} = 0 \qquad (\text{At } r=a) \qquad (a)$$

For equilibrium in the z direction, the transverse shear V_r must satisfy

$$V_r = -\frac{P}{2\pi r} \qquad (b)$$

and therefore approaches infinity near the concentrated load at $r=0$.

From Eq. (3.38),

$$w = w^{(p)} + A_0 + B_0\ln r + C_0 r^2 + D_0 r^2 \ln r \qquad (c)$$

The term in $\ln r$ in Eq. (c) yields an infinite displacement at $r=0$, which requires that

$$B_0 = 0 \qquad (d)$$

With $w^{(p)} = 0$ as a particular solution, Eq. (c) becomes

$$w = A_0 + C_0 r^2 + D_0 r^2 \ln r \qquad (e)$$

where the three constants A_0, C_0, and D_0 can be evaluated from the three conditions (a, b).

From Eqs. (3.28a) and (3.29a),

$$V_r = -D\frac{d}{dr}\left(\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}\right) \qquad (f)$$

which can be substituted into the equilibrium condition (b), which becomes

$$D\frac{d}{dr}\left(\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}\right) = \frac{P}{2\pi r} \qquad (g)$$

Substituting Eq. (e) into Eq. (g) and simplifying now yields

$$D_0 = \frac{P}{8\pi D} \qquad (h)$$

which shows that the singular term $r^2 \ln r$ in Eqs. (c) and (e) corresponds to a concentrated load applied at the plate's center.

Substitution of Eq. (e) into the two conditions (a), with D_0 given by Eq. (h), results in

$$\begin{aligned} A_0 + C_0 a^2 + \frac{Pa^2 \ln a}{8\pi D} &= 0 \\ 2C_0 + \frac{P}{8\pi D}(2\ln a + 1) &= 0 \end{aligned} \qquad (i)$$

which are solved to obtain

$$A_0 = \frac{Pa^2}{16\pi D} \quad ; \quad C_0 = -\frac{P}{16\pi D}(2\ln a + 1) \qquad (j)$$

With the results (h) and (j), Eq. (e) can now be written

$$w(r) = \frac{Pa^2}{16\pi D}\left\{1-\left(\frac{r}{a}\right)^2\left[1-2\ln\left(\frac{r}{a}\right)\right]\right\} \qquad (k)$$

The maximum displacement, at the center of the plate, is

$$w(0) = \frac{Pa^2}{16\pi D} \qquad (l)$$

3.5.2 General Solution by Fourier Series

For a general deformation with radial and circumferential dependence, the displacement $w(r,\theta)$ must be periodic in θ. The complementary solution $w^{(c)}$ can therefore be taken in the Fourier series form

$$w^{(c)}(r,\theta) = \sum_{m=0}^{\infty} f_m(r)\cos m\theta + \sum_{m=1}^{\infty} g_m(r)\sin m\theta \qquad (3.39)$$

where $f_m(r)$ and $g_m(r)$ are functions to be determined.

Substituting Eq. (3.39) into Eq. (3.35) and equating the coefficient of each term in $\cos m\theta$ to zero yields the following set of ordinary differential equations for the functions f_m:

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}\right)\left(\frac{d^2 f_m}{dr^2}+\frac{1}{r}\frac{df_m}{dr}-\frac{m^2 f_m}{r^2}\right) = 0 \qquad (3.40)$$

An identical equation for the functions g_m is obtained by equating the coefficient of each term in $\sin m\theta$ to zero.

Equation (3.40) may be solved by taking the function f_m in the form

$$f_m(r) = Ar^{\alpha} \qquad (3.41)$$

where A is an arbitrary constant. To determine α, Eq. (3.41) is substituted into Eq. (3.40), which then becomes

$$A(\alpha^2 - m^2)[(\alpha-2)^2 - m^2]\, r^{\alpha-4} = 0 \qquad (3.42)$$

For $m > 1$, Eq. (3.42) has the four distinct roots

$$\alpha_1 = m \quad ; \quad \alpha_2 = -m \qquad (3.43a,b)$$

$$\alpha_3 = 2+m \quad ; \quad \alpha_4 = 2-m \qquad (3.43c,d)$$

which correspond to four independent solutions of the form (3.41).

For $m = 0$ and $m = 1$, however, the four roots (3.43) are not distinct. In order to investigate these cases of repeated roots, the left-hand side of Eq. (3.42) can be represented by $\mathcal{L}(f_m)$ where $\mathcal{L}$ is the fourth-order differential operator of Eq. (3.40) and f_m is given by Eq. (3.41). If α^* is a repeated root of Eq. (3.42), it follows that

$$\frac{\partial}{\partial\alpha}[\mathcal{L}(f_m)]\,\Big|_{\alpha=\alpha^*} = 0 \qquad (3.44)$$

which, due to the linearity of the differential operator $\mathcal{L}$, can be written

$$\mathcal{L}\left(\frac{\partial f_m}{\partial\alpha}\Big|_{\alpha=\alpha^*}\right) = 0 \qquad (3.45)$$

We thus have an additional solution f_m^* of the form

$$f_m^* = \frac{\partial f_m}{\partial\alpha}\Big|_{\alpha=\alpha^*} = Ar^{\alpha^*}\ln r = f_m \ln r \qquad (3.46)$$

for each repeated root α^*.

The functions f_m which satisfy Eq. (3.40) are therefore given by

$$f_0 = A_0 + B_0 \ln r + C_0 r^2 + D_0 r^2 \ln r \quad (3.47a)$$
$$f_1 = A_1 r + B_1 r^{-1} + C_1 r^3 + D_1 r \ln r \quad (3.47b)$$
$$f_m = A_m r^m + B_m r^{-m} + C_m r^{2+m} + D_m r^{2-m} \quad (m > 1) \quad (3.47c)$$

where A_m, B_m, C_m, and D_m are arbitrary constants. Note that $m = 0$ corresponds to the axisymmetric case. The functions g_m in Eq. (3.39) must satisfy the same equation (3.40) as the f_m, and hence have the forms

$$g_1 = A'_1 r + B'_1 r^{-1} + C'_1 r^3 + D'_1 r \ln r \quad (3.48a)$$
$$g_m = A'_m r^m + B'_m r^{-m} + C'_m r^{m+2} + D'_m r^{-m+2} \quad (m > 1) \quad (3.48b)$$

For each value of m, the four arbitrary constants in the functions (3.47) or (3.48) are determined by four boundary conditions for the m'th Fourier harmonic ($\cos m\theta$ or $\sin m\theta$ term).

§3.6 Stiffness Matrix for Edge Bending of a Circular Plate

The general Fourier series solutions presented in the preceding section can be used to derive a *stiffness matrix* which relates circumferential harmonics of displacement and rotation of the edge of a circular plate to harmonics of edge shear and moment. For a circular plate of radius a, the edge rotation, transverse displacement, edge moment, and transverse edge shear at $r = a$ can be expressed in the Fourier series form

$$\begin{bmatrix} \chi_r \\ w \\ M_r \\ V_r \end{bmatrix}_{r=a} = \sum_{m=0}^{\infty} \begin{bmatrix} \chi_m \\ w_m \\ M_m \\ V_m \end{bmatrix} \cos m\theta + \sum_{m=1}^{\infty} \begin{bmatrix} \bar{\chi}_m \\ \bar{w}_m \\ \bar{M}_m \\ \bar{V}_m \end{bmatrix} \sin m\theta \quad (3.49)$$

where χ_r, M_r, and V_r can be written

$$\chi_r = -\frac{\partial w}{\partial r} \quad (3.50a)$$
$$M_r = -D\left[\frac{\partial^2 w}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right] \quad (3.50b)$$
$$V_r = Q_r + \frac{1}{r}\frac{\partial M_{r\theta}}{\partial \theta} = -D\left[\frac{\partial}{\partial r}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) + \frac{1-\nu}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial^2 w}{\partial \theta^2}\right)\right] \quad (3.50c)$$

in terms of the transverse displacement w.

For problems involving edge loading, the m'th circumferential cosine harmonic of the displacement w, from Eqs. (3.47), has the form

$$w = (A_m r^m + C_m r^{2+m}) \cos m\theta \tag{3.51}$$

where the singular terms of Eqs. (3.47) are omitted in order to satisfy regularity at $r = 0$.

Substituting Eq. (3.51) into Eqs. (3.50) and evaluating at $r = a$ provides the relations

$$\begin{bmatrix} \chi_m \\ w_m \end{bmatrix} = \begin{bmatrix} \mathbf{M}_D^{(m)} \end{bmatrix} \begin{bmatrix} A_m \\ C_m \end{bmatrix} \tag{3.52a}$$

$$\begin{bmatrix} M_m \\ V_m \end{bmatrix} = \begin{bmatrix} \mathbf{M}_F^{(m)} \end{bmatrix} \begin{bmatrix} A_m \\ C_m \end{bmatrix} \tag{3.52b}$$

where the 2×2 matrices $[\mathbf{M}_D^{(m)}]$ and $[\mathbf{M}_F^{(m)}]$ are given by

$$[\mathbf{M}_D^{(m)}] = a^m \begin{bmatrix} -\frac{m}{a} & -(m+2)a \\ 1 & a^2 \end{bmatrix} \tag{3.53a}$$

$$[\mathbf{M}_F^{(m)}] = Da^m \begin{bmatrix} -\frac{(1-\nu)m(m-1)}{a^2} & -(m+1)[(1-\nu)m+2(1+\nu)] \\ \frac{(1-\nu)m^2(m-1)}{a^3} & \frac{m(m+1)[(1-\nu)m-4]}{a} \end{bmatrix} \tag{3.53b}$$

Solving Eq. (3.52a) for A_m and C_m and substituting the result into Eq. (3.52b) yields the stiffness relation

$$\begin{bmatrix} M_m \\ V_m \end{bmatrix} = \begin{bmatrix} \mathbf{K}^{(m)} \end{bmatrix} \begin{bmatrix} \chi_m \\ w_m \end{bmatrix} \tag{3.54}$$

where

$$[\mathbf{K}^{(m)}] = [\mathbf{M}_F^{(m)}][\mathbf{M}_D^{(m)}]^{-1} = D \begin{bmatrix} \frac{2m+1+\nu}{a} & \frac{m^2(1+\nu)+2m}{a^2} \\ \frac{m^2(1+\nu)+2m}{a^2} & \frac{m^2(2m+1+\nu)}{a^3} \end{bmatrix} \tag{3.55}$$

is the stiffness matrix for edge bending of the circular plate. The Fourier harmonics are decoupled from one another, with a stiffness matrix of the form (3.55) for each circumferential cosine or sine harmonic m.

Example 3.3: Circular Plate with Discrete Supports

Figure 3-8 shows a circular table of radius a which rests on four supports and is subjected to the uniform downward load p_0. The supports are equally spaced, and subtend an angle β. We anticipate that the plate will deform non-axisymmetrically, sagging between the supports.

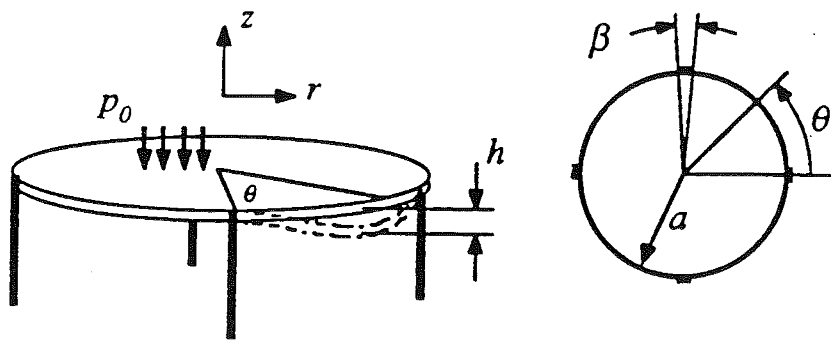

Figure 3-8. Uniformly loaded circular plate of radius a with four equally spaced support regions, each of which subtends an angle β.

The boundary conditions can be represented approximately as

$$M_r|_{r=a} = 0$$
$$V_r|_{r=a} = \begin{cases} 0 & \text{between supports} \\ V_s & \text{over supports} \end{cases} \qquad (a)$$

where V_s is a constant. Although the transverse shear is not in general constant over the support, this assumption will not appreciably affect the solution away from the supports. Overall vertical equilibrium for the uniformly loaded plate with n_s supports requires the value V_s in (a) to be

$$V_s = \frac{\pi p_0 a}{n_s \beta} \qquad (b)$$

where $n_s \geq 2$.

The solution for a uniformly loaded simply-supported plate, from Example 3.1, can be used as a particular solution:

$$w^{(p)} = -\frac{p_0 a^4}{64D}\left[\left(\frac{r}{a}\right)^4 - 2\left(\frac{3+\nu}{1+\nu}\right)\left(\frac{r}{a}\right)^2 + \frac{5+\nu}{1+\nu}\right] \qquad (c)$$

This particular solution satisfies the boundary conditions

$$M_r^{(p)}|_{r=a} = 0 \quad ; \quad V_r^{(p)}|_{r=a} = p_0 a/2 \qquad (d)$$

It is therefore necessary to find a complementary solution $w^{(c)}$ which satisfies

$$M_r^{(c)}|_{r=a} = 0$$
$$V_r^{(c)}|_{r=a} = \begin{cases} -\frac{p_0 a}{2} & \text{between supports} \\ \frac{\pi p_0 a}{n_s \beta} - \frac{p_0 a}{2} = \frac{p_0 a}{2}\left(\frac{2\pi}{n_s \beta} - 1\right) & \text{over supports} \end{cases} \qquad (e)$$

so that the total solution

$$w = w^{(p)} + w^{(c)} \tag{f}$$

satisfies the conditions (a) at the edge $r = a$.

The Fourier coefficients of the moment and shear of the complementary solution at $r = a$ can now be calculated from

$$M_m = \frac{1}{\pi}\int_0^{2\pi} M_r^{(c)}(a,\theta)\cos m\theta\, d\theta = 0 \tag{g}$$

$$V_m = \frac{1}{\pi}\int_0^{2\pi} V_r^{(c)}(a,\theta)\cos m\theta\, d\theta \tag{h}$$

Note that due to the symmetry, only the terms with $\cos n\theta$ dependence participate.

Also by symmetry, only one interval between the centers of two adjacent support regions need be considered, as shown in Fig. 3-9. From Eq. (h), we then have

$$V_m = \begin{cases} \frac{n_s}{\pi}\int_0^{\frac{2\pi}{n_s}} V_r^{(c)}(a,\theta)\cos m\theta\, d\theta & \text{for } m = 0, n_s, 2n_s, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{i}$$

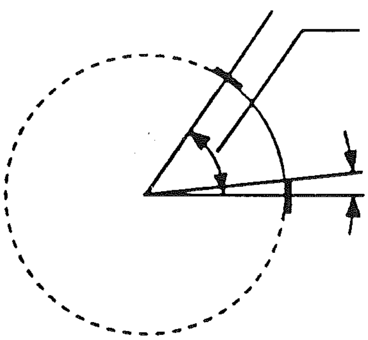

Figure 3-9. Circular plate with n_s equally spaced supports.

The integral in (i) can be evaluated in a piecewise manner for the shear distribution given by (e) as follows:

$$\begin{aligned}\int_0^{\frac{2\pi}{n_s}} V_r^{(c)}(a,\theta)\cos m\theta\, d\theta &= \frac{p_0 a}{2}\Bigg[\left(\frac{2\pi}{n_s\beta}-1\right)\int_0^{\frac{\beta}{2}}\cos m\theta\, d\theta \\ &- \int_{\frac{\beta}{2}}^{\frac{2\pi}{n_s}-\frac{\beta}{2}}\cos m\theta\, d\theta + \left(\frac{2\pi}{n_s\beta}-1\right)\int_{\frac{2\pi}{n_s}-\frac{\beta}{2}}^{\frac{2\pi}{n_s}}\cos m\theta\, d\theta\Bigg]\end{aligned} \tag{j}$$

For $m = 0, n_s, 2n_s, \ldots$, the expression (j) reduces to

$$\int_0^{\frac{2\pi}{n_s}} V_r^{(c)}(a,\theta)\cos m\theta\, d\theta = \begin{cases} 0 & m = 0 \\ \dfrac{\pi p_0 a}{n_s}\dfrac{2}{m\beta}\sin\dfrac{m\beta}{2} & m = n_s, 2n_s, \ldots \end{cases} \tag{k}$$

and Eq. (i) therefore becomes

$$V_m = \begin{cases} \dfrac{2p_0 a}{m\beta}\sin\dfrac{m\beta}{2} & m = n_s, 2n_s, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{l}$$

For $m \neq 0$, the stiffness relation (3.54) can be inverted to obtain w_m:

$$w_m = \frac{K_{11}^{(m)}V_m - K_{12}^{(m)}M_m}{K_{11}^{(m)}K_{22}^{(m)} - K_{12}^{(m)^2}} \tag{m}$$

where M_m and V_m are given by Eqs. (g) and (l). For $m = 0$, Eqs. (g) and (l) give $M_0 = V_0 = 0$, which is satisfied by the rigid body mode $w(r,\theta) = w_0 =$ Constant. The coefficient w_0 of the rigid body mode can be determined from the condition

$$\sum_{m=0}^{\infty} w_m = 0 \quad \Rightarrow \quad w_0 = -\sum_{m=1}^{\infty} w_m \tag{n}$$

which enforces zero displacement over the centers of the supports.

In particular, for $n_s = 4$ as in Fig. 3-8, Eqs. (l) and (m) with $m = 4$ yield

$$w_4 = \frac{p_0 a^4}{D}\frac{\sin 2\beta}{2\beta}\frac{9+\nu}{16[(9+\nu)^2 - 4(3+2\nu)^2]} \tag{o}$$

which provides the one-term Fourier series approximation

$$h \approx 2w_4 \tag{p}$$

for the maximum edge displacement h (see Fig. 3-8).

From Eq. (n),

$$w_0 \approx -w_4 \tag{q}$$

thus the displacement at the center of the plate is

$$\begin{aligned} w|_{r=0} &= w^{(p)}|_{r=0} + w_0 \\ &\approx w^{(p)}|_{r=0} - w_4 \end{aligned} \tag{r}$$

Note that the coefficients A_m and C_m, if they are required, can be recovered from one of Eqs. (3.52).

§3.7 Tensor Form of the Equilibrium Equations

The vector expressions

$$\mathbf{e}_x \otimes \mathbf{e}_x \quad , \quad \mathbf{e}_x \otimes \mathbf{e}_y \quad , \quad \mathbf{e}_y \otimes \mathbf{e}_x \quad , \quad \mathbf{e}_y \otimes \mathbf{e}_y \tag{3.56}$$

are referred to as *dyads*. The vectors on either side of the operator $\otimes$, which is called the *tensor product*, are to be taken as a pair. A dyad can be thought of as an ordered pair of vectors, analogous to an ordered pair of numbers (x, y).

The meaning of one of the dyads (3.56) is best understood by considering the dot product of the dyad and a vector. For example,

$$\mathbf{e}_x \cdot (\mathbf{e}_x \otimes \mathbf{e}_x) = (\mathbf{e}_x \cdot \mathbf{e}_x)\mathbf{e}_x = \mathbf{e}_x \tag{3.57a}$$

$$\mathbf{e}_x \cdot (\mathbf{e}_x \otimes \mathbf{e}_y) = (\mathbf{e}_x \cdot \mathbf{e}_x)\mathbf{e}_y = \mathbf{e}_y \tag{3.57b}$$

$$(\mathbf{e}_x \otimes \mathbf{e}_y) \cdot \mathbf{e}_x = \mathbf{e}_x(\mathbf{e}_y \cdot \mathbf{e}_x) = 0 \tag{3.57c}$$

Note that when a vector is "dotted" with a dyad, the result is another vector, because the product of the two vectors adjacent to the dot yields a scalar, and one of the vectors of the dyad "survives." Also note that the dot product between a dyad and a vector is in general not symmetric. (Compare Eqs. (3.57*b*) and (3.57*c*).)

The dyad is a useful vector construction for the representation of states of stress. Consider the vector expression

$$\mathrm{N} = N_x\, \mathbf{e}_x \otimes \mathbf{e}_x + N_{xy}\, \mathbf{e}_x \otimes \mathbf{e}_y + N_{yx}\, \mathbf{e}_y \otimes \mathbf{e}_x + N_y\, \mathbf{e}_y \otimes \mathbf{e}_y \tag{3.58}$$

in which N_x, N_{xy}, N_{yx}, and N_y are the stress resultants at a particular point in a plate. Equation (3.58) is referred to as a *tensor* representation of the in-plane stress state in the plate. The tensor N is of second order, because it consists of dyads.† A good way to understand the tensor N is to see what happens when it is dotted with other vectors.

First consider the dot products

$$\mathbf{e}_x \cdot \mathrm{N} = N_x \mathbf{e}_x + N_{xy} \mathbf{e}_y = \mathbf{N}_x \tag{3.59a}$$

$$\mathbf{e}_y \cdot \mathrm{N} = N_{yx} \mathbf{e}_x + N_y \mathbf{e}_y = \mathbf{N}_y \tag{3.59b}$$

The vector $\mathbf{N}_x$ is the *stress resultant vector* (force per unit length) which acts on edges with outward normals which point in the x direction (see Fig. 3-10a). The x component of the vector $\mathbf{N}_x$ is the stress resultant N_x, and the y component of the vector $\mathbf{N}_x$ is the shear stress resultant N_{xy}. Similarly, the vector $\mathbf{N}_y$ has components (N_{yx}, N_y), and is the stress resultant vector which acts on edges with outward normals which point in the y direction.

† Higher order tensors can be constructed from triads, tetrads, etc., which involve three or more vectors linked by the $\otimes$ operator.

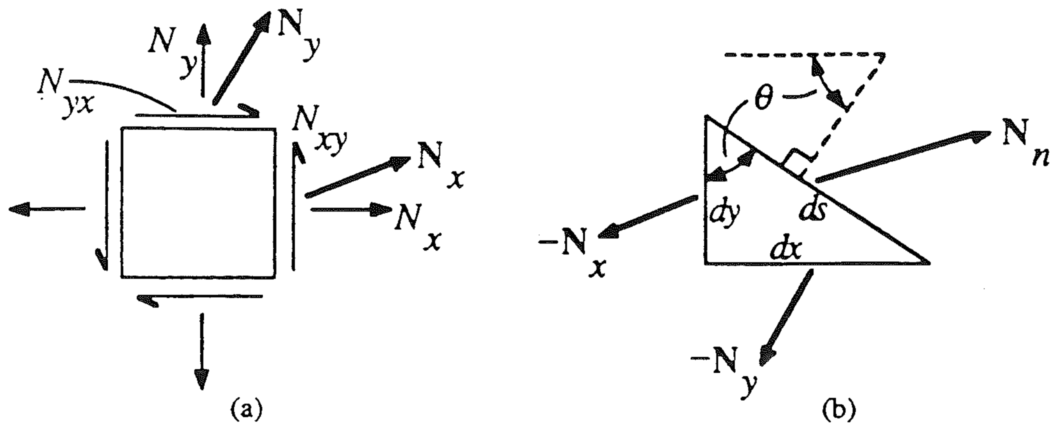

Figure 3-10. (a) The stress resultant vectors $\mathbf{N}_x$ and $\mathbf{N}_y$ act on edges with outward normals which point in the x and y directions, respectively. (b) The stress resultant vector $\mathbf{N}_n$ acts on an edge whose normal lies at an angle θ relative to the x axis.

Now consider the equilibrium of a triangular element as depicted in Fig. 3-10b. The stress resultant vector which acts on an edge whose normal lies at an angle θ relative to the x axis is denoted by $\mathbf{N}_n$. The stress resultant vectors which act on the edges of the element with lengths dx and dy are $-\mathbf{N}_y$ and $-\mathbf{N}_x$. The forces acting on the triangular element's edges must balance, which leads to the vector equation

$$\mathbf{N}_n\, ds \;-\; \mathbf{N}_y\, dx \;-\; \mathbf{N}_x\, dy \;=\; 0 \tag{3.60}$$

Substituting

$$dx = ds \sin\theta \quad ; \quad dy = ds \cos\theta \tag{3.61}$$

into Eq. (3.60) yields

$$\begin{aligned} \mathbf{N}_n &= \mathbf{N}_x \cos\theta + \mathbf{N}_y \sin\theta & (3.62a)\\ &= (\cos\theta\, \mathbf{e}_x + \sin\theta\, \mathbf{e}_y) \cdot \mathbf{N} & (3.62b)\\ &= \mathbf{e}_n \cdot \mathbf{N} & (3.62c) \end{aligned}$$

where Eqs. (3.59) are utilized in obtaining Eq. (3.62b) from Eq. (3.62a), and $\mathbf{e}_n$ is the unit vector which lies at an angle θ relative to the x axis.

If an imaginary cut is made in a plate, the normal $\mathbf{e}_n$ to one of the edges and the resultant $\mathbf{N}_n$ which acts on that edge are vectors with definite magnitude and direction. Equation (3.62c) tells us the relationship between these two vectors. The tensor N is therefore a vector operator which, if dotted with the normal to a surface, yields the stress resultant vector which acts on that surface. Just like Eq. (3.9), Eq. (3.62c) is a vector equation, and can be represented in any coordinate system which happens to be convenient.

The plane stress equilibrium equations can be written in vector form as

$$\nabla \cdot \mathbf{N} + \mathbf{p} = \mathbf{0} \tag{3.63}$$

in which

$$\mathbf{p} = p_x \mathbf{e}_x + p_y \mathbf{e}_y \tag{3.64}$$

is the in-plane load vector. The two equilibrium equations (1.7a, b) are recovered by substituting the expression (3.10a) for the gradient operator and the form (3.58) for the tensor $\mathbf{N}$ into Eq. (3.63), and equating to zero the coefficients of the vectors $\mathbf{e}_x$ and $\mathbf{e}_y$.

In order to write the equilibrium equations for plate bending in tensor form, it is necessary to introduce the quantities

$$\begin{aligned}
\mathbf{Q} &= Q_x \mathbf{e}_x + Q_y \mathbf{e}_y && (3.65a)\\
\mathbf{M} &= M_x \mathbf{e}_x \otimes \mathbf{e}_x + M_{xy} \mathbf{e}_x \otimes \mathbf{e}_y + M_{yx} \mathbf{e}_y \otimes \mathbf{e}_x + M_y \mathbf{e}_y \otimes \mathbf{e}_y && (3.65b)\\
\mathbf{m} &= m_x \mathbf{e}_x + m_y \mathbf{e}_y && (3.65c)
\end{aligned}$$

where $\mathbf{Q}$ and $\mathbf{M}$ are transverse shear and bending moment tensors, and $\mathbf{m}$ is the distributed moment load vector.

With the definitions (3.65) and the expression (3.10a) for the gradient operator, the equation (1.7c) for vertical equilibrium can be written

$$\nabla \cdot \mathbf{Q} + p_z = 0 \tag{3.66a}$$

and the two equations (1.7d, e) for moment equilibrium become

$$\nabla \cdot \mathbf{M} - \mathbf{Q} + \mathbf{m} = 0 \tag{3.66b}$$

The five equations (1.7a–e) of equilibrium for plate behavior are therefore equivalent to the three vector equations (3.66a, b) and (3.63).

§3.8 The Curvature Tensor

The curvatures and twist of the midplane are related to the midplane displacement w by Eqs. (1.21), which can be organized in vector form as follows:

$$\begin{aligned}
\boldsymbol{\kappa} &= \kappa_x\, \mathbf{e}_x \otimes \mathbf{e}_x + \kappa_{xy}\, \mathbf{e}_x \otimes \mathbf{e}_y + \kappa_{yx}\, \mathbf{e}_y \otimes \mathbf{e}_x + \kappa_y\, \mathbf{e}_y \otimes \mathbf{e}_y && (3.67a)\\
&= -\frac{\partial^2 w}{\partial x^2}\mathbf{e}_x \otimes \mathbf{e}_x - \frac{\partial^2 w}{\partial x \partial y}\mathbf{e}_x \otimes \mathbf{e}_y - \frac{\partial^2 w}{\partial y \partial x}\mathbf{e}_y \otimes \mathbf{e}_x - \frac{\partial^2 w}{\partial y^2}\mathbf{e}_y \otimes \mathbf{e}_y && (3.67b)\\
&= -\left(\mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y}\right) \otimes \left(\mathbf{e}_x \frac{\partial w}{\partial x} + \mathbf{e}_y \frac{\partial w}{\partial y}\right) && (3.67c)\\
&= -\nabla \otimes \nabla w && (3.67d)
\end{aligned}$$

Equation (3.67a) shows that the midplane curvatures κ_x, κ_y, and κ_{xy} can be interpreted as components of a ***curvature tensor***, denoted by $\boldsymbol{\kappa}$. From Eq. (3.67d), we see that $\boldsymbol{\kappa}$ is a vector function of the displacement w, and can therefore be represented in terms of any convenient set of basis vectors.

In the case of polar coordinates, the operator ∇ is given by Eq. (3.10b), and Eq. (3.67d) becomes

$$\boldsymbol{\kappa} = -\left(\mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\right)\otimes\left(\mathbf{e}_r\frac{\partial w}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial w}{\partial\theta}\right) \tag{3.68}$$

which can be expanded, with derivatives of $\mathbf{e}_r$ and $\mathbf{e}_\theta$ evaluated according to Eqs. (3.15). The result is

$$\boldsymbol{\kappa} = \kappa_r\,\mathbf{e}_r\otimes\mathbf{e}_r + \kappa_{r\theta}\,\mathbf{e}_r\otimes\mathbf{e}_\theta + \kappa_{\theta r}\,\mathbf{e}_\theta\otimes\mathbf{e}_r + \kappa_\theta\,\mathbf{e}_\theta\otimes\mathbf{e}_\theta \tag{3.69}$$

where κ_r, κ_θ, $\kappa_{r\theta}$, and $\kappa_{\theta r}$ are the same quantities (3.24) which were obtained by coordinate transformation.

For the plate bending problem, $\boldsymbol{\kappa}$ represents the plate midplane's deformed shape, which is the same regardless of the coordinate system employed. This deformed shape can be described by the components of $\boldsymbol{\kappa}$ relative to any convenient coordinate system. For the case of a polar coordinate system, the components κ_r and κ_θ are the curvatures in the directions of $\mathbf{e}_r$ and $\mathbf{e}_\theta$, and the component $\kappa_{r\theta}$ is the twist relative to these basis vectors.

§3.9 Plane Stress in Polar Coordinates

As discussed in Chapter 1, the governing equation (1.40) for the plane stress behavior of a plate, in the absence of distributed in-plane loads ($p_x = p_y = 0$), is the biharmonic equation

$$\Delta\Delta\phi = 0 \tag{3.70}$$

where ϕ is the Airy stress function.

The harmonic differential operator in terms of r and θ is given by Eq. (3.16b); thus Eq. (3.70) becomes

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right) = 0 \tag{3.71}$$

The expressions (1.39) which relate the stress resultants N_x, N_y, and N_{xy} to the stress function ϕ can be written in the tensor form

$$\begin{aligned}\mathrm{N} &= \frac{\partial^2\phi}{\partial y^2}\,\mathbf{e}_x\otimes\mathbf{e}_x - \frac{\partial^2\phi}{\partial y\partial x}\,\mathbf{e}_x\otimes\mathbf{e}_y - \frac{\partial^2\phi}{\partial x\partial y}\,\mathbf{e}_y\otimes\mathbf{e}_x + \frac{\partial^2\phi}{\partial x^2}\,\mathbf{e}_y\otimes\mathbf{e}_y \\ &= \left(\mathbf{e}_x\frac{\partial}{\partial y} - \mathbf{e}_y\frac{\partial}{\partial x}\right)\otimes\left(\mathbf{e}_x\frac{\partial\phi}{\partial y} - \mathbf{e}_y\frac{\partial\phi}{\partial x}\right)\end{aligned} \tag{3.72}$$

Equation (3.72) represents four equations, two of which are equivalent due to the fact that $N_{xy} = N_{yx} = -\frac{\partial^2\phi}{\partial x\partial y}$.

The differential operator $\mathbf{e}_x\frac{\partial}{\partial y} - \mathbf{e}_y\frac{\partial}{\partial x}$ which appears twice in Eq. (3.72) can be expressed

$$\mathbf{e}_x\frac{\partial}{\partial y} - \mathbf{e}_y\frac{\partial}{\partial x} = \boldsymbol{\epsilon}\cdot\left(\mathbf{e}_x\frac{\partial}{\partial x} + \mathbf{e}_y\frac{\partial}{\partial y}\right) = \boldsymbol{\epsilon}\cdot\boldsymbol{\nabla} \tag{3.73}$$

where the vector operator $\boldsymbol{\epsilon}$, defined by

$$\boldsymbol{\epsilon} = \mathbf{e}_x\otimes\mathbf{e}_y - \mathbf{e}_y\otimes\mathbf{e}_x \tag{3.74}$$

is a second-order tensor which, when dotted with a vector, yields a vector which is *rotated clockwise by ninety degrees* relative to the original vector. For example,

$$\begin{aligned}\boldsymbol{\epsilon}\cdot\mathbf{e}_x &= -\mathbf{e}_y \\ \boldsymbol{\epsilon}\cdot\mathbf{e}_y &= \mathbf{e}_x\end{aligned} \tag{3.75}$$

The "ninety degree rotation tensor" $\boldsymbol{\epsilon}$ can also be written in terms of the basis vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$:

$$\boldsymbol{\epsilon} = \mathbf{e}_r\otimes\mathbf{e}_\theta - \mathbf{e}_\theta\otimes\mathbf{e}_r \tag{3.76}$$

The differential operator $\boldsymbol{\epsilon}\cdot\boldsymbol{\nabla}$ can therefore be expressed

$$\begin{aligned}\boldsymbol{\epsilon}\cdot\boldsymbol{\nabla} &= (\mathbf{e}_r\otimes\mathbf{e}_\theta - \mathbf{e}_\theta\otimes\mathbf{e}_r)\cdot\left(\mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\right) \\ &= \mathbf{e}_r\frac{1}{r}\frac{\partial}{\partial\theta} - \mathbf{e}_\theta\frac{\partial}{\partial r}\end{aligned} \tag{3.77}$$

and the Eq. (3.72), in terms of polar coordinates, becomes

$$\mathbf{N} = (\boldsymbol{\epsilon}\cdot\boldsymbol{\nabla})\otimes(\boldsymbol{\epsilon}\cdot\boldsymbol{\nabla}\phi) \tag{3.78a}$$

$$= \left(\mathbf{e}_r\frac{1}{r}\frac{\partial}{\partial\theta} - \mathbf{e}_\theta\frac{\partial}{\partial r}\right)\otimes\left(\mathbf{e}_r\frac{1}{r}\frac{\partial\phi}{\partial\theta} - \mathbf{e}_\theta\frac{\partial\phi}{\partial r}\right) \tag{3.78b}$$

Equations (3.72) and (3.78*b*) employ different basis vectors, but represent the same tensor quantity $\mathbf{N}$, just as Eqs. (3.10*a*) and (3.10*b*) are equivalent representations for the gradient of a function.

Equation (3.78) can now be expanded, similarly to Eq. (3.68), to arrive at

$$\mathbf{N} = N_r\,\mathbf{e}_r\otimes\mathbf{e}_r + N_{r\theta}\,\mathbf{e}_r\otimes\mathbf{e}_\theta + N_{\theta r}\,\mathbf{e}_\theta\otimes\mathbf{e}_r + N_\theta\,\mathbf{e}_\theta\otimes\mathbf{e}_\theta \tag{3.79}$$

where

$$N_r = \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} \tag{3.80a}$$

$$N_\theta = \frac{\partial^2\phi}{\partial r^2} \tag{3.80b}$$

$$N_{r\theta} = N_{\theta r} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) \tag{3.80c}$$

The state of plane stress, represented by **N**, is the same regardless of the coordinate system which is employed, but the components of **N** depend on the basis vectors. Note that relative to the basis vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ the components of **N** and $\boldsymbol{\kappa}$ have similar forms.

§3.10 Circular Plates under In-plane Edge Loads

For the plane stress problem of a circular or annular plate subjected to in-plane edge loads, the governing equation in the stress function ϕ, in terms of polar coordinates, is given by Eq. (3.71). If the stress function $\phi(r, \theta)$ is taken in the Fourier series form

$$\phi(r,\theta) = \sum_{m=0}^{\infty} f_m(r)\cos m\theta + \sum_{m=1}^{\infty} g_m(r)\sin m\theta \tag{3.81}$$

it is evident that $f_m(r)$ and $g_m(r)$ must have the exact same form as in the complementary solution (3.39) to the plate bending problem. The functions $f_m(r)$ and $g_m(r)$ in Eq. (3.81) are therefore given by the same equations (3.47) and (3.48).

The axisymmetric case, in which ϕ is independent of θ, corresponds to the $m = 0$ term in Eq. (3.81). This is completely analogous to the plate bending problem. With $f_0(r)$ given by Eq. (3.47*a*), the solution to the axisymmetric plane stress problem is therefore

$$\phi(r) = A_0 + B_0\ln r + C_0 r^2 + D_0 r^2\ln r \tag{3.82a}$$

where A_0, B_0, C_0, and D_0 are constants to be determined. From Eqs. (3.80), it is evident that the constant term A_0 of Eq. (3.82) contributes no stresses, and can be be taken equal to zero with no loss of generality; thus

$$\phi(r) = B_0\ln r + C_0 r^2 + D_0 r^2\ln r \tag{3.82b}$$

It can be shown that the third term in Eq. (3.82*b*) corresponds to a plane stress deformation which involves a discontinuity in the displacements

— a dislocation — at some value of θ.† For axisymmetric problems of complete circular and annular plates, we therefore have

$$D_0 = 0 \tag{3.83}$$

and

$$\phi(r) = B_0 \ln r + C_0 r^2 \tag{3.84}$$

where B_0 and C_0 are evaluated by applying two axisymmetric boundary conditions.

Example 3.4: Annular Plate with Axisymmetric Edge Loads

Figure 3-11 shows an annular plate with outer radius a and inner radius b, subjected to the edge loads

$$N_r(a) = N_a \quad ; \quad N_r(b) = N_b \tag{a}$$

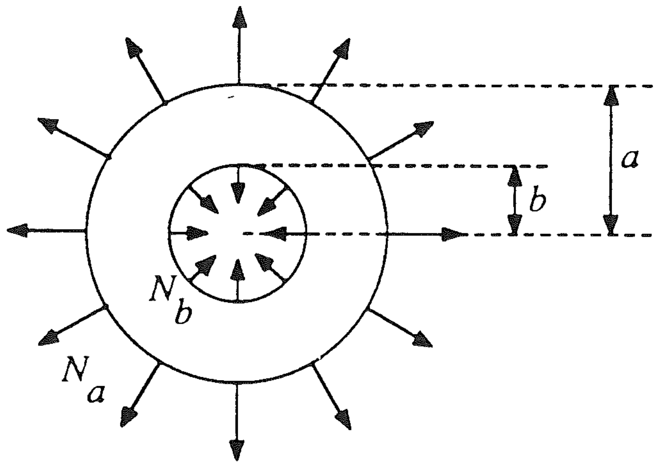

Figure 3-11. Annular plate subjected to the axisymmetric edge loads $N_r = N_a$ at $r = a$ and $N_r = N_b$ at $r = b$.

Substitution of the stress function (3.84) into Eqs. (3.80) yields

$$\begin{aligned} N_r &= 2C_0 + \frac{B_0}{r^2} \\ N_\theta &= 2C_0 - \frac{B_0}{r^2} \\ N_{r\theta} &= 0 \end{aligned} \tag{b}$$

The boundary conditions (a) can therefore be written

$$2C_0 + \frac{B_0}{a^2} = N_a \quad ; \quad 2C_0 + \frac{B_0}{b^2} = N_b \tag{c}$$

† See Timoshenko and Goodier (1970), pp. 78–9, for a more detailed discussion of this point.

and the two equations (c) are readily solved to obtain

$$B_0 = -\frac{a^2b^2(N_a - N_b)}{a^2 - b^2} \quad ; \quad C_0 = \frac{a^2N_a - b^2N_b}{2(a^2 - b^2)} \tag{d}$$

The stress resultants, from Eqs. (b) and (d), are

$$\begin{aligned} N_r(r) &= \frac{a^2}{a^2 - b^2}\left[N_a - \left(\frac{b}{a}\right)^2 N_b - (N_a - N_b)\left(\frac{b}{r}\right)^2\right] \\ N_\theta(r) &= \frac{a^2}{a^2 - b^2}\left[N_a - \left(\frac{b}{a}\right)^2 N_b + (N_a - N_b)\left(\frac{b}{r}\right)^2\right] \end{aligned} \tag{e}$$

Note that for equal loads along the inner and outer edges, that is

$$N_a = N_b = N_0 \tag{f}$$

Equations (e) yield the uniform, isotropic stress distribution

$$N_r(r) = N_\theta(r) = N_0 \tag{g}$$

Now imagine that a very small hole is drilled in a plate which is under uniform isotropic tension. In this case, we have

$$N_a = N_0 \quad ; \quad N_b = 0 \quad ; \quad b/a \approx 0 \tag{h}$$

and Eqs. (e) reduce to

$$N_r(r) = N_0\left[1 - \left(\frac{b}{r}\right)^2\right] \quad ; \quad N_\theta(r) = N_0\left[1 + \left(\frac{b}{r}\right)^2\right] \tag{i}$$

from which it is seen that the maximum stress acts in the θ direction at $r = b$ (the edge of the hole), and has the value

$$\sigma_{\theta\theta}(b) = N_\theta(b)/t = 2N_0/t \tag{j}$$

The hole therefore introduces a *stress concentration factor* of two.

§3.11 Additional Solutions for Circular and Annular Plates

The preceding sections presented Fourier series solutions to the homogeneous biharmonic equations (3.71) and (3.35). In addition to the Fourier series solutions, which are periodic in the circumferential coordinate θ, the four functions

$$F_1(r,\theta) = a_1\theta \tag{3.85a}$$
$$F_2(r,\theta) = a_2r^2\theta \tag{3.85b}$$
$$F_3(r,\theta) = a_3r\theta\sin\theta \tag{3.85c}$$
$$F_4(r,\theta) = a_4r\theta\cos\theta \tag{3.85d}$$

where a_1, a_2, a_3, and a_4 are arbitrary constants, can also be verified to satisfy Eq. (3.71) or Eq. (3.35). Note that the solutions (3.85) are *not* periodic in θ.

For the plane stress problem, the physical interpretations of the solutions (3.85) are as follows:

F_1: Uniform shear stress $N_{r\theta}$ (torque in the z direction).
F_2: Discontinuity in stress at $r = 0$.
F_3: Concentrated load in the x direction at $r = 0$.
F_4: Concentrated load in the y direction at $r = 0$.

For the plate bending problem, the physical interpretations of the solutions (3.85) are:

F_1: Helical deformation of an annular strip.
F_2: Discontinuity in moment at $r = 0$.
F_3: Concentrated moment load in the y direction at $r = 0$.
F_4: Concentrated moment load in the x direction at $r = 0$.