Introduction to the Theory of Plates 1
Charles R. Steele 1
Stanford University 1

Chapter 1.0 Stretching and Bending of Plates - Fundamentals

1.1 Introduction

A plate is a structural element which is thin and flat. By "thin," it is meant that the plate's transverse dimension, or thickness, is small compared to the length and width dimensions. A mathematical expression of this idea is:

$$t/L \ll 1 \tag{1.1}$$

where t represents the plate's thickness, and L represents a representative length or width dimension. (See Fig. 1.1.) More exactly, L represents the minimum wave length of deformation, which can be much smaller than the plate minimum lateral dimension for problems of localized loading, dynamics and stability. Plates might be classified as very thin if $L/t > 100$, moderately thin if $20 < L/t < 100$, thick if $3 < L/t < 20$, and very thick if $L/t < 3$. The "classical" theory of plates is applicable to very thin and moderately thin plates, while "higher order theories" for thick plates are useful. For the very thick plates, however, it becomes more difficult and less useful to view the structural element as a plate – a description based on the three-dimensional theory of elasticity is required.

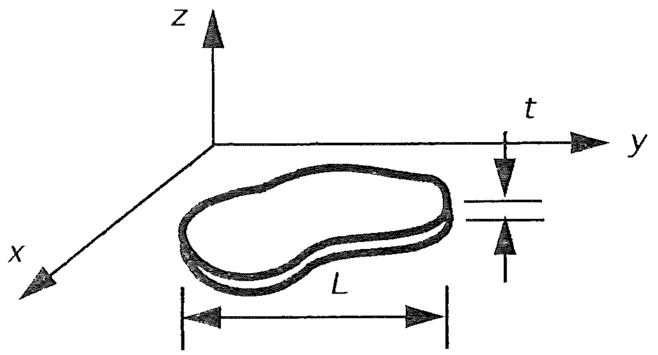

FIGURE 1.1. Plate and associated (x, y, z) coordinate system.

In this chapter, we derive the basic equations which describe the behavior of plates taking advantage of the plate's thin, planar character. The approach is a generalization of the one-dimensional Euler-Bernoulli beam theory, which exploits the slender shape of a beam. We will develop a two-dimensional plate theory which employs the in-plane coordinates x and y in Fig. 1.1 as independent variables. Of particular interest are the plate's "stretching" behavior – associated with in-plane loads in the x- and y-directions, and the plate's bending behavior – associated with moment and shear loads applied to the plate's edges, and transverse loads in the z- direction.

Figure 1.1 depicts a plate and an associated (x, y, z) coordinate system. The top and bottom surfaces lie at $z = \pm t/2$. The flat surface $z = 0$ is the plate midsurface, which provides a convenient reference plane for the derivation of the governing equations for the plate.

1.2 Three-Dimensional Considerations

Many things are easier if we begin with the three-dimensional equations of linear elasticity. The solutions for three-dimensional problems is generally difficult and time-consuming for a computation. In many respects, however, the derivation of the three-dimensional theory is more straightforward than the reduced approximate beam, plate and shell theories. Indeed, the three-dimensional theory is the basis for all approximate theories. The equations can be found in many texts, including Timoshenko and Goodier, 1970[14].

1.2.1 Stress components

The application of external forces to a body produces an internal state of stress. Stress is measured in units of force per unit area, and can be thought of as the intensity of the internal forces acting at a particular point in the body. Figure 1.2 depicts the stresses which act on the surfaces of a three-dimensional element of a solid body.

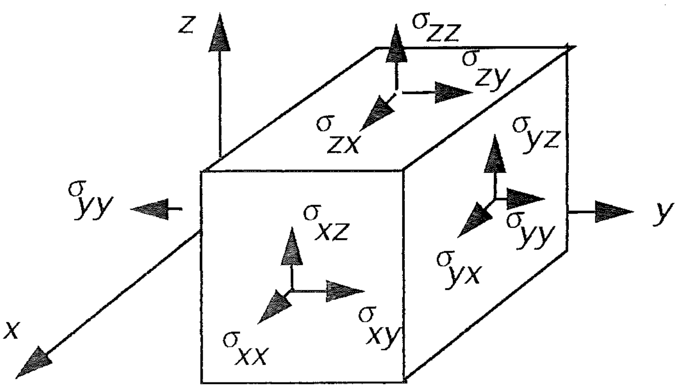

FIGURE 1.2. Stress components acting on the surfaces of a three-dimensional element in cartesian coordinates.

The components in Fig. 1.2 are identified by two indices. The first index denotes the *direction of the outward normal* to the surface being acted upon, and the second index denotes the *direction in which the force acts*. For example, σ_{zx} is the force per unit area of the surface with outward normal in the positive z-direction and acting in the positive x-direction. The stress components σ_{xx}, σ_{yy}, and σ_{zz} act normally to the element faces, while the remaining six components σ_{xy}, σ_{xz}, σ_{yx}, σ_{yz}, σ_{zx}, and σ_{zy} are the components of shear stresses, which act tangentially to the element faces. On the "back" faces not shown in Fig. 1.2, i.e., the faces with outward normals in the negative directions of the coordinate axes, the directions of all the components are reversed, as indicated by the component σ_{yy}. With this sign convention, tensile forces have positive values of stress and compressive forces negative values.

Some authors choose the opposite order for the indices, with the first index giving the direction of force and the second the face. However, the majority of publications on plates and shells use the notation in Fig. 1.2.

1.2.2 Equilibrium

1.2.2 .1 Divergence theorem. Consider the cube in Fig. 1.2 to be located in the region of a body with the coordinates:

$$x_1 \le x \le x_2 \ ; \ y_1 \le y \le y_2 \ ; \ z_1 \le z \le z_2 \tag{1.2}$$

The total force in the x-direction can be obtained by integrating the components in the x-direction on the six faces, with the result:

$$\int_{z_1}^{z_2}\int_{y_1}^{y_2}[\sigma_{xx}(x_2, y, z) - \sigma_{xx}(x_1, y, z)]dydz + \int_{x_1}^{x_2}\int_{z_1}^{z_2}[\sigma_{yx}(x, y_2, z) - \sigma_{yx}(x, y_1, z)]dzdx$$

$$\int_{y_1}^{y_2}\int_{x_1}^{x_2}[\sigma_{zx}(x, y, z_2) - \sigma_{zx}(x, y, z_1)]dxdy + \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} b_x(x, y, z)dzdydx = 0 \tag{1.3}$$

in which b_x is the component in the x-direction of the body force per unit of volume. However, the first surface integrals can be rewritten as volume integrals. (This is the divergence theorem for rectangular coordinates.) Thus the equation becomes:

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2}\left[\frac{\partial}{\partial x}\sigma_{xx}(x, y, z) + \frac{\partial}{\partial y}\sigma_{yx}(x, y, z) + \frac{\partial}{\partial z}\sigma_{zx}(x, y, z) + b_x(x, y, z)\right]dxdydz = 0 \tag{1.4}$$

Within the body, the subregion can be chosen arbitrarily. For the result of the integration to be zero for any arbitrary subregion, no matter how small or large, the integrand must be identically zero. Thus the partial differential equations for equilibrium are obtained:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + b_x = 0 \tag{1.5a}$$

$$\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + \frac{\partial \sigma_{xy}}{\partial x} + b_y = 0 \tag{1.5b}$$

$$\frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + b_z = 0 \tag{1.5c}$$

in which b_x, b_y, and b_z are the components of the body force per unit of volume.

Moment equilibrium of the three-dimensional element in Fig. 1.2 around each of the coordinate axes leads to the condition of symmetry of the shear stress components for the classical theory of elasticity:

$$\sigma_{xy} = \sigma_{yx} \ ; \ \sigma_{yz} = \sigma_{zy} \ ; \ \sigma_{zx} = \sigma_{xz} \tag{1.6}$$

We mention that for "nonclassical" elasticity, in addition to a force per unit area, the possibility of a torque per unit area acting on the body is included. For such theories, Eq. 1.6 does not hold, and the stress components are not symmetric.

1.2.2 .2 Expansion method. The above derivation is exact. However, a direct method which is often used is to consider the element in Fig. 1.3. The summation of the forces in the x-direction gives:

$$[\sigma_{xx}(x+\Delta x, y, z) - \sigma_{xx}(x, y, z)]\Delta y\Delta z + [\sigma_{yx}(x, y+\Delta y, z) - \sigma_{yx}(x, y, z)]\Delta z\Delta x$$

$$[\sigma_{zx}(x, y, z+\Delta z) - \sigma_{zx}(x, y, z)]\Delta x\Delta y + b_x\Delta x\Delta y\Delta z = 0 \tag{1.7}$$

Dividing by $\Delta x\Delta y\Delta z$ and taking the limit yields Eq. 1.5a. The same procedure yields the other two equations of equilibrium.

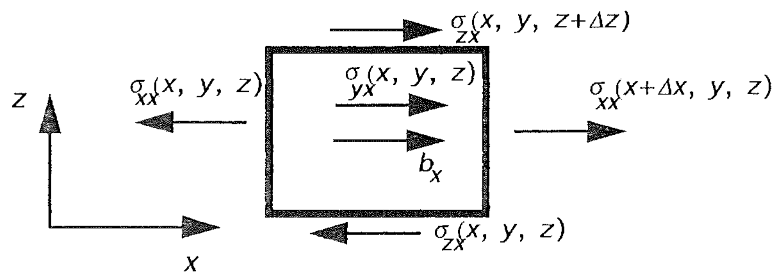

FIGURE 1.3. Increments of the components of stress in the x-direction.

1.2.3 Strain and Displacement

The components of displacement in the x-, y-, and z- directions is often denoted by u, v, and w. The strain (engineering) is the change in length divided by the original length, which gives the strain components in the three directions:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \; ; \; \varepsilon_{yy} = \frac{\partial v}{\partial y} \; ; \; \varepsilon_{zz} = \frac{\partial w}{\partial z} \tag{1.8a}$$

and the components of shear strain:

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \; ; \; \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \; ; \; \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \tag{1.8b}$$

1.2.4 Constitutive Relations

The stress components are related to the strain components by the generalized Hooke's law for an isotropic material:

$$\varepsilon_{xx} = \frac{1}{E}[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})] \quad (1.9a)$$

$$\varepsilon_{yy} = \frac{1}{E}[\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})] \quad (1.9b)$$

$$\varepsilon_{zz} = \frac{1}{E}[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})] \quad (1.9c)$$

where E is the Young's modulus and ν is the Poisson's ratio. The shearing strain components are:

$$\gamma_{xy} = \frac{\sigma_{xy}}{G} \quad (1.9d)$$

$$\gamma_{yz} = \frac{\sigma_{yz}}{G} \quad (1.9e)$$

$$\gamma_{zx} = \frac{\sigma_{zx}}{G} \quad (1.9f)$$

where G is the shear modulus. For the isotropic material, the shear modulus is determined by the Young's modulus and Poisson's ratio:

$$G = \frac{E}{2(1+\nu)} \quad (1.10)$$

So both the shear stress and shear strain components are symmetric in the two indices. For a general anisotropic material, all the stress and strain components are related.

1.3 Force and Moment Resultants

The nine stress components shown in Fig. 1.2 uniquely define the state of stress at a point in a three-dimensional body, and are in general functions of x, y, and z. In the theory for a thin plate, however, we would like to work with force quantities which depend on x and y alone. This can be achieved by integrating the stresses in the z-direction, through the plate's thickness, in order to obtain the following *stress resultant* quantities:

$$N_x = \int_{-t/2}^{t/2} \sigma_{xx} dz \ ; \ N_{xy} = \int_{-t/2}^{t/2} \sigma_{xy} dz \ ; Q_x = \int_{-t/2}^{t/2} \sigma_{xz} dz$$

$$N_y = \int_{-t/2}^{t/2} \sigma_{yy} dz \ ; \ N_{yx} = \int_{-t/2}^{t/2} \sigma_{yx} dz \ ; Q_y = \int_{-t/2}^{t/2} \sigma_{yz} dz \quad (1.11)$$

and the moment resultant quantities:

$$M_x = \int_{-t/2}^{t/2} \sigma_{xx} z dz \;;\; M_{xy} = \int_{-t/2}^{t/2} \sigma_{xy} z dz$$

$$M_y = \int_{-t/2}^{t/2} \sigma_{yy} z dz \;;\; M_{yx} = \int_{-t/2}^{t/2} \sigma_{yx} z dz \tag{1.12}$$

The stress resultants given by Eq. 1.11 have the units of force per unit length, and act in the directions shown in Fig. 1.4. N_x and N_y are in-plane tensile (when positive) stress resultants in the x- and y- directions; N_{xy} and N_{yx} are in-plane shear stress resultants; and Q_x and Q_y are transverse shear stress resultants.

Loads on the plate in the x-, y-, and z -directions are denoted by p_x, p_y, and p_z in Fig. 1.4. These are the *external* forces acting on the plate at a given point. External loads can be applied as body forces, such as gravity, and they can be applied as surface tractions, in which case they correspond to the stresses σ_{zx}, σ_{zy}, and σ_{zz}, evaluated at $z = \pm t/2$, as shown in the next section. The loads have the units of force per unit area and are functions of x and y.

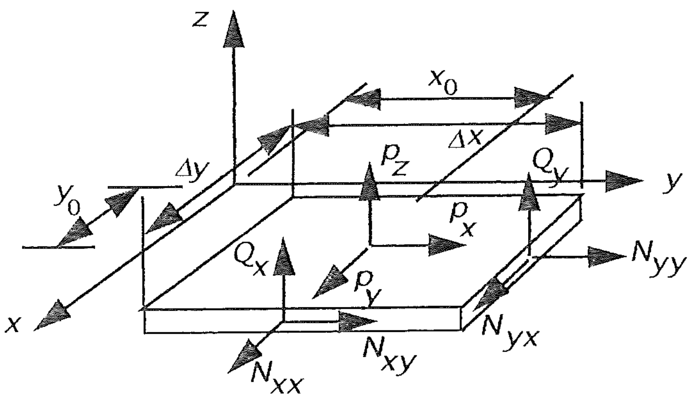

FIGURE 1.4. Force resultants acting on a two-dimensional plate element.

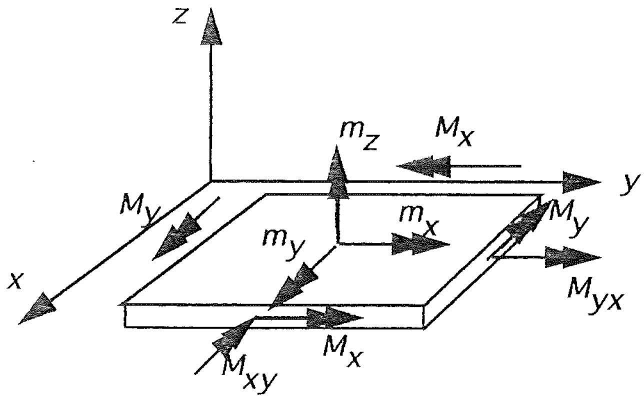

FIGURE 1.5. Moment resultants acting on the two-dimensional plate element.

The moment resultants given by Eq. 1.12 have the units of moment per unit length, and act in the directions shown in Fig. 1.5. M_x and M_y are bending moment resultants. The directions of these moments are determined from Eq. 1.12, so the right-hand vector representation is as seen in Fig. 1.5. Thus the vector M_x is in the y-direction, and M_y is in the negative x-direction. If one remembers the basic definitions Eqs. 1.11 and 1.12, then the directions of all the resultants can be easily obtained. A positive value of the stress component σ_{xx} at the upper plate surface ($z = + t/2$) causes a positive value of M_x. The vectors for M_x and M_y for the back faces are shown in Fig. 1.5, to emphasize that, unlike the shear stress resultants, the moment resultants 'chase each other around the plate' (W. Flügge). Earlier authors used various definitions for the moment resultants, but Eq. 1.12 seems to be standard today. The quantities M_{xy} and M_{yx} are the twisting moment resultants, which tend to twist the edge of the plate around the x- or y-axis. The vector directions of the twisting moments can be obtained by considering a positive value of σ_{xy} and σ_{yx} at the upper surface. For completeness, a prescribed moment per unit area acting on the surface is shown in Fig. 1.5, with the components m_x, m_y, and m_z.

The stress and moment resultants provide convenient force quantities for the analysis of plates, just as moment, shear, and net tensile force are convenient in the analysis of beams. Note that the edge of the plate has a force resultant with components in all three directions (x, y, z) but the moment resultant has components only in the in-plane directions (x, y). This is even with the moment component m_z applied to the plate surface.

1.4 Equilibrium

When a plate carries a static load, the plate must be in equilibrium, which means that the forces and moments acting on any arbitrary element of the plate must sum to zero. In general, there will be six equilibrium conditions: one force balance equation and one moment balance equation for each of the three coordinate directions. We consider two approaches to the derivation of the equations of equilibrium.

1.4.1 Direct from Resultants

The equations of equilibrium can be obtained from the Taylor series expansion of the resultants for the rectangular sub-region Fig. 1.4, as follows. The forces in the x-direction are:

$$\left(N_x + \frac{\partial N_x}{\partial x}\Delta x\right)\Delta y - N_x \Delta y$$

$$+\left(N_{yx} + \frac{\partial N_{yx}}{\partial y}\Delta y\right)\Delta x - N_{yx}\Delta x + p_x \Delta x \Delta y = 0 \qquad (1.13)$$

Dividing by $\Delta x \Delta y$ and taking the limit yields the equation for force equilibrium in the x-direction:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y} + p_x = 0 \qquad (1.14a)$$

Similarly, we obtain the equation for force equilibrium in the y-direction:

$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} + p_y = 0 \qquad (1.14b)$$

the force equilibrium in the z-direction:

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p_z = 0 \qquad (1.14c)$$

the moment equilibrium in the y-direction:

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} - Q_x + m_y = 0 \qquad (1.14d)$$

the moment equilibrium in the x-direction:

$$\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y - m_x = 0 \qquad (1.14e)$$

and the moment equilibrium in the z-direction:

$$N_{xy} - N_{yx} + m_z = 0 \qquad (1.14f)$$

However, for classical elasticity, the three-dimensional stresses are symmetric Eq. 1.6, so that the definition of the resultants Eq. 1.11 gives $N_{xy} = N_{yx}$, Consequently, Eq. 1.14f shows that $m_z = 0$ for a plate theory consistent with classical elasticity theory. Because of Eq. 1.6, it follows from Eq. 1.12 that the twisting moment is also symmetric:

$$M_{xy} = M_{yx} \tag{1.15}$$

In order to simplify the derivation of Eqs. 1.14a-1.14f, it has been assumed that the forces can be treated as acting on the *undeformed* plate element. The equations therefore apply only when the deformation of the plate is small, i.e., for strains small relative to unity and normal displacement small relative to the thickness t. The analysis for larger deformation is considered later.

1.4.2 Integrate 3-Dimensional Equations

An alternate method to obtain the equations of equilibrium is to integrate the three-dimensional equations directly. This works particularly well for the flat plate under present consideration. Integrating Eq. 1.5a yields Eq. 1.14a, in which the effective pressure is precisely related to the 3-dimensional stress on the surface and the body force:

$$p_x(x, y) = \sigma_{zx}(x, y, t/2) - \sigma_{zx}(x, y, -t/2) + \int_{-t/2}^{t/2} b_x dz \tag{1.16a}$$

Similarly, integrating Eq. 1.5b yields Eq. 1.14b in which:

$$p_y(x, y) = \sigma_{zy}(x, y, t/2) - \sigma_{zy}(x, y, -t/2) + \int_{-t/2}^{t/2} b_y dz \tag{1.17b}$$

Integrating Eq. 1.5c yields Eq. 1.14c, in which:

$$p_z(x, y) = \sigma_{zz}(x, y, t/2) - \sigma_{zz}(x, y, -t/2) + \int_{-t/2}^{t/2} b_z dz \tag{1.18c}$$

Multiplying Eq. 1.5a and Eq. 1.5b by z and integrating yields Eq. 1.14d and Eq. 1.14e in which:

$$m_y(x, y) = \frac{t}{2}[\sigma_{zx}(x, y, t/2) + \sigma_{zx}(x, y, -t/2)] + \int_{-t/2}^{t/2} b_x z dz \tag{1.19d}$$

$$m_x(x, y) = -\frac{t}{2}[\sigma_{zy}(x, y, t/2) + \sigma_{zy}(x, y, -t/2)] - \int_{-t/2}^{t/2} b_y z dz \tag{1.20e}$$

Something new is the result of multiplying Eq. 1.5c by z and integrating. The result is:

$$\frac{\partial M_{xz}}{\partial x} + \frac{\partial M_{yz}}{\partial y} - N_z + m_{z2} = 0 \tag{1.21a}$$

where the new "resultants" are:

$$M_{xz} = \int_{-t/2}^{t/2} \sigma_{xz} z dz \; ; \; M_{yz} = \int_{-t/2}^{t/2} \sigma_{yz} z dz \; ; \; N_z = \int_{-t/2}^{t/2} \sigma_{zz} dz \tag{1.22b}$$

which actually correspond to a self-equilibrating distribution of the 3-dimensional stress in the plate. Thus these are not actually resultants. The other new term in Eq. 1.21a is:

$$m_{z2}(x, y) = \frac{t}{2}[\sigma_{zz}(x, y, t/2) + \sigma_{zz}(x, y, -t/2)] + \int_{-t/2}^{t/2} b_z z dz \qquad (1.23c)$$

which is also a self-equilibrating quantity and not actually a resultant moment. So integrating the 3-dimensional equations directly yields only five equations Eqs. 1.14a-1.14e with actual resultants and a supplementary condition Eq. 1.21a, that is useful in understanding effects of higher order. In particular, from Eq. 1.23c with the normal stress from the two surfaces added, it is clear that Eq. 1.21a is a statement regarding the stretching of the plate in the z-direction.

1.5 Plate Kinematics and Constitutive Relations

The equilibrium equations (1.14a-1.14f) alone are insufficient to determine the response of a plate to a particular load, because they contain too many unknown resultant components. Eqs. 1.14a,1.14b are two equations in three unknown in-plane stress resultants; and Eqs. 1.14c-1.14e are three equations in three unknown moment resultants and two unknown transverse shear resultants. The plate is therefore statically indeterminate. In the analysis of beams and beam structures, there are many significant problems that are directly statically determinant, so that the state of stress can be obtained only from the equations of equilibrium. In contrast, there are few plate problems for which a useful solution can be obtained directly from the equilibrium equations. Hence generally, it is necessary to consider the displacements for the plate.

The kinematic relations Eqs. 1.8a and 1.8b and constitutive relations Eq. 1.9a-1.9f are expressed in terms of the three-dimensional strains and stresses, but the equilibrium equations are expressed in terms of stress and moment resultants that are functions of just x and y. Equations 1.8a, 1.8b and 1.9a-1.9f must therefore be simplified to fit with our formulation in terms of the midsurface coordinates (x, y). This requires the introduction of some approximations.

1.5.1 Basics

1.5.1 .1 Plane Stress. From our experience with beams, we know that a beam carries loads primarily through axial and bending stress components in the direction parallel to the beam's axis. Transverse shear stresses and stresses acting normal to the beam's axis are relatively small. The generalization of this behavior to the case of a thin plate suggests the following approximation:

$$|\sigma_{zz}|, |\sigma_{xz}|, |\sigma_{yz}| \ll |\sigma_{xx}|, |\sigma_{yy}|, |\sigma_{xy}| \qquad (1.24)$$

Thus the constitutive relations Eqs. 1.9a, 1.9b, and 1.9d reduce to the *plane strain* approximation:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} \cong \frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} \qquad (1.25)$$

which has the inverse:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} \tag{1.26}$$

and the additional conditions:

$$\varepsilon_{zz} \cong -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy}) \;\; ; \;\; \gamma_{yz} \cong 0 \;\; ; \;\; \gamma_{xz} \cong 0 \tag{1.27}$$

1.5.1 .2 Plane Strain. The approximation Eq. 1.27 is widely used for thin beam, plate and shell theories. However, in certain situations, the plate or shell wall consists of stiff reinforcement in the transverse direction. An example is the biological membrane. The appropriate approximation is then that the strain in the radial direction is small:

$$|\varepsilon_{zz}| \ll |\varepsilon_{xx}|, |\varepsilon_{yy}| \tag{1.28}$$

The three-dimensional equations Eqs. 1.9a-1.9c then reduce to the *plane strain* relations:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} \cong \frac{1+\nu}{E} \begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} \tag{1.29}$$

which has the inverse:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{(1-2\nu)(1+\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{(1-2\nu)}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} \tag{1.30}$$

1.5.1 .3 Expansion for displacements. The displacements may be expanded in a Taylor series in the z-direction:

$$u(x, y, z) = u(x, y, 0) + z\frac{\partial}{\partial z}u(x, y, 0) + \frac{z^2}{2}\frac{\partial^2}{\partial x^2}u(x, y, 0) + \ldots \tag{1.31a}$$

$$v(x, y, z) = v(x, y, 0) + z\frac{\partial}{\partial z}v(x, y, 0) + \frac{z^2}{2}\frac{\partial^2}{\partial x^2}v(x, y, 0) + \ldots \tag{1.31b}$$

$$w(x, y, z) = w(x, y, 0) + z\frac{\partial}{\partial z}w(x, y, 0) + \frac{z^2}{2}\frac{\partial^2}{\partial x^2}w(x, y, 0) + \ldots \tag{1.31c}$$

So different levels of approximation are obtained by retaining a different number of terms in the expansion in the z-direction.

1.5.2 Classical Theory (Kirchhoff)

The 'classical' theory is obtained by retaining the linear terms in the in-plane displacement components and only the constant in the normal displacement:

$$u(x, y, z) \cong u^0(x, y) + z\beta_x(x, y) \tag{1.32a}$$

$$v(x, y, z) \cong v^0(x, y) + z\beta_y(x, y) \tag{1.32b}$$

$$w(x, y, z) \cong w^0(x, y) \tag{1.32c}$$

in which the superscript 0 denotes the displacement of the midsurface ($z = 0$), and the coefficients of the linear terms β_x and β_y can be interpreted as the rotations of the midsurface in the x- and y-directions. From the condition Eq. 1.27 that the transverse shear strain components be small, the rotations of the midsurface are related to the rotations of the unit normal to the surface:

$$\frac{\partial w^0}{\partial x} = -\beta_x \quad ; \quad \frac{\partial w^0}{\partial y} = -\beta_y \tag{1.33}$$

Substituting Eqs. 1.32a-1.33 into Eqs. 1.8a,1.8b yields the in-plane strain components:

$$\varepsilon_{xx} = \frac{\partial u^0}{\partial x} + z\frac{\partial \beta_x}{\partial x} = \varepsilon^0{}_{xx} + z\kappa_x \tag{1.34a}$$

$$\varepsilon_{yy} = \frac{\partial v^0}{\partial y} + z\frac{\partial \beta_y}{\partial y} = \varepsilon^0{}_{yy} + z\kappa_y \tag{1.34b}$$

$$\gamma_{xy} = \frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x} + z\left(\frac{\partial \beta_y}{\partial x} + \frac{\partial \beta_x}{\partial y}\right) = \gamma^0{}_{xy} + 2z\kappa_{xy} \tag{1.34c}$$

So each strain component has a constant term, corresponding to the midsurface strain, and a term varying with z, corresponding to the change in curvature of the midsurface. Because of Eq. 1.33, the changes in curvature can be written in terms of the derivatives of the normal displacement:

$$\kappa_x = -\frac{\partial^2 w^0}{\partial x^2} \quad ; \quad \kappa_y = -\frac{\partial^2 w^0}{\partial y^2} \quad ; \quad \kappa_{xy} = -\frac{\partial}{\partial y}\frac{\partial}{\partial x}w^0 \tag{1.35}$$

With Eqs. 1.34a-1.34c, the three-dimensional strain is approximated by quantities dependent only on the displacement of the midsurface, referred to as strain measures. Therefore, with the constitutive relation Eq. 1.26 and the definitions Eqs. 1.11, 1.12, the stress resultants can be obtained in terms of the strain measures. For example, the x-components of force resultant is:

$$N_x = \int_{-t/2}^{t/2} \sigma_{xx} dz = \int_{-t/2}^{t/2} \frac{E}{1-\nu^2}[\varepsilon^0_{xx} + \nu\varepsilon^0_{yy} + z(\kappa_x + \nu\kappa_y)]dz \tag{1.36}$$

and the moment is:

$$M_x = \int_{-t/2}^{t/2} \sigma_{xx} z dz = \int_{-t/2}^{t/2} \frac{E}{1-\nu^2}[\varepsilon^0_{xx} + \nu\varepsilon^0_{yy} + z(\kappa_x + \nu\kappa_y)]z dz \tag{1.37}$$

So for the elastic properties E and ν independent of the thickness coordinate z, the integrations reduce substantially because the curvature terms drop out of the expression for the in-plane force resultant Eq. 1.36 and the in-plane strains drop out of the expression for the bending moment resultant Eq. 1.37:

$$N_x = \frac{Et}{1-\nu^2}[\varepsilon^0_{xx} + \nu\varepsilon^0_{yy}] \tag{1.38}$$

$$M_x = \frac{Et^3}{12(1-\nu^2)}[\kappa_x + \nu\kappa_y] \tag{1.39}$$

The complete system for the in-plane force resultants is:

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \frac{Et}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix}\begin{bmatrix} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \gamma^0_{xy} \end{bmatrix} \tag{1.40}$$

with the inverse:

$$\begin{bmatrix} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \gamma^0_{xy} \end{bmatrix} = \frac{1}{Et}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} \tag{1.41}$$

and for the bending resultants:

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \frac{Et^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu) \end{bmatrix} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \tag{1.42}$$

with the inverse:

$$\begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \frac{12}{Et^3} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & (1+\nu) \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} \tag{1.43}$$

The bending stiffness coefficient is often denoted by the symbol D:

$$D = \frac{Et^3}{12(1-\nu^2)} \tag{1.44}$$

or alternatively as:

$$D = Etc^2 \tag{1.45}$$

where c is a reduced thickness:

$$c = \frac{t}{\sqrt{12(1-\nu^2)}} \tag{1.46}$$

Some notes are the following:

(i) The shear strain γ^0_{xy} in Eq. 1.34c is the *total* while the twist κ_{xy} is *half*. This is convenient since the twist is just the mixed second derivative of the displacement Eq. 1.35.

(ii) The transverse shear strains and stresses are assumed to be small (Eq. 1.27); however, this does *not* mean that the transverse shear force resultants Q_x and Q_y should be set to zero. These are very significant in the equations of equilibrium (Eqs. 1.14d-1.14e).

(iii) The reduction is shown for constant E and ν. However, the key is the explicit form for the dependence on z of the strains (Eqs. 1.34a-1.34c). Thus the same procedure holds for general layered anisotropic plates, yielding the resultants in terms of the strain measures defined on the mid-surface. Generally, however, the in-plane quantities Eq. 1.40 and the bending quantities Eq. 1.42 will be coupled. In the design of layered composite plates, a "balanced" design is often used, for which the bending and stretching are uncoupled, and furthermore, a "quasi-isotropic" layup of the fibers is used, for which Eqs. 1.40 and 1.42 are valid with effective values of E and ν.

The assumptions made are summarized as:

(i) The displacements of the plate are small in comparison to the plate's thickness, and the strains and midsurface slopes are much less than unity.

(ii) The displacements u and v vary linearly through the thickness of the cross section Eq. 1.32a and Eq. 1.32b, or in other words, "normals remain straight." This is analogous to the assumption that "plane sections remain plane" in elementary beam theory (Euler-Bernoulli beam theory). However, the plate kinematic model permits warping of an initially plane section, so that plane sections do not necessarily remain plane.

(iii) The normal stress σ_{zz} is small in comparison with the in-plane components σ_{xx}, σ_{yy} and σ_{xy} . This yields the plane stress approximation Eq. 1.25.

(iv) The normal displacement w is essentially constant through the thickness Eq. 1.32c.

(v) The transverse shear stresses σ_{xz} and σ_{yz} are small in comparison with the in-plane components σ_{xx}, σ_{yy} and σ_{xy} . Hence the transverse shear strains γ_{xz} and γ_{yz} are small and the slope of the mid-surface is the same as the rotation of the normals Eq. 1.33. In other words, "normals remain normal."

1.5.3 Shear Deformation Theory (Mindlin-Reissner)

Substantially better results for the thicker plates can be obtained by including the effect of transverse shear deformation. Consistent treatments were introduced by Reissner (1945)[10] and Mindlin (1951)[7]. Eqs. 1.32a-1.32c remain valid. However, assumption (v) above, giving Eq. 1.33, is not made. Instead, the transverse shearing strains are computed from Eqs. 1.8a:

$$\gamma_{xz} = \beta_x + \frac{\partial w^0}{\partial x} \quad ; \quad \gamma_{yz} = \beta_y + \frac{\partial w^0}{\partial y} \tag{1.47}$$

and the transverse shearing force resultants from Eq. 1.11 are:

$$Q_y = \int_{-t/2}^{t/2} \sigma_{yz} dz = \frac{1}{\mu} \int_{-t/2}^{t/2} G\gamma_{yz} dz = \frac{Gt}{\mu}\left(\beta_y + \frac{\partial w^0}{\partial y}\right) \quad ; \quad Q_x = \frac{Gt}{\mu}\left(\beta_x + \frac{\partial w^0}{\partial x}\right) \tag{1.48}$$

A more careful consideration of the three-dimensional shear stress distribution, yields the correction factor μ =6/5 (Reissner, 1945[10]) to Eq. 1.48. The other relations Eqs. 1.40, 1.42 remain the same as in the Kirchhoff theory. For the Kirchhoff theory, the unknown quantities are the displacement components u^0, v^0, and w^0. For the shear deformation theory, the rotation components β_x and β_y are also unknown. For a fiber reinforced, composite plate it is usually the situation that the matrix is soft in comparison with the fibers. In this case the transverse shear stiffness is determined mainly by the matrix modulus, while the effective E and ν are determined by the fibers. If we keep the relation Eq. 1.10, then the reduced matrix modulus can be taken into account by an increased value of μ. Typically for composites, $5 < \mu < 50$.

1.5.4 Higher Order Theories

There are a variety of techniques for including additional terms in the expansion for the displacements, such as discussed by Lo, Christensen, and Wu (1977)[5], and in countless subsequent publications. For the layered plate, however, the best approach is to use a "zig-zag" displacement field, in which the preceding shear deformation theory is used for each laminate, with the continuity of displacement enforced at each interface. When the elastic properties of the laminates are substantially different, this approach is necessary. For all such theories, the degree of complexity is substantially increased. For problems involving constraint conditions at the plate surface, the normal stress σ_{zz} can be significant, and a more refined representation is required for the normal displacement w. (See, e.g., Essenburg, 1975[2].)

1.6 Stresses in Plates

After the solution for the resultants is obtained, it is necessary to determine the three-dimensional stresses. The relation is obtained by substituting the strain distributions Eqs. 1.34a-1.34c into the stress-strain relations Eq. 1.26, and make use of the resultants Eqs. 1.38, 1.39, with the result:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{1}{t}\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} + \frac{12z}{t^3}\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} \tag{1.49}$$

Within the framework of the present assumptions, two contributions to the in-plane stresses are identified:

(i) *Direct stresses*, which are proportional to the in-plane stress resultants N_x, N_y, and N_{xy}, and are uniformly distributed through the thickness.

(ii) *Bending stresses*, which are proportional to the moment resultants M_x, M_y, and M_{xy}, and vary linearly through the thickness.

Equation 1.49 is the two-dimensional analog of the formula for the stress in a beam subjected to combined tension and bending. For extremely thin plates, or membranes, loads are carried predominantly by means of the direct stresses. Thus the direct stresses are also referred to as *membrane stresses*.

In addition to the three stress components given by Eq. 1.49, it may be required to know the values of the transverse shear stresses σ_{xz} and σ_{yz}, as well as the normal stress σ_{zz}. Recall that these stresses are generally much smaller than σ_{xx}, σ_{yy}, and σ_{xy}. It is interesting that the smaller quantities require more effort to estimate. After substitution for σ_{xx}, σ_{yy}, and σ_{xy} from Eq. 1.49, the first two equations Eqs 1.5a and 1.5b may be integrated with respect to z to obtain:

$$\sigma_{xz} = \sigma^0_{xz} - \frac{z}{t}\left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y}\right) - \frac{6z^2}{t^3}\left(\frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y}\right) \tag{1.50a}$$

$$\sigma_{yz} = \sigma^0{}_{yz} - \frac{z}{t}\left(\frac{\partial N_y}{\partial y} + \frac{\partial N_{yx}}{\partial x}\right) - \frac{6z^2}{t^3}\left(\frac{\partial M_y}{\partial y} + \frac{\partial M_{yx}}{\partial x}\right) \qquad (1.50b)$$

where $\sigma^0{}_{xz}$ and $\sigma^0{}_{yz}$ are integration constants. The equilibrium equations (Eqs. 1.14a, 1.14b, 1.14d, and 1.14e) can be used to rewrite Eqs. 1.50a, 1.50b as:

$$\sigma_{xz}(x, y, z) = \frac{3Q_x}{2t}\left[1 - \left(2\frac{z}{t}\right)^2\right] + R_x \qquad (1.51a)$$

$$\sigma_{yz}(x, y, z) = \frac{3Q_y}{2t}\left[1 - \left(2\frac{z}{t}\right)^2\right] + R_y \qquad (1.51b)$$

where the terms with zero average value are collected into R_x and R_y:

$$R_x = \sigma_{xz}\left(x, y, -\frac{t}{2}\right) - \frac{3m_y}{2t}\left[1 - \left(2\frac{z}{t}\right)^2\right] + \frac{1}{2}\left(1 + \frac{2z}{t}\right)p_x - \int_{-t/2}^{z} b_x dz \qquad (1.52a)$$

$$R_y = \sigma_{yz}\left(x, y, -\frac{t}{2}\right) + \frac{3m_x}{2t}\left[1 - \left(2\frac{z}{t}\right)^2\right] + \frac{1}{2}\left(1 + \frac{2z}{t}\right)p_y - \int_{-t/2}^{z} b_y dz \qquad (1.53b)$$

where the constants of integration have been chosen to satisfy the surface conditions at $z = \pm t/2$. The significant term is that due to the transverse shear resultant, which causes a transverse shear stress that varies parabolically through the thickness and has a maximum at the midsurface $z = 0$.

The normal stress is calculated from the third equation of equilibrium Eq. 1.5c in a similar fashion. The shear resultants Q_x and Q_y can be eliminated with Eq. 1.14c. The result is completely independent of the resultants! So we have a "statically determinate" result of a stress component determined directly from the surface loading. The distribution that satisfies the conditions on the faces $z = \pm t/2$ is:

$$\sigma_{zz}(x, y, z) = \sigma_{zz}\left(x, y, -\frac{t}{2}\right) + \frac{3}{4}\left[\frac{2}{3} + \frac{2z}{t} - \frac{1}{3}\left(\frac{2z}{t}\right)^3\right]p_z - \int_{-t/2}^{z} b_z dz - \int_{-t/2}^{z}\left(\frac{\partial R_x}{\partial x} + \frac{\partial R_y}{\partial y}\right)dz \qquad (1.54)$$

The significant effect is the loading in the z-direction, which according to Eq. 1.54 causes a normal stress σ_{zz} that varies as a cubic polynomial in z. Because the total integrals of R_x and R_y are zero, the last term in Eq. 1.54 is zero at both faces.

The approximate kinematic conditions and the constitutive relations are based on the assumption that the stresses Eqs. 1.51a, 1.51b, and 1.54 are small in comparison with Eq. 1.49. Therefore these can be calculated for a verification.

1.7 Plate Differential Equations

The preceding equations can be reduced to partial differential equations of a standard form. Until this point the partial derivatives have been written fully; subsequently the following shorter notation will be used, in which the comma before the index indicates the partial derivative:

$$\frac{\partial F}{\partial x} = F_{,x} \tag{1.55}$$

The particular advantage of the linear solution for the homogeneous, isotropic plate, or the balanced composite plate, is that the in-plane and the out-of-plane bending problems are uncoupled.

1.7.1 In-Plane Equation

A convenient approach to the in-plane problem is to introduce the *Airy stress function* ϕ. In terms of the scalar function $\phi = \phi(x, y)$ the in-plane resultants are:

$$N_x = \varphi_{,yy} - \int_a^x p_x dx \ ; \ N_y = \varphi_{,xx} - \int_b^y p_y dy \ ; \ N_{xy} = -\varphi_{,xy} \tag{1.56}$$

It is easily verified that the relations Eq. 1.56 satisfy identically the equations for in-plane equilibrium Eqs. 1.14a and 1.14b. the strains are then easily computed from Eq. 1.41:

$$\varepsilon^0_{xx} = \frac{1}{Et}\left(\varphi_{,yy} - \nu\varphi_{,xx} - \int_a^x p_x dx + \nu\int_b^y p_y dy\right) \tag{1.57a}$$

$$\varepsilon^0_{yy} = \frac{1}{Et}\left(\varphi_{,xx} - \nu\varphi_{,yy} - \int_b^y p_y dy + \nu\int_a^x p_x dx\right) \tag{1.57b}$$

$$\gamma^0_{xy} = -\frac{2(1+\nu)}{Et}\varphi_{,xy} \tag{1.57c}$$

The function ϕ is however, restricted since the strains must yield a consistent displacement distribution. The method for this is to consider the strain-displacement relations from Eqs. 1.34a-1.34c:

$$\varepsilon^0_{xx} = u^0_{,x} \ ; \ \varepsilon^0_{yy} = v^0_{,y} \ ; \ \gamma^0_{xy} = u^0_{,y} + v^0_{,x} \tag{1.58}$$

For the strains prescribed, Eq. 1.58 gives three equations for the two displacement components u and v. Consequently, there must be a condition on the strain components to permit a solution for the displacements. The condition, referred to as *compatibility of strain*, is:

$$\varepsilon^0_{xx,yy} + \varepsilon^0_{yy,xx} - \gamma^0_{xy,xy} = 0 \tag{1.59}$$

It is easily seen that substitution of the strains in terms of displacements Eq. 1.58 satisfies Eq. 1.59. Now, substituting the strains in terms of the stress function Eqs. 1.57a-1.57c into Eq. 1.59 yields the equation for ϕ:

$$\Delta\Delta\varphi = \int_a^x (p_{x,yy} - \nu p_{x,xx})dx + \int_b^y (p_{y,xx} - \nu p_{y,yy})dy \tag{1.60}$$

in which Δ is the *Laplacian*, or *harmonic* operator:

$$\Delta\varphi = \varphi_{,xx} + \varphi_{,yy} \tag{1.61}$$

The differential operator $\Delta\Delta$ is often referred to as the *biharmonic* operator. Since the Laplacian can be interpreted as the divergence $\nabla\bullet$ of the gradient ∇, the operators are also written as:

$$\Delta = \nabla\bullet\nabla = \nabla^2 \tag{1.62}$$

$$\Delta^2 = \nabla^2\nabla^2 = \nabla^4 \tag{1.63}$$

1.7.2 Bending Equation (Kirchhoff)

The basic approximation that normals remain normal Eq. 1.33 gives the rotations in terms of the displacement of the midsurface. Equations 1.34a-1.34c give the curvature change measures in terms of the rotations, and Eqs. 1.42 give the moments in terms of the curvature changes. Therefore, the transverse shear Eq. 1.14d can be written in terms of the displacement:

$$Q_x = m_y + M_{x,x} + M_{yx,y} \tag{1.64a}$$

$$= m_y + D\left[(\beta_{x,x} + \nu\beta_{y,y})_{,x} + \frac{(1-\nu)}{2}(\beta_{x,y} + \beta_{x,y})_{,y}\right] \tag{1.64b}$$

$$= m_y - D[(w_{,xx} + \nu w_{,yy})_{,x} + (1-\nu)w_{,xy,y}] \tag{1.64c}$$

$$= m_y - D(w_{,xx} + w_{,yy})_{,x} \tag{1.64d}$$

$$= m_y - D\Delta w_{,x} \tag{1.64e}$$

For this reduction, the plate bending stiffness D is constant. Substituting Eq. 1.64e and the similar equation for the y-component into the equation of equilibrium Eq. 1.14c yields the final equation:

$$(m_y - D\Delta w_{,x})_{,x} + (-m_x - D\Delta w_{,y})_{,y} + p_z = 0 \tag{1.65}$$

which can be rewritten as:

$$D\Delta\Delta w = p_z + m_{y,x} - m_{x,y} \tag{1.66}$$

Thus both the in-plane and the out-of-plane problems reduce to the biharmonic operator.

1.7.3 Plate Bending with Transverse Shear Deformation (Mindlin-Reissner)

For the consideration of the transverse shear deformation, the components of the rotation of the normal β_x and β_y can be rewritten in terms of two functions Φ and Ψ:

$$\beta_x = -\Phi_{,x} + \Psi_{,y} \quad (1.67a)$$

$$\beta_y = -\Phi_{,y} - \Psi_{,x} \quad (1.67b)$$

In terms of vector analysis, we are using the standard procedure of writing the rotation vector as the sum of the gradient of the scalar Φ and the curl of a vector with the z-component Ψ. Since we are dealing with a linear system of equations, the equations for each component can be handled separately. It is also necessary to split the load terms into two parts:

$$m_y = m_{G,x} + m_{C,y} \quad (1.68a)$$

$$m_x = -m_{G,y} + m_{C,x} \quad (1.68b)$$

where m_G and m_C are two scalar functions. Note that the curl part m_C drops out of the equation Eq. 1.66.

1.7.3 .1 Gradient part. The substitution of the Φ part of Eqs. 1.67a and 1.67b into Eq. 1.64b yields instead of Eq. 1.64e the results:

$$M_x = -D(\Phi_{,xx} + \nu\Phi_{,yy}) \quad (1.69a)$$

$$M_y = -D(\Phi_{,yy} + \nu\Phi_{,xx}) \quad (1.69b)$$

$$M_{xy} = -D(1-\nu)\Phi_{,xy} \quad (1.69c)$$

$$Q_x = m_{G,x} - D\Delta\Phi_{,x} \quad (1.69d)$$

$$Q_y = m_{G,y} - D\Delta\Phi_{,y} \quad (1.69e)$$

So, instead of Eq. 1.66 the result is exactly the same equation for Φ:

$$D\Delta\Delta\Phi = p_z + m_{y,x} - m_{x,y} = p_z + \Delta m_G \quad (1.70)$$

The equation for the normal displacement w comes from Eq. 1.48:

$$w_{,x} = \Phi_{,x} + \frac{\mu Q_x}{Gt} \quad (1.71a)$$

so from Eq. 1.69a:

$$w_{,x} = \Phi_{,x} + \frac{\mu}{Gt}(m_{G,x} - D\Delta\Phi_{,x}) \quad (1.71b)$$

$$w_{,y} = \Phi_{,y} + \frac{\mu}{Gt}(m_{G,y} - D\Delta\Phi_{,y}) \tag{1.71c}$$

so the integral of each is the same:

$$w = \Phi + \frac{\mu}{Gt}(m_G - D\Delta\Phi) \tag{1.71d}$$

By Eq. 1.71d the effect of the transverse shear deformation can be estimated. The Laplacian term has the coefficient $\delta_G{}^2$, where:

$$\delta_G = \sqrt{\frac{\mu}{Gt}D} = t\sqrt{\frac{\mu}{6(1-\nu^2)}} \tag{1.72}$$

If the solution Φ of Eq. 1.70 varies significantly over the distance L, then each derivative has the magnitude:

$$|\Phi_{,x}|, |\Phi_{,x}| < \frac{|\Phi|}{L} \tag{1.73}$$

The Laplacian term in Eq. 1.71d is small for $L >> \delta_G$, e.g., for a deformation with the variation distance L that is large in comparison with the plate thickness. A similar argument can be made for the m_G term. When both terms are neglected in Eq. 1.71d, we have the Kirchhoff result Eq. 1.66 with $\Phi = w$.

1.7.3 .2 Curl Part. the component Ψ in Eqs. 1.67a and 1.67b leads to a rather different type of equation. The bending moment and transverse shears components are:

$$M_x = (1-\nu)D\Psi_{,xy} \tag{1.74a}$$

$$M_y = -(1-\nu)D\Psi_{,xy} \tag{1.74b}$$

$$M_{xy} = \frac{1}{2}D(1-\nu)(\Psi_{,yy} - \Psi_{,xx}) \tag{1.74c}$$

$$Q_x = \left[D\frac{(1-\nu)}{2}\Delta\Psi + m_C\right]_{,y} \tag{1.74d}$$

$$Q_y = -\left[D\frac{(1-\nu)}{2}\Delta\Psi + m_C\right]_{,x} \tag{1.74e}$$

With these shear components, the equation of transverse force equilibrium Eq. 1.14c is identically satisfied, and the derivatives of the normal displacement are:

$$w_{,x} = \left[-\Psi + \frac{\mu}{Gt}\left(D\frac{(1-\nu)}{2}\Delta\Psi + m_C\right)\right]_{,y} \tag{1.75a}$$

$$w_{,y} = -\left[-\Psi + \frac{\mu}{Gt}\left(D\frac{(1-\nu)}{2}\Delta\Psi + m_C\right)\right]_{,x} \tag{1.75b}$$

So, the only solution for these two equations is for w to be identically zero, and for Ψ to satisfy the partial differential equation:

$$-\frac{\mu}{Gt}D\frac{(1-\nu)}{2}\Delta\Psi + \Psi = \frac{\mu}{Gt}m_C \tag{1.75c}$$

The solution of this equation consists of a particular solution, due to the nonhomogeneous load term on the right-hand side, and a complementary solution of the homogeneous equation, (with zero right-hand side). The coefficient of the derivative terms is δ_C^2, where:

$$\delta_C = \sqrt{\frac{\mu}{Gt}D\frac{(1-\nu)}{2}} = t\sqrt{\frac{\mu}{12(1-\nu^2)}} \tag{1.76}$$

So δ_C is equal to a length less than the thickness. For a plate with lateral dimensions large in comparison with the thickness, this is called a *singular perturbation* problem. The solution of the homogeneous equation consists of terms which decrease exponentially within the distance δ_C from the boundary of the plate, and so are referred to as *boundary layer* solutions. Therefore, the effect of this extra work in including the transverse shear deformation consists of a little correction to the displacement w in the potential solution Eq. 1.71d and a boundary layer correction within a thickness of the boundary from the curl part Eq. 1.75c. Consequently, an excellent understanding of the general behavior of the plate can be obtained by considering only the reduced Kirchhoff theory.

1.7.4 Stretching of Normal

Equation 1.54 provides the solution for the distribution of the stress in the normal direction in terms of the surface loading. However, insight into the effects of higher order can be obtained by considering the stretching of the normal. Instead of Eq. 1.32a-1.32c, we add a term to the normal displacement:

$$u(x, y, z) \cong u^0(x, y) + z\beta_x(x, y) \tag{1.77a}$$

$$v(x, y, z) \cong v^0(x, y) + z\beta_y(x, y) \tag{1.77b}$$

$$w(x, y, z) \cong w^0(x, y) + z\beta_z(x, y) \tag{1.77c}$$

The strains from Eqs. 1.8a and 1.8b are:

$$\varepsilon_{zz} = w_{,z} = \beta_z \tag{1.78a}$$

$$\gamma_{xz} = w_{,x} + u_{,z} = z\beta_{z,x} + \beta_x + w^0_{,x} \tag{1.78b}$$

$$\gamma_{yz} = w_{,y} + v_{,z} = z\beta_{z,y} + \beta_y + w^0_{,y} \tag{1.78c}$$

which give the resultants from Eqs. 1.22b:

$$M_{xz} = \int_{-t/2}^{t/2} \sigma_{xz} z dz = \frac{Gt^3}{12}\beta_{z,x} \tag{1.79a}$$

$$M_{yz} = \int_{-t/2}^{t/2} \sigma_{yz} z dz = \frac{Gt^3}{12}\beta_{z,y} \tag{1.79b}$$

$$N_z = \int_{-t/2}^{t/2} \sigma_{zz} dz = Et\beta_z + \nu(N_x + N_y) \tag{1.79c}$$

Thus Eq. 1.21a yields the equation for β_z:

$$-\frac{Gt^3}{12Et}\Delta\beta_z + \beta_z = \frac{1}{Et}[m_{z2} - \nu(N_x + N_y)] \tag{1.80}$$

which is a second order equation for the average normal strain β_z.The coefficient of the Laplacian operator is δ_z^2, where:

$$\delta_z = \frac{t}{\sqrt{12}} \quad \leftarrow \quad \frac{t}{\sqrt{24(1+\nu)}} \tag{1.81}$$

Once again, for a solution with the variation on the distance $L >> \delta_z$, the Laplacian term is small and the average strain is:

$$\beta_z \cong \frac{1}{Et}[m_{z2} - \nu(N_x + N_y)] \tag{1.82}$$

which is the condition that the normal stress be small. The solution of Eq. 1.80 with zero right-hand side, however, permits the prescription of the normal strain on the boundary. This is again a boundary layer effect, with the decay distance δ_z. This decay distance is about the same as for the transverse shear layer Eq. 1.76. Therefore, in general, the solution for the plate separates into global effects described by the classical Kirchhoff theory and three dimensional effects that are restricted to the region of about one thickness near the boundaries.

1.8 Boundary Conditions

The boundary conditions are perhaps most easily understood for the rectangular, three-dimensional body in Fig. 1.2. For the face with outward normal in the positive x-direction, the tractions, or components of stress, that can be prescribed are σ_{xx}, σ_{xy}, and σ_{xz}. The displacements in the directions of the stress components are u, v, and w. So the differential work that the boundary tractions perform on the body is the product of the stress components and the corresponding differential displacements. Therefore, the rate of work on the face is:

$$\dot{W} = \int_{y_1}^{y_2}\int_{-t/2}^{t/2} [\sigma_{xx}\dot{u} + \sigma_{xy}\dot{v} + \sigma_{xz}\dot{w}]dzdy \tag{1.83a}$$

where the dots denote the time derivative. (Alternately, the dots could denote the virtual displacements.) With the approximation for the displacements Eq. 1.77a-1.77c, the expression for the work becomes:

$$\dot{W} = \int_{y_1}^{y_2}\int_{-t/2}^{t/2} [\sigma_{xx}(\dot{u}^0 + z\dot{\beta}_x) + \sigma_{xy}(\dot{v}^0 + z\dot{\beta}_y) + \sigma_{xz}(\dot{w}^0 + z\dot{\beta}_z)]dzdy \tag{1.83b}$$

which from the definitions of the resultants Eqs. 1.11 and 1.12 reduces to the plate quantities:

$$\dot{W} = \int_{y_1}^{y_2} [N_x\dot{u}^0 + M_x\dot{\beta}_x + N_{xy}\dot{v}^0 + M_{xy}\dot{\beta}_y + Q_x\dot{w}^0 + N_z\dot{\beta}_z]dy \tag{1.83c}$$

Hence, the edge tractions on the face of the three-dimensional body are reduced to the plate resultants which can be prescribed on the edge of the plate:

$$N_x,\ N_{xy},\ M_z,\ M_x,\ M_{xy},\ Q_x \tag{1.84a}$$

and the corresponding displacement and rotation quantities:

$$u^0,\ v^0,\ \beta_z,\ \beta_x,\ \beta_y,\ w^0 \tag{1.84b}$$

For boundary conditions, either the force quantities Eq. 1.84a, or the displacement quantities Eq. 1.84b, may be prescribed on the edge. Another possibility is to prescribe a linear combination of the force and displacement quantities which includes a mix of some displacement and some force quantities. For the isotropic plate, the in-plane and bending problems are uncoupled.

1.8.1 Isotropic Plate In-Plane

Equation 1.60 governs the in-plane behavior of the plate, also referred to as the *plane stress* problem. This is a fourth order partial differential equation for the stress function φ. In the classification of partial differential equations, this is referred to as *elliptic*. Consequently, exactly two scalar boundary conditions must be prescribed at each point of the boundary. The two force quantities are obviously:

$$N_x,\ N_{xy} \tag{1.85a}$$

and the two corresponding displacement quantities are:

$$u^0,\ v^0 \tag{1.86}$$

From the relation of the stress resultants and the stress function Eq. 1.56, prescribing the resultants Eq. 1.85a is equivalent to prescribing the value of φ and its normal derivative on each point of the boundary. This convenient treatment for prescribed forces on the boundary is an advantage of the stress function approach. Displacements require more effort, since the strain displacement relations Eq. 1.58 must be integrated, and no simple relation between the stress function and the displacement components can be found. However, a combination of displacements can be considered, namely the strain along the boundary and the change in curvature of the boundary. These are quantities that are independent of rigid body translation and rotation. For the positive x-edge, the strain along the edge is:

$$\varepsilon^0_{yy} = v_{,y} = \frac{1}{Et}\left(\varphi_{,xx} - \nu\varphi_{,yy} - \int_b^y p_y dy + \nu\int_a^x p_x dx\right) \tag{1.86a}$$

and the curvature change of the edge is:

$$\kappa_g = -u^0_{,yy} = -(u^0_{,y} + v^0_{,x})_{,y} + (v^0_{,y})_{,x} = -\gamma_{,y} + \varepsilon^0_{yy,x}$$

Writing the strain quantities in terms of the stress function yields:

$$\kappa_g = \frac{1}{Et}\left[\Delta\varphi_{,x} + (1+\nu)\varphi_{,xyy} - \int_b^y p_{y,x} dy + \nu p_x\right] \tag{1.86b}$$

Thus the derivatives of the displacements along the boundary yield results in terms of the derivatives of the stress function, which is convenient for many problems.

The effect of stretching of the normal can be added. Equation 1.80 is a second order equation for β_z, once the resultants N_x and N_y are known. Thus one scalar condition is required on each point of the boundary, either the average strain β_z or the "resultant" M_{xz} in terms of the derivative of β_z given by Eq. 1.79a.

1.8.2 Isotropic Plate Bending with Mindlin-Reissner Theory

After deleting the in-plane quantities, three force and three displacement quantities remain to be satisfied Eqs. 1.84a and 1.84b. The gradient part of plate bending Eq. 1.71a is a fourth order equation and the curl part Eq. 1.75c is a second order equation. So together, three conditions on each point of the boundary can be prescribed. The in-plane resultants and the stretching, discussed in the previous section, can be handled separately. It is unfortunate that the gradient and curl parts of the bending problem are mixed together. The displacement quantities that can be prescribed on the positive x-edge are repeated from Eqs. 1.67a, 1.67b, and 1.72:

$$\beta_x = -\Phi_{,x} + \Psi_{,y} \tag{1.87a}$$

$$\beta_y = -\Phi_{,y} - \Psi_{,x} \tag{1.87b}$$

$$w = \Phi + \frac{\mu}{Gt}(m_G - D\Delta\Phi) \tag{1.87c}$$

and the corresponding resultants are from Eqs. 1.69a, 1.69c, 1.69d, 1.74a, 1.74c, and 1.74d:

$$M_x = -D[\Phi_{,xx} + \nu\Phi_{,yy} - (1-\nu)\Psi_{,xy}] \tag{1.88a}$$

$$M_{xy} = -D(1-\nu)\left[\Phi_{,xy} - \frac{1}{2}(\Psi_{,yy} - \Psi_{,xx})\right] \tag{1.88b}$$

$$Q_x = m_{G,x} + m_{C,y} - D\Delta\left[\Phi_{,x} - \frac{(1-\nu)}{2}\Psi_{,y}\right] \tag{1.88c}$$

Apparently no exact separation of the Φ and Ψ contributions can be made. However, because of the boundary layer nature of the Ψ contribution, an approximate uncoupling is possible. For a computational finite element approach, a separation of the problem into the Φ and Ψ contributions is not made, and the formulation in terms of the basic quantities Eq. 1.83c is straightforward.

1.8.3 Isotropic Plate Bending for Kirchhoff Theory

The preceding shear deformation theory is satisfying from several aspects. However, it is also more elaborate than is necessary for the general problem of plate bending. The simpler classical theory of Kirchhoff (1850)[4] is widely used. The biharmonic Eq. 1.66 with the displacement w governs the problem. This is a fourth order equation, so that two independent quantities must be prescribed at each point of the boundary. A problem emerges, however, since the three displacement quantities (the displacement w, and the two rotation components β_x and β_y) or the three force quantities (Q_x, M_x, and M_{xy}) can be prescribed in the preceding shear deformation theory. the proper reduction to only two boundary conditions can be obtained from the work on the edge Eq. 1.83c. For the Kirchhoff theory the rotation components are related directly to the derivatives of the normal displacement Eq. 1.33. Thus the edge work rate for the bending problem becomes:

$$\dot{W} = \int_{y_1}^{y_2} [M_x\dot{\beta}_x + M_{xy}\dot{\beta}_y + Q_x\dot{w}]dy \tag{1.89a}$$

$$= \int_{y_1}^{y_2} [-M_x\dot{w}_{,x} - M_{xy}\dot{w}_{,y} + Q_x\dot{w}]dy \tag{1.89a}$$

However, the second term may be integrated by parts, giving:

$$= (-M_{xy}\dot{w})\Big|_{y_1}^{y_2} + \int_{y_1}^{y_2} [-M_x\dot{w}_{,x} + (M_{xy,y} + Q_x)\dot{w}]dy \tag{1.89a}$$

Thus the work done by the normal displacement w depends on the magnitude of the combined term $V_x = Q_x + M_{xy,y}$. This represents an effective transverse shear force on the edge. Thus the Kirchhoff theory uses the force boundary quantities of only:

$$M_x \text{ and } V_x = Q_x + M_{xy,y} \tag{1.90a}$$

and the corresponding displacement and rotation quantities:

$$\beta_x = -w_{,x} \text{ and } w \tag{1.90b}$$

Writing the boundary quantities Eq. 1.90a in terms of the displacement gives:

$$M_x = -D(w_{,xx} + \nu w_{,yy}) \tag{1.91a}$$

$$V_x = m_{G,x} + m_{C,y} - D(\Delta w_{,x} + (1-\nu) w_{,xyy}) \tag{1.91b}$$

So it is interesting that without surface loads, the in-plane stress problem, with the boundary conditions on the strain Eq. 1.86a and curvature change Eq. 1.86b, is exactly analogous to the Kirchoff plate bending problem, with the boundary conditions on moment M_x Eq. 1.91a and effective transverse shear V_x. Eq. 1.91b, except for a change in the sign of Poisson's ratio. Similarly, the in-plane problem, with boundary conditions on both stress components, is analogous to the bending problem of prescribed displacement and rotation.

The extra term in the work Eq. 1.89c from the integration is of considerable importance. This shows that the twisting moments M_{xy} at the corners $y = y_1$ and $y = y_2$ has the effect of a concentrated force. This is called the *corner force*. For the rectangular plate, There are also contribution from the edges $x = x_1$ and x_2. So the total corner force is equal to $2M_{xy}$ and, for positive M_{xy}, acts in the negative z-direction on the corners (x_1, y_1) and (x_2, y_2). On the corners (x_1, y_2) and (x_2, y_1) the force $2M_{xy}$ is in the positive z-direction, as shown in Fig. 1.6.

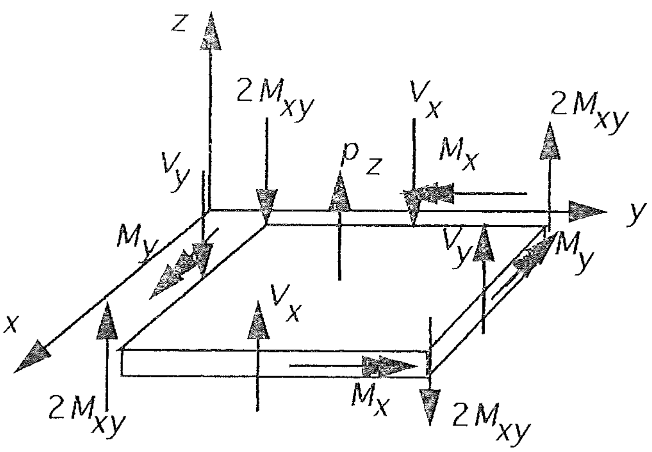

FIGURE 1.6. Effective resultants from Kirchhoff theory acting on the rectangular plate.

Chapter 2 Cylindrical Bending of Plates

The plate equations are two-dimensional, in the x-y plane. However, it is of value to consider the situation for which the solution depends on only one variable, that we take to be the x-coordinate. The displacement in the y-direction is taken as identically zero. Thus the exact solution for the three-dimensional body reduces to the plane strain problem in the x-z plane. The plate equation reduces to a one-dimensional equation in the x-coordinate, exactly that of a straight beam, but with the plate stiffness term which has dependence on Poisson's ratio. The following problems are of interest because an exact solution of the plane strain equation can be compared with the approximate plate solution. For convenience, the plate equations necessary for the cylindrical bending are summarized here. The equilibrium equations Eq. 1.14c and Eq. 1.14d reduce to:

$$Q_{x,x} = -p_z \tag{2.1a}$$

$$M_{x,x} = Q_x - m_y \tag{2.1b}$$

and the moment-curvature relations Eq. 1.42 give:

$$M_x = D\kappa_x \tag{2.1c}$$

$$M_y = \nu D\kappa_x \tag{2.1d}$$

The curvature-rotation relation Eq. 1.34a is:

$$\kappa_x = \beta_{x,x} \tag{2.1e}$$

and the rotation- displacement relation from the Kirchhoff theory Eq. 1.33 is:

$$w^0{}_{,x} = -\beta_x \tag{2.1f}$$

while that from the Mindlin-Reissner theory Eq. 1.48 is:

$$w^0{}_{,x} = -\beta_x + \frac{\mu Q_x}{Gt} \tag{2.1g}$$

For *statically determinate* problems, Q_x and M_x are known at some point. In this case, the *shear diagram* is the integral of Eq. 2.1a and the *moment diagram* is the integral of Eq. 2.1b. The rotation is obtained from the integral of Eq. 2.1e and the displacement from the integral of Eq. 2.1f, or Eq. 2.1g for more accuracy. For statically indeterminate problems, the boundary conditions may be in terms of the displacement and rotation. The elimination of the stress resultants in the system of equations leads to the equations for Mindlin-Reissner theory:

$$(D\beta_{x,x})_{,xx} = -p_z - m_{y,x} \tag{2.2a}$$

$$w_{,x} = -\beta_x + \frac{\mu}{Gt}[(D\beta_{x,x})_{,x} + m_y] \tag{2.2b}$$

while for the Kirchhoff theory the final equation reduces to:

$$(Dw_{,xx})_{,xx} = p_z + m_{y,x} \tag{2.3a}$$

For constant bending stiffness D, this is the same as Eq. 1.66 with the dependence on the y-coordinate removed.

2.1 Pure Bending

This is the most simple problem in beam and plate bending.

In Fig. 2. 1 is a sketch of a plate clamped to a wall at $x = -L$ and loaded at the end $x = 0$ by a moment M_L. The z-coordinate is chosen in the downward direction, so that the positive moment from Eq. 1.12 is in the direction corresponding to tension in the x-direction at positive values of z.

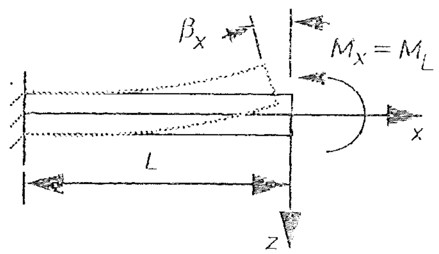

Figure 2. 1 Pure bending of plate.

2.1.1 Plate Solution

For zero transverse loading p_z, zero distributed moment m_y, and zero end shear Q_x, the solution of Eq. 2.1a is Q_x identically zero, and from Eq. 2.1b that M_x is constant. Thus:

$$M_x \equiv M_L = D\kappa_x = D\beta_{x,x} \tag{2.4a}$$

In addition, Eq. 2.1d shows that the moment in the y-direction is nonzero:

$$M_y = \nu D\kappa_x \tag{2.4b}$$

Therefore, the stress components from Eq. 1.49, Eq. 1.51a, and Eq. 1.54 are:

$$\sigma_{xx} = \frac{12M_L}{t^2}z \tag{2.5a}$$

$$\sigma_{xz} = \sigma_{zz} = 0 \tag{2.5b}$$

Since the transverse shear Q_x is zero, the rotation is related to the derivative of the displacement in both the Kirchhoff theory Eq. 2.1f and in the shear deformation theory Eq. 2.1g. Thus the integral of Eq. 2.4a gives the rotation:

$$\beta_x = \frac{M_L}{D}(x + L) = -w^0_{,x} \tag{2.6a}$$

with the constant of integration chosen so that the rotation is zero at the clamped end. So Eq. 1.34a gives the tangential displacement:

$$u = \frac{M_L}{D}(x+L)z \tag{2.6b}$$

and the integration of Eq. 2.6a gives the transverse displacement:

$$w^0 = -\frac{M_L}{2D}(x+L)^2 \tag{2.6c}$$

2.1.2 Exact Solution

The body in Fig. 2.1 can be considered as a plane strain in the x-z plane. So the Eq. 1.60 is valid with the y-coordinate replaced with z. The exact solution has the stress function:

$$\varphi = \frac{2M_L}{t^3}z^3 \tag{2.7}$$

which yields *exactly* the stress components Eq. 2.5a and Eq. 2.5b. The displacements are:

$$u = \frac{M_L L}{D}\left(1 + \frac{x}{L}\right)z \tag{2.8a}$$

$$w = -\frac{M_L L}{2D}\left[\left(1+\frac{x}{L}\right)^2 + \frac{t^2}{L^2}\frac{\nu}{4(1+\nu)}\left(2\frac{z}{t}\right)^2\right] \tag{2.8b}$$

Therefore in the case of pure bending, the plate solution gives the exact stress distribution and the exact tangential displacement u. The exact transverse displacement w Eq. 2.8b agrees with the plate solution on the midsurface $z = 0$ Eq. 2.6c, but changes with the distance from the midsurface. Note also that the exact condition of zero w at every point of the clamped end is not satisfied by this exact solution. (In fact, to satisfy this condition a singular stress state is required at the clamped corners.) The maximum error at the end $x = 0$ is however of the magnitude:

$$\text{Error} = \frac{\nu}{4(1+\nu)}\left(\frac{t}{L}\right)^2 \tag{2.9}$$

So for a beam longer than the thickness, the additional displacement in Eq. 2.8b is rather small.

2.2 Pure Shear Loading

Now we consider the next most basic problem in beam and plate bending. In Fig. 2. 2 is a sketch of a plate clamped to a wall at $x = -L$ and loaded at the end $x = 0$ by a shear force Q_L. The solution is that the shear is constant but that the moment varies along the beam.

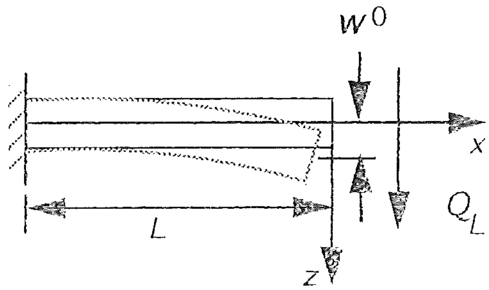

Figure 2. 2 Pure shear loading of plate.

2.2.1 Plate Solution

For zero p_z and m_y, and nonzero shear at $z = 0$, Eq. 2.1a and Eq. 2.1b are satisfied by a constant transverse shear and a linearly varying bending moment:

$$Q_x \equiv Q_L \tag{2.10a}$$

$$M_x = D\kappa_x = D\beta_{x,x} = Q_L x \tag{2.10b}$$

The constant of integration is chosen to give the moment M_x that is zero at the end $x = 0$. As before, Eq. 2.1d shows that the moment in the y-direction is nonzero:

$$M_y = \nu D\kappa_x \tag{2.10c}$$

Therefore, the tangential stress component is from Eq. 1.49:

$$\sigma_{xx} = \frac{12Q_L}{t^3}zx \tag{2.11a}$$

while the transverse shear stress is computed from Eq. 1.51a:

$$\sigma_{xz} = \frac{Q_L}{t}\frac{3}{2}\left[1 - \left(\frac{2z}{t}\right)^2\right] \tag{2.11b}$$

and the normal stress is zero from Eq. 1.54:

$$\sigma_{zz} = 0 \tag{2.11c}$$

The rotation is the integral of Eq. 2.10b, with the constant of integration chosen to give zero rotation at $x = -L$:

$$\beta_x = -\frac{Q_L L^2}{2D}\left(1 - \frac{x^2}{L^2}\right) \tag{2.12a}$$

so Eq. 1.34a gives the tangential displacement:

$$u = -\frac{Q_L L^2}{2D}\left(1 - \frac{x^2}{L^2}\right)z \tag{2.12b}$$

Another integration gives the transverse displacement. For the Kirchhoff theory Eq. 1.33 gives:

$$w^0 = \frac{Q_L L^3}{3D}\left(\frac{x}{L} + 1\right)\left(1 + \frac{x}{2L} - \frac{x^2}{2L^2}\right) \tag{2.12c}$$

while the Mindlin-Reissner theory Eq. 1.48 gives an additional displacement contribution:

$$w^0 = \frac{Q_L L^3}{3D}\left(\frac{x}{L} + 1\right)\left(1 + \frac{x}{2L} - \frac{x^2}{2L^2} + \frac{3\mu D}{GtL^2}\right) \tag{2.12d}$$

Writing out the additional contribution gives:

$$\frac{3\mu D}{GtL^2} = \frac{\mu t^2}{2(1-\nu)L^2} \tag{2.12e}$$

So for the isotropic plate, with $\mu = 6/5$, the shear deformation correction is small when the length is greater than the thickness.

2.2.2 Exact Solution

The body in Fig. 2. 2 can be considered as a plane strain in the x-z plane. So Eq. 1.60 is valid with the y-coordinate replaced with z. The exact solution has the stress function:

$$\varphi = \frac{12Q_L}{t^2}\left[z^2 - 3\left(\frac{t}{2}\right)^2\right]xz \tag{2.13}$$

which yields exactly the stress components Eq. 2.11a-Eq. 2.11c. The displacement in the tangential direction is:

$$u = \frac{Q_L L^2}{2D}\left[\frac{x^2}{L^2} - 1 + \frac{(2-\nu)t^2}{4(1-\nu)L^2}\left\{\frac{1}{5} - \frac{1}{3}\left(\frac{2z}{t}\right)^2\right\}\right]z \tag{2.14}$$

A two-dimensional effect appears, in the additional z-dependence in the exact solution compared with the plate solution Eq. 2.12b. Since the behavior is cubic in the z-coordinate, this is referred to as *warping* of the normal. The effect is, however, also small when the length L is greater than the thickness t. However, using Eq. 2.14 to compute an averaged tangential displacement $\bar{u}$,defined by:

$$\bar{u}(x) = \frac{1}{M_x}\int_{-\frac{t}{2}}^{\frac{t}{2}} u(x, z)\sigma_{xx}(x, z)dz \tag{2.15}$$

yields exactly the same as the plate solution Eq. 2.12b. The exact transverse displacement is:

$$w = -\frac{Q_l L^3}{3D}\left[\left(\frac{x}{L}+1\right)\left(1+\frac{x}{2L}-\frac{x^2}{2L^2}\right)-\frac{t^2 x}{L^3}\left[\left(\frac{2z}{t}\right)^2-\frac{1}{5}\right]\frac{3\nu}{8(1-\nu)}\right] \tag{2.16a}$$

However, the averaged displacement $\bar{w}$, defined by:

$$\bar{w}(x) = \frac{1}{Q_x}\int_{-\frac{t}{2}}^{\frac{t}{2}} w(x, z)\sigma_{xz}(x, z)dz \tag{2.16b}$$

yields:

$$\bar{w} = \frac{Q_l L^3}{3D}\left(\frac{x}{L}+1\right)\left(1+\frac{x}{2L}-\frac{x^2}{2L^2}+\frac{t^2}{L^2}\frac{3}{5(1-\nu)}\right) \tag{2.16c}$$

which is exactly the plate solution Eq. 2.12d for $\mu = 6/5$, which is a justification for the use of this value of μ. So for the pure shear problem, as well as the pure bending problem, the difference between the Kirchhoff theory, the Mindlin-Reissner theory and the exact solutions is in terms that have the size of t^2/L^2. Also to note is the ratio of the maximum of the transverse shear stress Eq. 2.11b to the maximum of the tangential stress Eq. 2.11a:

$$\frac{|\sigma_{xy}|_{\max}}{|\sigma_{xx}|_{\max}} = \frac{t}{4L} \tag{2.17}$$

So for the thin plate, the transverse stress is small in comparison with the tangential. This is the justification for the Kirchhoff approximation Eq. 1.27 that the transverse shear strain in small in comparison with the tangential.

2.3 Transverse Loading

Now we consider a problem with a uniform transverse loading of the plate with the nonzero magnitude p_z. In Fig. 2. 3 is the sketch of the plate clamped to a wall at $x = -L$ and free at the end $x = 0$. The solution has a resultant shear that varies linearly and a resultant moment that varies quadratically along the beam. The exact solution has a nonzero transverse normal stress σ_{zz} that is of interest to compare with the plate approximation. To obtain a simple exact solution, however, a special distribution of body force is used:

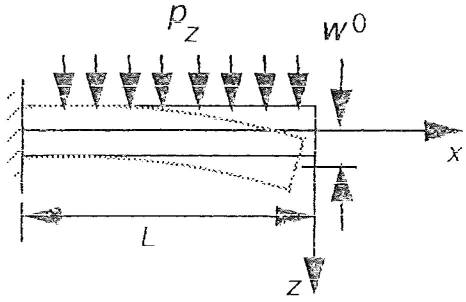

Figure 2. 3 Uniform loading of plate.

$$b_z = \frac{p_z}{t^3}\frac{12(1-\nu)}{\nu}z^2 \tag{2.18a}$$

with the surface tractions:

$$\sigma_{zz}\left(x, \frac{t}{2}\right) = -p_z\left(\frac{1}{2\nu} - 1\right) \tag{2.18b}$$

$$\sigma_{zz}\left(x, -\frac{t}{2}\right) = p_z\left(\frac{1}{2\nu} - 1\right) \tag{2.18c}$$

It can be verified that substituting Eq. 2.18a-Eq. 2.18c into Eq. 1.18c yields p_z, the resultant force per unit length of the midsurface. The dependence of the body force and tractions on Poisson's ratio is strange, but it works.

2.3.1 Plate Solution

The equilibrium equations Eq. 2.1a and Eq. 2.1b are satisfied by the distribution of transverse shear and bending moment:

$$Q_x = -p_z x \tag{2.19a}$$

$$M_x = D\kappa_x = D\beta_{x,x} = -\frac{p_z x^2}{2} \tag{2.19b}$$

The constants of integration are chosen so that both the moment M_x and the transverse shear are zero at the end $x = 0$. As before, Eq. 2.1d shows that the moment in the y-direction is nonzero:

$$M_y = \nu D\kappa_x \tag{2.19c}$$

Therefore, the tangential stress component is from Eq. 1.49:

$$\sigma_{xx} = -\frac{6p_z}{t^3}zx^2 \tag{2.20a}$$

while the transverse shear stress is computed from Eq. 1.51a:

$$\sigma_{xz} = -\frac{3p_z}{2t}\left[1-\left(\frac{2z}{t}\right)^2\right]x \tag{2.20b}$$

For this case, the normal stress from Eq. 1.54 is not equal to zero:

$$\sigma_{zz} = -\frac{p_z}{4}\left(\frac{2z}{t}\right)\left[\left(\frac{2}{\nu}-1\right)\left(\frac{2z}{t}\right)^2-3\right] \tag{2.20c}$$

The rotation is the integral of Eq. 2.10b, with the constant of integration chosen to give zero rotation at $x = -L$:

$$\beta_x = -\frac{p_z L^3}{6D}\left(1+\frac{x^3}{L^3}\right) \tag{2.21a}$$

so Eq. 1.34a gives the tangential displacement:

$$u = -\frac{p_z L^3}{6D}\left(1+\frac{x^3}{L^3}\right)z \tag{2.21b}$$

Another integration gives the transverse displacement. For the Kirchhoff theory Eq. 1.33 gives:

$$w^0 = \frac{p_z L^4}{8D}\left(\frac{x}{L}+1\right)\left[1+\frac{x}{3L}-\frac{x^2}{3L^2}+\frac{x^3}{3L^3}\right] \tag{2.21c}$$

while the Mindlin-Reissner theory Eq. 1.48 gives an additional displacement contribution, again of the relative size of t^2/L^2:

$$w^0 = \frac{p_z L^4}{8D}\left(\frac{x}{L}+1\right)\left[1+\frac{x}{3L}-\frac{x^2}{3L^2}+\frac{x^3}{3L^3}+\frac{t^2}{L^2}\frac{2\mu}{3(1-\nu)}\left(1-\frac{x}{L}\right)\right] \tag{2.21d}$$

2.3.2 Exact Solution

The body in Fig. 2. 3 with the body force Eq. 2.18a and surface tractions Eq. 2.18b and Eq. 2.18c has the plane strain solution given by the stress function:

$$\varphi = -\frac{p_z}{t^3}\left[z^2-3\left(\frac{t}{2}\right)^2\right]x^2 z \tag{2.22}$$

which yields exactly the stress components Eq. 2.20a-Eq. 2.20c. The averaged tangential displacement $\bar{u}$,defined by Eq. 2.15 yields the result:

$$\bar{u} = -\frac{p_z L^3}{6D}\left[1 + \frac{x^3}{L^3} - \frac{3(1-3\nu)}{10(1-\nu)}\frac{t^2}{L^2}\left(1 + \frac{x}{L}\right)\right]z \tag{2.23a}$$

Thus for this problem, the averaged tangential displacement differs from the plate result Eq. 2.21b by terms that are the size of t^2/L^2. Note that the difference is zero for $\nu = 1/3$, a typical value for many materials. The total displacement in the tangential direction is:

$$u = \bar{u} + \frac{p_z L^3}{6D}\left[\frac{t^2}{L^2}\frac{(2-\nu)x}{4(1-\nu)L}\left\{\frac{3}{5} - \left(\frac{2z}{t}\right)^2\right\}\right]z \tag{2.23b}$$

which has the cubic variation in the z-direction. The averaged displacement in the z-direction $\bar{w}$, defined by Eq. 2.16b yields:

$$\bar{w} = \frac{p_z L^4}{8D}\left(\frac{x}{L} + 1\right)\left(1 + \frac{x}{3L} - \frac{x^2}{3L^2} + \frac{x^3}{3L^3} + \frac{t^2}{L^2}\frac{1}{5(1-\nu)}\left[(5-3\nu)\left(1 - \frac{x}{L}\right) - 2(1-3\nu)\right]\right) \tag{2.23c}$$

Which is exactly the plate solution Eq. 2.21d for $\mu = 6/5$, when $\nu = 1/3$. The exact transverse displacement has additional terms:

$$w = \bar{w} + \frac{p_z L^4}{8D}\left\{\frac{t^2}{L^2}\frac{\nu}{2(1-\nu)}\frac{x^2}{L^2}\left\{\left(\frac{2z}{t}\right)^2 - \frac{1}{5}\right\}\right\}$$

$$+ \frac{p_z L^4}{8D}\left\{\frac{t^4}{16L^4}\left[\frac{1}{3}\left(\frac{2}{\nu} - 1\right)\left\{\left(\frac{2z}{t}\right)^4 - \frac{3}{35}\right\} - 2\left\{\left(\frac{2z}{t}\right)^2 - \frac{1}{5}\right\}\right]\right\} \tag{2.23d}$$

So for this particular transverse load problem, as well as the pure bending and pure shear problems, the difference between the Kirchhoff theory, the Mindlin-Reissner theory and the exact solution is in terms that have the size of t^2/L^2.

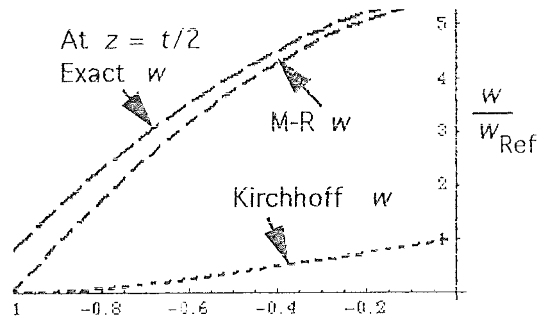

Figure 2. 4 Transverse load, $L/t = 0.5$.

In Fig. 2. 4 is shown a comparison of the Kirchhoff solution Eq. 2.21c, the Mindlin-Reissner solution Eq. 2.21d, and the exact solution Eq. 2.23d at the surface $z = t/2$, for a very short beam, $L/t = 0.5$. Even for this short beam, the Mindlin-Reissner correction gives a reasonable approximation for the normal displacement. In Fig. 2. 5 are the results for $L/t = 2$, for which the Mindlin-Reissner approximation is fairly good. For $L/t = 5$ in Fig. 2. 6, the Kirchhoff solution is not too far in error. For higher values of L/t the three are indistinguishable.

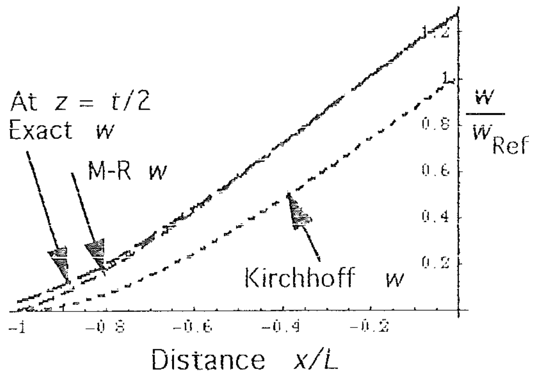

Figure 2. 5 Transverse load, L/t = 2.

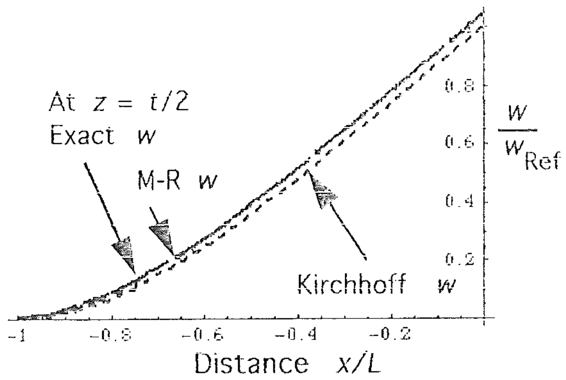

Figure 2. 6 Transverse load, L/t = 5.

Also note that the ratio of maximum shear stress to maximum tangential stress is:

$$\frac{|\sigma_{xz}|_{max}}{|\sigma_{xx}|_{max}} = \frac{t}{2L} \tag{2.24a}$$

and the ratio of maximum transverse stress to maximum tangential stress, for $\nu = 1/3$, is:

$$\frac{|\sigma_{zz}|_{max}}{|\sigma_{xx}|_{max}} = \frac{t^2}{6L^2} \tag{2.25a}$$

These results of relatively small transverse shear and normal stress are consistent with the kinematic approximation of the Kirchhoff theory.

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