# CRITERIO DI HUBER-HENCKY-VON MISES

GRANDEZZA INDICE DEL PERICOLO: MASSIMO LAVORO DEL DEVIATORE DI FORMA

SCOMPOSIZIONE DEL TENSORE DEGLI SFORZI $\underline{\underline{\sigma}}$ (SCRITTO IN TERMINI DI SFORZI PRINCIPALI):

$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} = \underbrace{p \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{PARTE IDROSTATICA}} + \underbrace{\begin{bmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & S_3 \end{bmatrix}}_{\text{PARTE DEVIATORICA}}$$

DOVE $p = \dfrac{\sigma_1 + \sigma_2 + \sigma_3}{3}$ É LO SFORZO MEDIO PRINCIPALE: $p = \dfrac{I_1}{3}$ CON $I_1 =$ INVARIANTE PRIMO DI $\underline{\underline{\sigma}}$

E
$$S_1 = \sigma_1 - p = \frac{2\sigma_1 - \sigma_2 - \sigma_3}{3}$$
$$S_2 = \sigma_2 - p = \frac{2\sigma_2 - \sigma_1 - \sigma_3}{3}$$
$$S_3 = \sigma_3 - p = \frac{2\sigma_3 - \sigma_1 - \sigma_2}{3}$$

RICORDANDO CHE $I_2 = -(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3)$ É L'INVARIANTE SECONDO DI $\underline{\underline{\sigma}}$

ANALOGAMENTE POSSIAMO DEFINIRE GLI INVARIANTI DEL DEVIATORE DEGLI SFORZI:

$$J_1 = S_1 + S_2 + S_3 = \sigma_1 + \sigma_2 + \sigma_3 - 3p = 0$$

$$J_2 = -(S_1S_2 + S_1S_3 + S_2S_3) = -\left[(\sigma_1 - p)(\sigma_2 - p) + (\sigma_1 - p)(\sigma_3 - p) + (\sigma_2 - p)(\sigma_3 - p)\right] =$$

$$= -\left[\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3 - p(\sigma_1 + \sigma_2 + \sigma_1 + \sigma_3 + \sigma_2 + \sigma_3) + 3p^2\right] =$$

$$= -\left[\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3 - 2p(\sigma_1 + \sigma_2 + \sigma_3) + 3p^2\right] =$$

$$= -\left[\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3 - 2p(3p) + 3p^2\right] =$$

$$= -\left[\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3 - 6p^2 + 3p^2\right] =$$

$$= -\left[\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3\right] + 3p^2 = I_2 + \frac{1}{3}I_1^2$$

SCOMPOSIZIONE DEL TENSORE DELLE DEFORMAZIONI $\underline{\underline{\varepsilon}}$ (SCRITTO IN TERMINI DI DILATAZIONI PRINCIPALI):

$$\underline{\underline{\varepsilon}} = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix} = \underbrace{\frac{\varepsilon_V}{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{PARTE VOLUMETRICA}} + \underbrace{\begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}}_{\text{PARTE DEVIATORICA}}$$

DOVE $\varepsilon_V = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$ É LA DEFORMAZIONE DI UN VOLUMETTO CUBICO RITAGLIATO SECONDO LE DIREZIONI PRINCIPALI. $\varepsilon_V = I_1'$, CON $I_1' =$ INVARIANTE PRIMO DI $\underline{\underline{\varepsilon}}$

E $\quad E_1 = \varepsilon_1 - \frac{\varepsilon_v}{3} = \frac{2\varepsilon_1 - (\varepsilon_2 + \varepsilon_3)}{3}$

$$E_2 = \varepsilon_2 - \frac{\varepsilon_v}{3} = \frac{2\varepsilon_2 - (\varepsilon_1 + \varepsilon_3)}{3}$$

$$E_3 = \varepsilon_3 - \frac{\varepsilon_v}{3} = \frac{2\varepsilon_3 - (\varepsilon_1 + \varepsilon_2)}{3}$$

RICORDANDO CHE $I_2' = -(\varepsilon_1\varepsilon_2 + \varepsilon_1\varepsilon_3 + \varepsilon_2\varepsilon_3)$ È L'INVARIANTE SECONDO DI $\underline{\varepsilon}$

ANALOGAMENTE POSSIAMO DEFINIRE GLI INVARIANTI DEL DEVIATORE DELLE DEFORMAZIONI:

$$J_1' = E_1 + E_2 + E_3 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 - 3\frac{\varepsilon_v}{3} = 0$$

$$J_2' = -[E_1E_2 + E_1E_3 + E_2E_3] = -\left[(\varepsilon_1 - \tfrac{\varepsilon_v}{3})(\varepsilon_2 - \tfrac{\varepsilon_v}{3}) + (\varepsilon_1 - \tfrac{\varepsilon_v}{3})(\varepsilon_3 - \tfrac{\varepsilon_v}{3}) + (\varepsilon_2 - \tfrac{\varepsilon_v}{3})(\varepsilon_3 - \tfrac{\varepsilon_v}{3})\right] =$$

$$= -\left[\varepsilon_1\varepsilon_2 + \varepsilon_1\varepsilon_3 + \varepsilon_2\varepsilon_3 - \frac{\varepsilon_v}{3}(2\varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_3) + 3\left(\frac{\varepsilon_v}{3}\right)^2\right] =$$

$$= -\left[\varepsilon_1\varepsilon_2 + \varepsilon_1\varepsilon_3 + \varepsilon_2\varepsilon_3 - \frac{2}{3}\varepsilon_v^2 + \frac{1}{3}\varepsilon_v^2\right] =$$

$$= -[\varepsilon_1\varepsilon_2 + \varepsilon_1\varepsilon_3 + \varepsilon_2\varepsilon_3] + \frac{1}{3}\varepsilon_v^2 = I_2' + \frac{1}{3} I_1'^2$$

PER IL LEGAME ELASTICO:

$$\varepsilon_1 = \frac{1+\nu}{E}\sigma_1 - \frac{\nu}{E}(\sigma_1 + \sigma_2 + \sigma_3)$$

$$\varepsilon_2 = \frac{1+\nu}{E}\sigma_2 - \frac{\nu}{E}(\sigma_1 + \sigma_2 + \sigma_3)$$

$$\varepsilon_3 = \frac{1+\nu}{E}\sigma_3 - \frac{\nu}{E}(\sigma_1 + \sigma_2 + \sigma_3)$$

ALLORA: $\varepsilon_v = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \frac{1+\nu}{E}(\sigma_1 + \sigma_2 + \sigma_3) - \frac{3\nu}{E}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1-2\nu}{E}(\sigma_1 + \sigma_2 + \sigma_3)$

MA $(\sigma_1 + \sigma_2 + \sigma_3) = I_1 = 3p$

QUINDI: $\varepsilon_v = \frac{1-2\nu}{E} 3p$

PERTANTO:

$$E_1 = \varepsilon_1 - \frac{\varepsilon_v}{3} = \frac{1+\nu}{E}\sigma_1 - \frac{\nu}{E}(\sigma_1 + \sigma_2 + \sigma_3) - \frac{1-2\nu}{E}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1+\nu}{E}\sigma_1 - \left[\frac{1-2\nu+\nu}{E}\right](\sigma_1 + \sigma_2 + \sigma_3) =$$

$$= \frac{1+\nu}{E}\sigma_1 - \left[\frac{1-\nu}{E}(\sigma_1 + \sigma_2 + \sigma_3)\right]$$

$$E_2 = \varepsilon_2 - \frac{\varepsilon_v}{3} = \frac{1+\nu}{E}\sigma_2 - \frac{\nu}{E}(\sigma_1 + \sigma_2 + \sigma_3) - \frac{1-2\nu}{E}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1+\nu}{E}\sigma_2 - \left[\frac{1-\nu}{E}(\sigma_1 + \sigma_2 + \sigma_3)\right]$$

$$E_3 = \varepsilon_3 - \frac{\varepsilon_v}{3} = \frac{1+\nu}{E}\sigma_3 - \frac{\nu}{E}(\sigma_1 + \sigma_2 + \sigma_3) - \frac{1-2\nu}{E}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1+\nu}{E}\sigma_3 - \left[\frac{1-\nu}{E}(\sigma_1 + \sigma_2 + \sigma_3)\right]$$

L'ENERGIA TOTALE (LAVORO IMMAGAZZINATO DURANTE LA DEFORMAZIONE) VALE:

$$\mathcal{E}_{TOT} = \frac{1}{2}(\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3) =$$

$$= \frac{1}{2}\left\{ (S_1+p)\left(\varepsilon_1+\frac{\varepsilon_v}{3}\right)+(S_2+p)\left(\varepsilon_2+\frac{\varepsilon_v}{3}\right)+(S_3+p)\left(\varepsilon_3+\frac{\varepsilon_v}{3}\right)\right\} =$$

$$= \frac{1}{2}\left\{ \left(S_1\varepsilon_1+p\frac{\varepsilon_v}{3}+p\varepsilon_1+S_1\frac{\varepsilon_v}{3}\right)+\left(S_2\varepsilon_2+p\frac{\varepsilon_v}{3}+p\varepsilon_2+S_2\frac{\varepsilon_v}{3}\right)+\left(S_3\varepsilon_3+p\frac{\varepsilon_v}{3}+p\varepsilon_3+S_3\frac{\varepsilon_v}{3}\right)\right\} =$$

$$= \frac{1}{2}\left\{S_1\varepsilon_1+S_2\varepsilon_2+S_3\varepsilon_3\right\}+\frac{1}{2}\left\{p\cdot\varepsilon_v\right\}+\frac{1}{2}\left\{p(\varepsilon_1+\varepsilon_2+\varepsilon_3)\right\}+\frac{1}{2}\left\{(S_1+S_2+S_3)\frac{\varepsilon_v}{3}\right\}$$

MA: $(\varepsilon_1+\varepsilon_2+\varepsilon_3) = J_1' = 0$ E $(S_1+S_2+S_3) = J_1 = 0$

$\Rightarrow$ $\frac{1}{2}\left\{S_1\varepsilon_1+S_2\varepsilon_2+S_3\varepsilon_3\right\} = \mathcal{E}_{DEV}$ É L'ENERGIA DI DEFORMAZIONE PER VARIAZIONE DI FORMA

$\Rightarrow$ $\frac{1}{2}\left\{p\cdot\varepsilon_v\right\} = \mathcal{E}_{vol}$ É L'ENERGIA DI DEFORMAZIONE PER VARIAZIONE DI VOLUME

PER IL CRITERIO DI HUBER-HENCKY-VON MISES, CONSIDERIAMO $\mathcal{E}_{DEV}$

$$\mathcal{E}_{DEV} = \frac{1}{2}\left\{S_1\varepsilon_1+S_2\varepsilon_2+S_3\varepsilon_3\right\} = \frac{1}{2}\left\{\left(\varepsilon_1-\frac{\varepsilon_v}{3}\right)(\sigma_1-p)+\left(\varepsilon_2-\frac{\varepsilon_v}{3}\right)(\sigma_2-p)+\left(\varepsilon_3-\frac{\varepsilon_v}{3}\right)(\sigma_3-p)\right\} =$$

$$= \frac{1}{2}\left\{\varepsilon_1\sigma_1+\varepsilon_2\sigma_2+\varepsilon_3\sigma_3-\frac{\varepsilon_v}{3}(\sigma_1+\sigma_2+\sigma_3)-p(\varepsilon_1+\varepsilon_2+\varepsilon_3)+3\frac{\varepsilon_v}{3}p\right\} =$$

$$= \frac{1}{2}\left\{\varepsilon_1\sigma_1+\varepsilon_2\sigma_2+\varepsilon_3\sigma_3-\varepsilon_v\overbrace{\left(\frac{\sigma_1+\sigma_2+\sigma_3}{3}\right)}^{p}-\cancel{p\varepsilon_v}+\cancel{\varepsilon_v p}\right\} =$$

$$\mathcal{E}_{DEV} = \frac{1}{2}\left\{\varepsilon_1\sigma_1+\varepsilon_2\sigma_2+\varepsilon_3\sigma_3-\varepsilon_v p\right\}$$

TENENDO CONTO DEL LEGAME:

$$\mathcal{E}_{DEV} = \frac{1}{2}\left\{\frac{1+\nu}{E}\sigma_1^2-\frac{\nu}{E}\sigma_1(\sigma_1+\sigma_2+\sigma_3)+\frac{1+\nu}{E}\sigma_2^2-\frac{\nu}{E}\sigma_2(\sigma_1+\sigma_2+\sigma_3)+\frac{1+\nu}{E}\sigma_3^2-\frac{\nu}{E}\sigma_3(\sigma_1+\sigma_2+\sigma_3)\right.$$

$$\left.-\frac{1-2\nu}{E}3p^2\right\} =$$

N.B. $p = \frac{\sigma_1+\sigma_2+\sigma_3}{3}$

$$= \frac{1}{2}\left\{\frac{1+\nu}{E}(\sigma_1^2+\sigma_2^2+\sigma_3^2)-\frac{\nu}{E}(\sigma_1+\sigma_2+\sigma_3)^2-\frac{1-2\nu}{E}\frac{(\sigma_1+\sigma_2+\sigma_3)^2}{3}\right\} =$$

$$= \frac{1}{2}\left\{\frac{1+\nu}{E}(\sigma_1^2+\sigma_2^2+\sigma_3^2)-\frac{3\nu}{E}\frac{(\sigma_1+\sigma_2+\sigma_3)^2}{3}-\frac{1-2\nu}{E}\frac{(\sigma_1+\sigma_2+\sigma_3)^2}{3}\right\} =$$

$$= \frac{1}{2}\left\{\frac{1+\nu}{E}(\sigma_1^2+\sigma_2^2+\sigma_3^2)-\frac{(\sigma_1+\sigma_2+\sigma_3)^2}{3}\left[\frac{3\nu}{E}+\frac{1-2\nu}{E}\right]\right\} =$$

$$= \frac{1}{2}\left\{\frac{1+\nu}{E}(\sigma_1^2+\sigma_2^2+\sigma_3^2)-\frac{(\sigma_1+\sigma_2+\sigma_3)^2}{3}\left[\frac{1+\nu}{E}\right]\right\} =$$

RICORDANDO CHE $I_1 = (\sigma_1 + \sigma_2 + \sigma_3)$ E $I_2 = -(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3)$ ALLORA:

$$\varepsilon_{DEV} = \frac{1}{2}\frac{1+\nu}{E}\left[(\sigma_1^2+\sigma_2^2+\sigma_3^2) + \frac{(\sigma_1+\sigma_2+\sigma_3)^2}{3}\right] =$$

$$= \frac{1}{2}\frac{1+\nu}{E}\left[I_1^2 + 2I_2 - \frac{1}{3}I_1^2\right] = \frac{1}{2}\frac{1+\nu}{E}\left[\frac{2}{3}I_1^2 + 2I_2\right] = \frac{1}{2}\frac{2(1+\nu)}{E}\left[\frac{1}{3}I_1^2 + I_2\right] =$$

$$= \frac{1}{2}\frac{1}{G}\left[I_2 + \frac{1}{3}I_1^2\right] = \frac{1}{2}\frac{1}{G}J_2$$

$$\varepsilon_{DEV} = \frac{1}{2G}J_2 = \frac{1}{2G}\left\{-\sigma_1\sigma_2 - \sigma_1\sigma_3 - \sigma_2\sigma_3 + \frac{1}{3}(\sigma_1+\sigma_2+\sigma_3)^2\right\} =$$

$$= \frac{1}{2G}\left\{-(\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_2\sigma_3) + \frac{1}{3}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2+2\sigma_1\sigma_2+2\sigma_1\sigma_3+2\sigma_2\sigma_3\right)\right\} =$$

$$= \frac{1}{2G}\left\{-\frac{3}{3}(\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_2\sigma_3) + \frac{1}{3}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2+2\sigma_1\sigma_2+2\sigma_2\sigma_3+2\sigma_2\sigma_3\right)\right\} =$$

$$= \frac{1}{2G}\left\{\frac{1}{3}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2-\sigma_1\sigma_2-\sigma_1\sigma_3-\sigma_2\sigma_3\right)\right\} = \frac{1}{6G}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2-\sigma_1\sigma_2-\sigma_1\sigma_3-\sigma_2\sigma_3\right)$$

IN CONDIZIONE DI SFORZO MONOASSIALE: $\sigma_1 = K$, $\sigma_2 = 0$, $\sigma_3 = 0$

$$\varepsilon_{DEV} = \frac{1}{6G}K^2 = \overline{\varepsilon}_{DEV}$$

PER LA VERIFICA DEVE ESSERE $\varepsilon_{DEV} \leq \overline{\varepsilon}_{DEV} \Rightarrow \frac{1}{6G}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2-\sigma_1\sigma_2-\sigma_1\sigma_3-\sigma_2\sigma_3\right) \leq \frac{1}{6G}K^2$

E QUINDI: $\sqrt{\sigma_1^2+\sigma_2^2+\sigma_3^2-\sigma_1\sigma_2-\sigma_1\sigma_3-\sigma_2\sigma_3} \leq K$