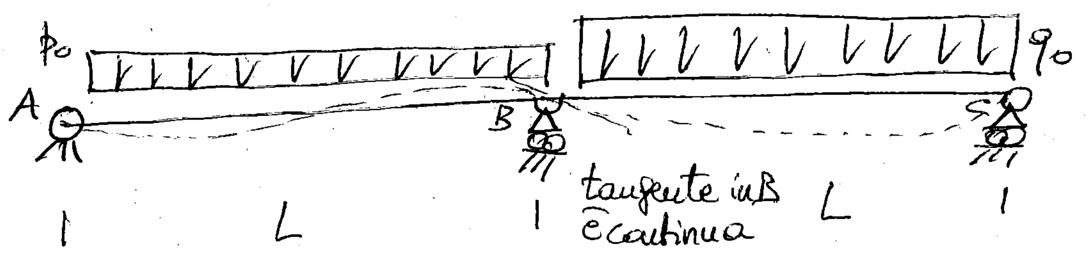
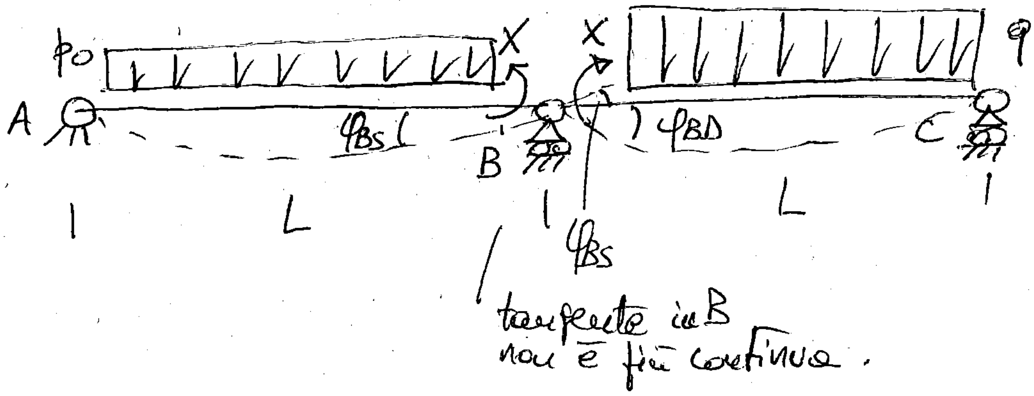
Principio dei lavori virtuali per travi iperstatiche

Applicazione al caso di travi continue.

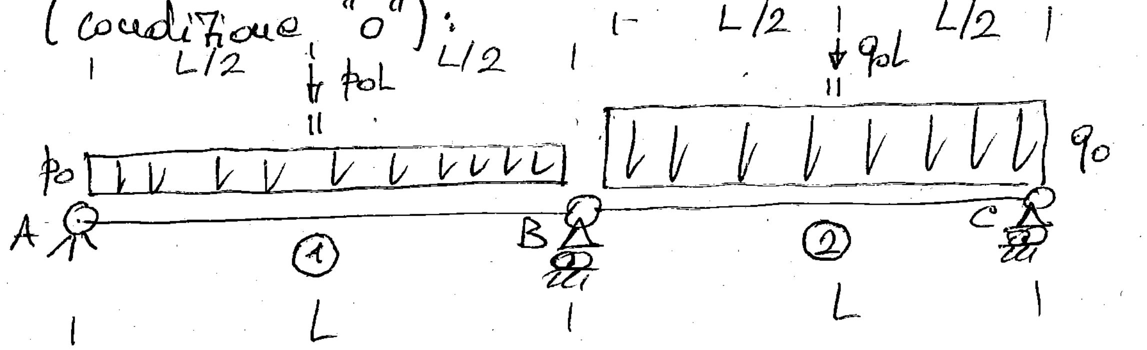


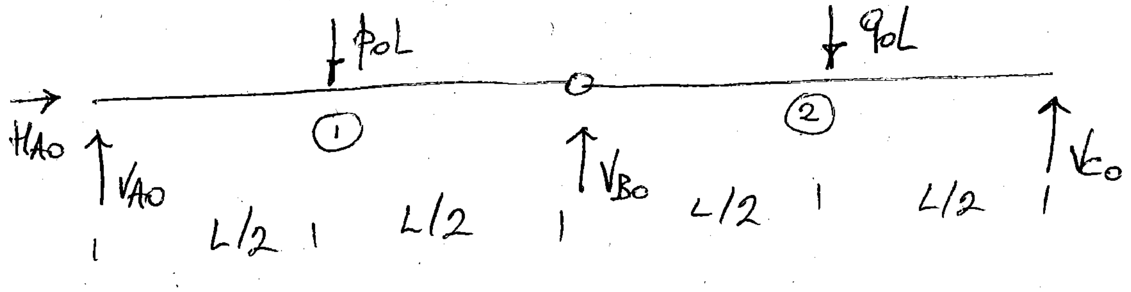
Struttura iperstatica: $\Delta \varphi_B = 0$



Struttura resa isostatica: $\Delta\varphi_B = \varphi_{Bs} + \varphi_{BD} \neq 0$

A) Struttura resa isostatica soggetta a sole azioni esterne (condizione "0"):





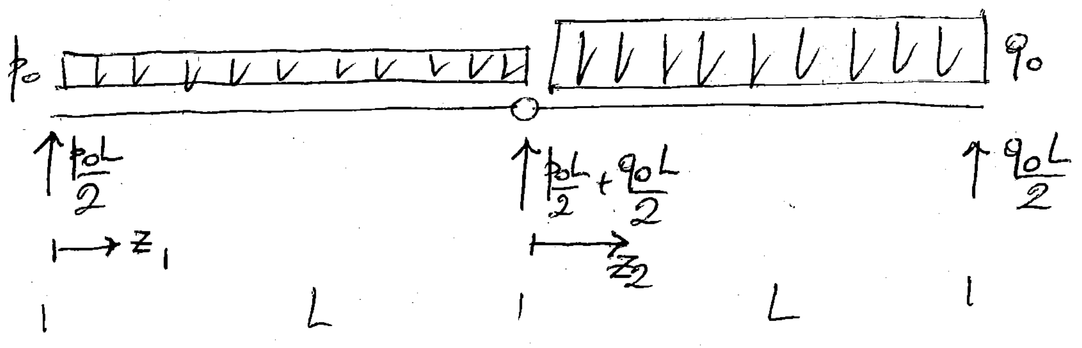
$\rightarrow R_x = 0 \qquad H_{A0} = 0$

$\uparrow R_y = 0 \qquad V_{A0} + V_{B0} + V_{C0} - p_0 L - q_0 L = 0$ } eq. cardinali

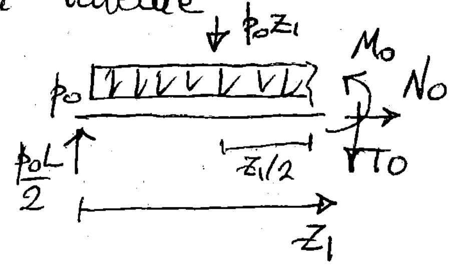
$\circlearrowleft M_{z(A)} = 0 \qquad -\frac{p_0 L^2}{2} + V_{B0} L - \frac{3}{2} q_0 L^2 + V_{C0} \cdot 2L = 0$

$\circlearrowleft M^{(2)}_{z(B)} = 0 \qquad -\frac{q_0 L^2}{2} + V_{C0} L = 0$ eq. ausiliaria

Segue: $H_{A0} = 0$; $V_{C0} = \frac{q_0 L}{2}$; $V_{B0} = \frac{p_0 L}{2} + \frac{q_0 L}{2}$; $V_{A0} = \frac{p_0 L}{2}$



Azioni interne



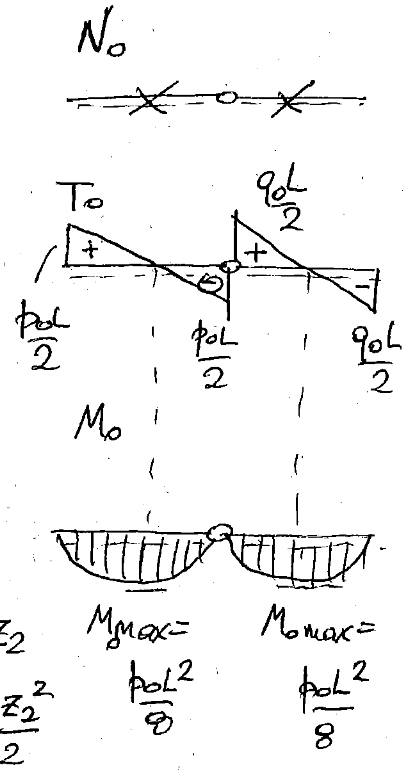
A → B

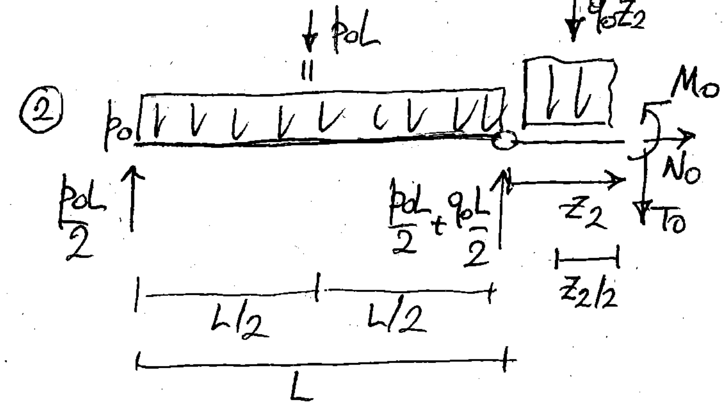
$0 \le z_1 < L$

$N_0(z_1)=0$

$T_0(z_1)=\frac{p_0L}{2}-p_0z_1$

$M_0(z_1)=\frac{p_0L}{2}z_1-\frac{p_0z_1^2}{2}$





B → C

$0 < z_2 \le L$

$N_0(z_2)=0$

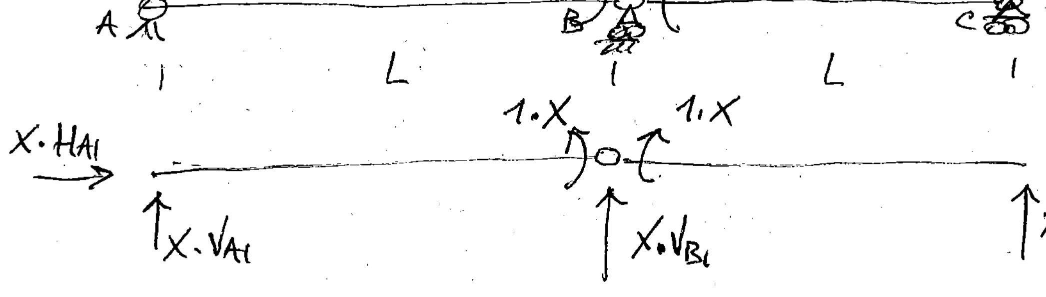
$T_0(z_2)=\frac{q_0L}{2}-q_0z_2$

$M_0(z_2)=\frac{q_0Lz_2}{2}-q_0\frac{z_2^2}{2}$

infatti ↺ $M_{(z_2)}=0 \quad -\frac{p_0L}{2}(L+z_2)+p_0L\left(\frac{L}{2}+z_2\right)-\left(\frac{p_0L}{2}+\frac{q_0L}{2}\right)z_2+q_0\frac{z_2^2}{2}+M_0(z_2)=0$

$M_0(z_2)=\cancel{\frac{p_0L^2}{2}}+\cancel{\frac{p_0L}{2}z_2}-\cancel{\frac{p_0L^2}{2}}-\cancel{p_0Lz_2}+\cancel{\frac{p_0L}{2}z_2}+\frac{q_0L}{2}z_2-q_0\frac{z_2^2}{2}=\frac{q_0L}{2}z_2-q_0\frac{z_2^2}{2}$

B) Struttura resa isostatica soggetta alla sola azione della iperstatica $X=1\cdot X$ (condizione "1"$\cdot X$)



$\rightarrow R_x=0 \quad XH_{A1}=0$

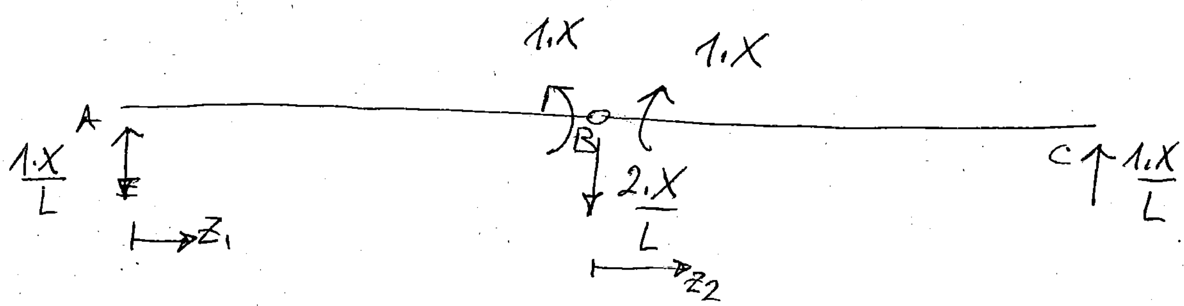
$\uparrow R_y=0 \quad XV_{A1}+XV_{B1}+XV_{C1}=0$

↺ $M_{z(A)}=0 \quad XV_{B1}\cdot L+X\cdot V_{C1}\cdot 2L=0$

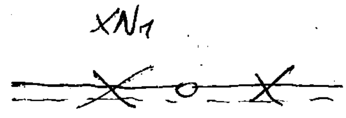
} eq. cardinali

5. $M_{z(B)}^{(2)} = 0 \qquad -1\cdot X + X\cdot V_{c_1} L = 0$ eq. ausiliaria

ne segue: $XH_{A1} = 0$; $XV_{C1} = \frac{1\cdot X}{L}$; $XV_{B1} = -2\frac{1\cdot X}{L}$; $XV_{A1} = \frac{1\cdot X}{L}$



Azioni interne:



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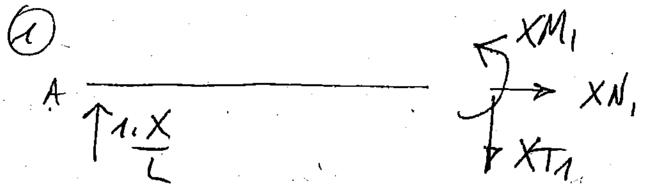


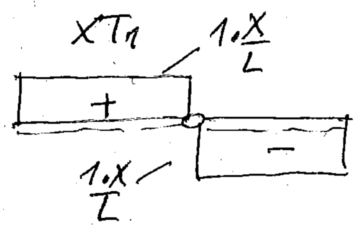
$A \rightarrow B$

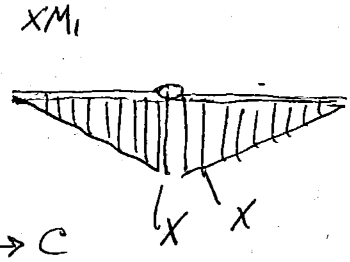
$0 \leq z_1 < L$

$XN_1(z_1) = 0$

$XT_1(z_1) = \frac{1\cdot X}{L}$

$XM_1(z_1) = \frac{1\cdot X}{L} z_1$





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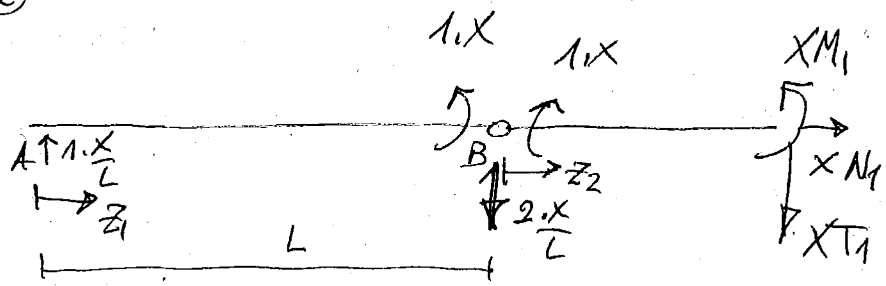


$B \rightarrow C$

$0 \leq z_2 \leq L$

$XN_1(z_1) = 0$

$XT_1(z_1) = -\frac{1\cdot X}{L}$

$XM_1(z_1) = X\left(1 - \frac{z_2}{L}\right)$

infatti $\quad \Sigma M_{(z_2)} = 0 \quad -\frac{1\cdot X}{L}(L + z_2) + 1\cdot X - 1\cdot X + \frac{2\cdot X}{L} z_2 + XM_1(z_2) = 0$

$$XM_1(z_2) = \frac{1\cdot X}{L} L + \frac{1\cdot X}{L} z_2 - \cancel{1\cdot X} + \cancel{1\cdot X} - \frac{2X}{L} z_2 = 1\cdot X - \frac{1\cdot X}{L} z_2$$

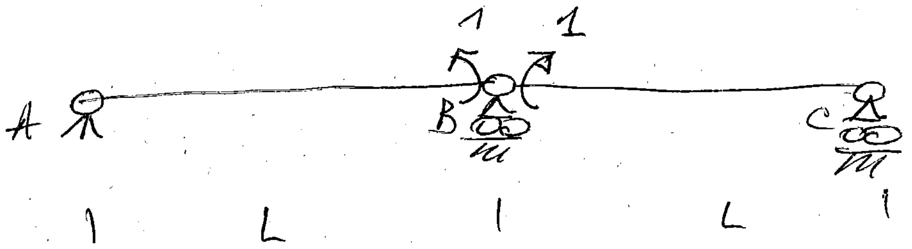
c) Sistema reale: spostamenti e deformazioni congruenti

$$\varepsilon = \frac{N_0 + XN_1}{EA} = 0 \qquad \forall z_1, \forall z_2$$

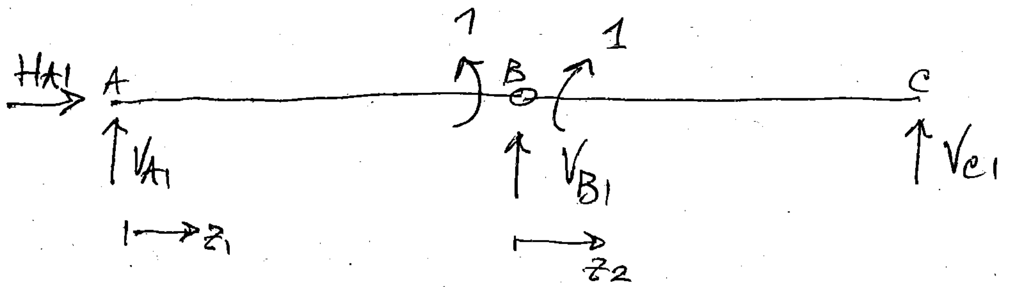
$$\gamma = \frac{T_0 + XT_1}{GA_T} = \begin{cases} \dfrac{\frac{p_0 L}{2} - p_0 z_1 + X\cdot\frac{1}{L}}{GA_T} & A \rightarrow B \quad 0 \leq z_1 < L \\[2ex] \dfrac{\frac{q_0 L}{2} - q_0 z_2 - X\frac{1}{L}}{GA_T} & B \rightarrow C \quad 0 \leq z_2 < L \end{cases}$$

$$\chi = \frac{M_0 + X M_1}{EJ} = \begin{cases} \dfrac{\frac{p_0 L}{2} z_1 - p_0 \frac{z_1^2}{2} + X \frac{z}{L}}{EJ} & A \to B \quad 0 \leq z_1 < L \\[2ex] \dfrac{\frac{q_0 L}{2} z_2 - q_0 \frac{z_2^2}{2} + X\left(1 - \frac{z_2}{L}\right)}{EJ} & B \to C \quad 0 < z_2 \leq L \end{cases}$$

D) Sistema ausiliario: forze e sforzi in equilibrio



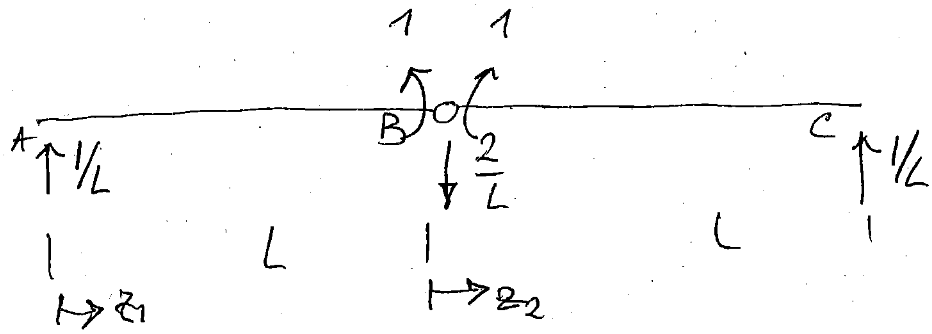
NB: Le 2 coppie applicate compiono lavoro rispettivamente per φ_{BS} e φ_{BD}



NOTA: È EQUIVALENTE alla struttura resa isostatica soggetta alla sola azione dell'iperstatica quando si sceglie $X=1$.

Pertanto:

$H_{A1} = 0 \;;\; V_{A1} = \frac{1}{L} \;;\; V_{B1} = -\frac{2}{L} \;;\; V_{C1} = \frac{1}{L}$



e analogamente per le azioni interne

$$\left.\begin{aligned} N_1(z_1) &= 0 \\ T_1(z_1) &= \frac{1}{L} \\ M_1(z_1) &= \frac{1}{L} z_1 \end{aligned}\right\} \quad \begin{aligned} &A \to B \\ &0 \le z_1 < L \end{aligned}$$

$$\left.\begin{aligned} N_1(z_2) &= 0 \\ T_1(z_2) &= -\frac{1}{L} \\ M_1(z_2) &= 1\left(1-\frac{z_2}{L}\right) \end{aligned}\right\} \quad \begin{aligned} &B \to C \\ &0 \le z_2 < L \end{aligned}$$

E) APPLICAZIONE DEL P.L.V. (trascurando deformazione a taglio)

$$L_e = 1 \cdot \varphi_{BS} + 1 \cdot \varphi_{BD} = 1 \cdot \Delta\varphi_B = 0$$

$\uparrow$ $\Delta\varphi_B = 0$ per congruenza.

$$L_i = \cancel{\int_0^L N_1 \cdot \varepsilon \, dz_1} + \cancel{\int_0^L N_1 \cdot \varepsilon \, dz_2} + \int_0^L M_1 \chi \, dz_1 + \int_0^L M_2 \chi \, dz_2 .$$

$$L_i = \int_0^L 1 \cdot \frac{z_1}{L} \cdot \left(\frac{p_0 \frac{L}{2} z_1 - p_0 \frac{z_1^2}{2} + X \frac{z_1}{L}}{EJ} \right) dz_1 + \int_0^L 1\left(1-\frac{z_2}{L}\right) \left[\frac{q_0 \frac{L}{2} z_2 - q_0 \frac{z_2^2}{2} + X\left(1-\frac{z_2}{L}\right)}{EJ} \right] dz_2$$

$$L_e = L_i \quad \Rightarrow \quad L_i = 0 \qquad \text{ovvero} \quad C_{10} + X C_{11} = 0$$

$$L_i = \int_0^L 1 \cdot \frac{z_1}{L} \, \frac{p_0 \frac{L}{2} z_1 - p_0 \frac{z_1^2}{2}}{EJ} dz_1 + X \int_0^L 1 \cdot \frac{z_1}{L} \, \frac{\frac{z_1}{L}}{EJ} dz_1 +$$

$$+ \int_0^L 1\left(1-\frac{z_2}{L}\right) \frac{q_0 \frac{L}{2} z_2 - q_0 \frac{z_2^2}{2}}{EJ} dz_2 + X \int_0^L 1 \cdot \left(1-\frac{z_2}{L}\right) \frac{\left(1-\frac{z_2}{L}\right)}{EJ} dz_2$$

$$L_i = \underbrace{\left\{ \int_0^L \frac{p_0 \frac{L}{2} \frac{z_1^2}{L} - p_0 \frac{z_1^3}{2L}}{EJ} dz_1 + \int_0^L \frac{q_0 \frac{L}{2} z_2 - \frac{q_0 z_2^2}{2} - \frac{q_0 z_2^2}{2} + \frac{q_0 z_2^3}{2L}}{EJ} dz_2 \right\}}_{C_{10}} +$$

$$+ X \underbrace{\left\{ \int_0^L \frac{\frac{z_1^2}{L^2}}{EJ} dz_1 + \int_0^L \frac{\left(1-\frac{z_2}{L}\right)^2}{EJ} dz_2 \right\}}_{C_{11}}$$

$$C_{10} = \frac{1}{EJ}\left\{\int_0^L \left(\frac{p_0}{2} z_1^2 - \frac{p_0 z_1^3}{2L}\right) dz_1 + \int_0^L \left(\frac{q_0 L}{2} z_2 - q_0 z_2^2 + \frac{q_0 z_2^3}{2L}\right) dz_2\right\}$$

$$C_{10} = \frac{1}{EJ}\left[\frac{p_0 z_1^3}{6} - \frac{p_0 z_1^4}{8L}\right]_0^L + \frac{1}{EJ}\left[\frac{q_0 L z_2^2}{4} - \frac{q_0 z_2^3}{3} + \frac{q_0 z_2^4}{8L}\right]_0^L$$

$$C_{10} = \frac{p_0 L^3}{6EJ} - \frac{p_0 L^3}{8EJ} + \frac{q_0 L^3}{4EJ} - \frac{q_0 L^3}{3EJ} + \frac{q_0 L^3}{8EJ} = \frac{(4-3)p_0 L^3}{24EJ} + \frac{(6-8+3)q_0 L^3}{24EJ}$$

$$C_{10} = \frac{p_0 L^3}{24EJ} + \frac{q_0 L^3}{24EJ}$$

$$C_{11} = \frac{1}{EJ}\left\{\int_0^L \frac{z_1^2}{L^2} dz_1 + \int_0^L \left(\frac{L-z_2}{L}\right)^2 dz_2\right\}$$

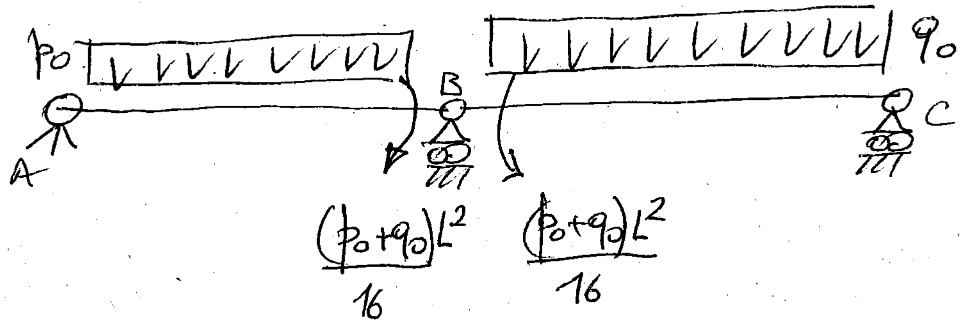
$$C_{11} = \frac{1}{EJ}\left[\frac{z_1^3}{3L^2}\right]_0^L + \frac{1}{EJ}\left[\left(1-\frac{z_2}{L}\right)^3 \cdot \left(-\frac{L}{3}\right)\right]_0^L$$

$$C_{11} = \frac{L}{3EJ} + \frac{L}{3EJ} = \frac{2L}{3EJ}$$

$$C_{10} + X C_{11} = 0 \Rightarrow X = -\frac{C_{10}}{C_{11}} = -\frac{\frac{p_0 L^3}{24EJ} + \frac{q_0 L^3}{24EJ}}{\frac{2}{3}\frac{L}{EJ}}$$

ovvero $\boxed{X = -\frac{p_0 L^2}{16} - \frac{q_0 L^2}{16}}$

F) SOSTITUZIONE DEL VALORE DELL'IPERSTATICA



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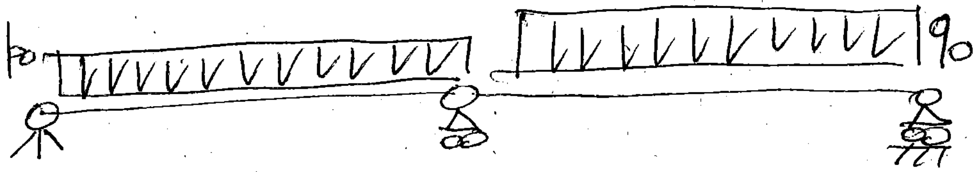

(STATO "0")

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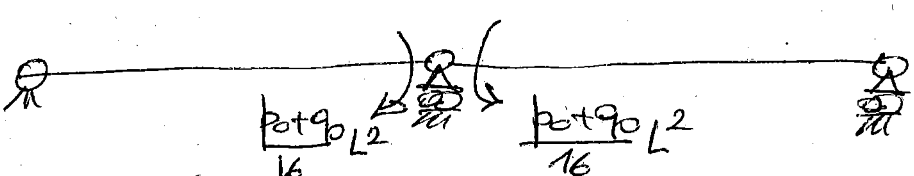
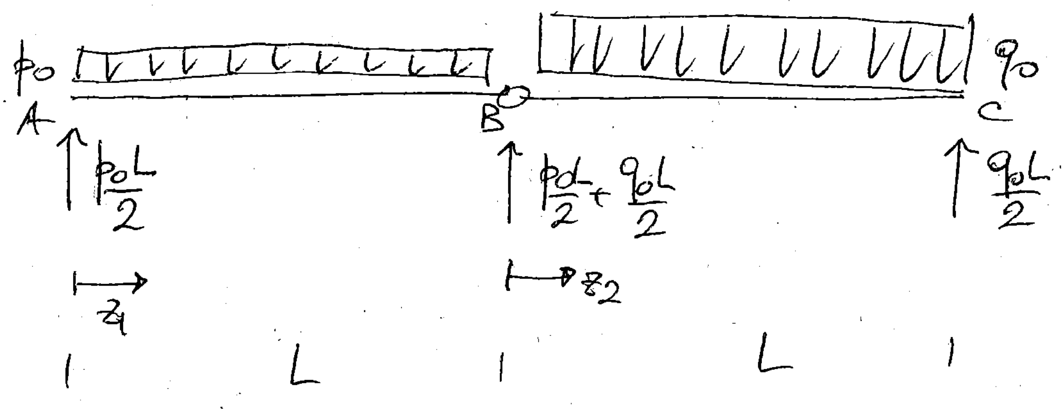


(STATO "1·X")



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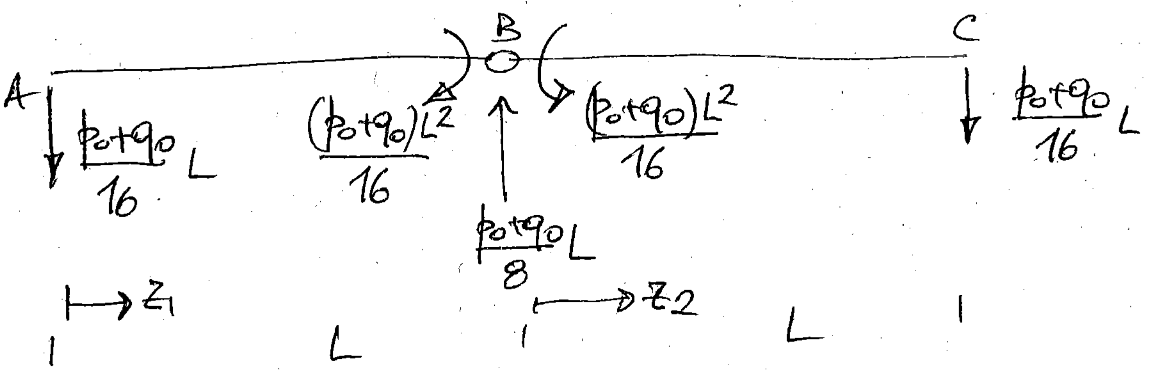

Pertanto

$$N = N_0 - \frac{(p_0+q_0)L^2}{16} N_1 = \begin{cases} 0 & A \to B \quad 0 \le z_1 < L \\ 0 & B \to C \quad 0 < z_2 \le L \end{cases}$$

$$T = T_0 - \left(\frac{p_0+q_0}{16}\right) L^2 T_1 = \begin{cases} \frac{7}{16} p_0 L - p_0 z_1 - \frac{1}{16} q_0 L & A \to B \quad 0 \le z_1 < L \\ \frac{9}{16} q_0 L - q_0 z_2 + \frac{1}{16} p_0 L & B \to C \quad 0 < z_2 \le L \end{cases}$$

$$M = M_0 - \left(\frac{p_0+q_0}{16}\right) L^2 M_1 = \begin{cases} \frac{7}{16} p_0 L z_1 - \frac{p_0 z_1^2}{2} - \frac{1}{16} q_0 L z_1 & A \to B \quad 0 \le z_1 < L \\ -\frac{q_0 L^2}{16} + \frac{9}{16} q_0 L z_2 - \frac{q_0 z_2^2}{2} - \frac{p_0 L^2}{16} + \frac{p_0 L z_2}{16} & B \to C \quad 0 < z_2 \le L \end{cases}$$