Institut für Regelungs- und Automatisierungstechnik
Graz University of Technology

Sliding Mode Control: Basic Theory, Advances and Applications

Elio USAI

eusai@diee.unica.it

Dept of Electrical and Electronic Eng.
University of Cagliari

Lecture 4

State Estimation and Fault Detection via Sliding Modes

- Real-time differentiation via higher-order sliding modes
- State estimation in dynamical systems
- HOSM Algebraic Observers
- HOSM Dynamic Observers
- Model-Based FDI via VSS

L4 – Real-time differentiation

- Digital real-time differentiators can avoid the use of additional measurement devices.

- Differentiators are widely used in output feedback control of non linear systems with uncertainties.

- Differentiation of a smooth signal z is a well-known ill-posed problem whose solution has been attained by regularisation via integration.

- Robustness with respect measurement noise and digital discrete-time implementation is required.

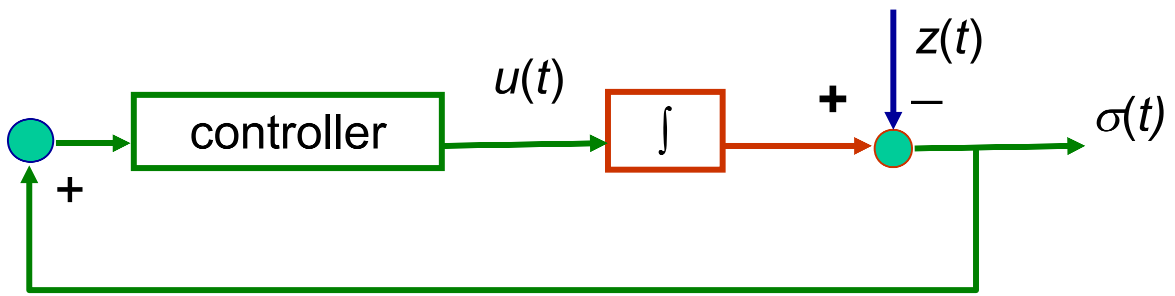
- Bandwidth and robustness are conflicting requirements.

L4 – Real-time differentiation

Most effective and used existing algorithms are based on:

- Linear filters approximating the ideal differentiator – High Gain observers with completely uncertain system model

- Non linear filters based on Higher-Order sliding modes –
  Super-Twisting differentiator
  Discontinuous High Gain differentiator



Usually it is required that the signal *z* is "sufficiently smooth"
for 2-SMD it means that the derivative of *z* is Lipshtz

L4 – Real-time differentiation

High-Gain Differentiator

$$\begin{cases} \dot{x}_1 = x_2 - \gamma_0 k (x_1 - z) \\ \dot{x}_2 = -\gamma_1 k^2 (x_1 - z) \end{cases} \qquad (\gamma_0, \gamma_1, k) > 0 \qquad \hat{\dot{z}} = x_2$$

- Cut-off characteristic depends on parameters γ_0 and γ_1
- Filter bandwidth depends on parameter *k*
- Differentiation accuracy is almost constant within the filter bandwidth

- Phase distortion mainly depends on parameters γ_0 and γ_1
- Accuracy depends on parameter *k*
- Estimation error never vanishes

L4 – Real-time differentiation

Super-Twisting Differentiator

$$\begin{cases} \dot{x}_1 = x_2 - \kappa_0 \sqrt{|x_1 - z|}\,\mathrm{sgn}(x_1 - z) \\ \dot{x}_2 = -\kappa_1\,\mathrm{sgn}(x_1 - z) \end{cases} \qquad \kappa_0 = 1.5\sqrt{|\ddot{z}|_\infty};\ \ \kappa_1 = 1.1|\ddot{z}|_\infty \qquad \hat{\dot{z}} = \dot{x}_1$$

- Estimation error vanishes in finite time
- Convergence time depends on κ_0 and κ_1
- Noise propagation is limited

- Convergence cannot be assured if the spectrum of *z* is enriched, even at low frequency
- Accuracy in discrete-time implementation depends on the product among the parameters κ_0 and κ_1 and the sampling time T_s

L4 – Real-time differentiation

Discontinuous High-Gain Differentiator

$$\begin{cases} \dot{x}_1 = x_2 - \gamma_0 k (x_1 - z) \\ \dot{x}_2 = -\gamma_1 k^2 (x_1 - z) - W \operatorname{sgn}(x_1 - z) \end{cases}$$

$$(\gamma_0, \gamma_1, k, W) > 0 \qquad \hat{\dot{z}} = x_2$$

$$W > |\ddot{z}|_\infty + 2\frac{|\dddot{z}|_\infty}{\gamma_0 k}; \qquad 4\gamma_1 > \gamma_0$$

- Asymptotic theoretic exactness of the differentiator is assured.
- The γ_0, γ_1, and k parameters can be tuned to have the best shape and bandwidth, since the discontinuous term compensate for the phase distortion.
- Limiting k, and W, implies that the differentiator bandwidth is not much larger than needed, therefore high frequency noise propagation is limited.
- Simulations showed that enriching the signal spectrum at low frequency can cause the differentiator not to converge, but its sensitivity with respect to the signal spectrum richness is less than that of the Super-Twisting differentiator.
- z must have limited second and third derivative.

L4 – Real-time differentiation

$z = \sin(t)$

$\gamma_1 = \gamma_0 = 1$

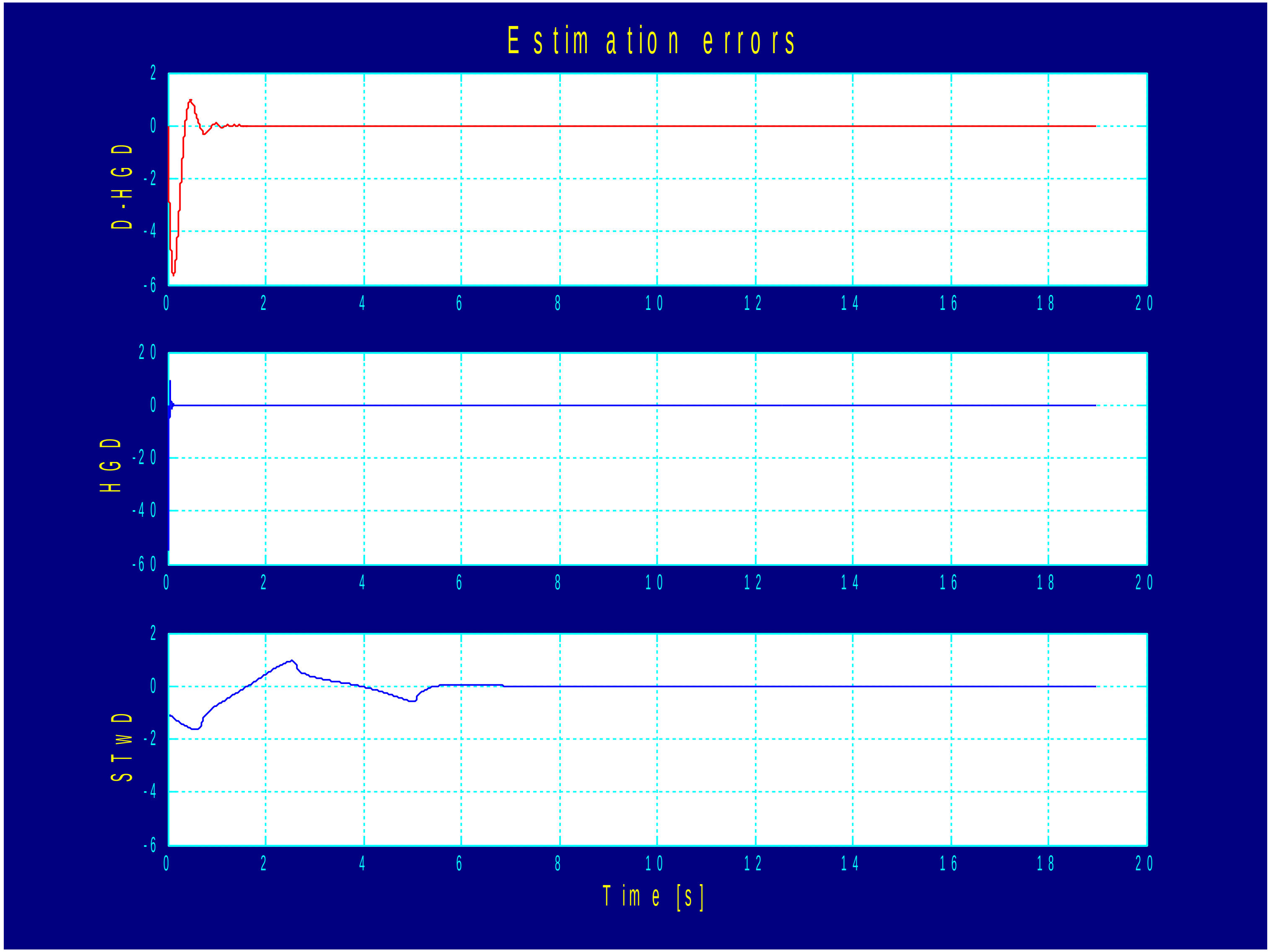
$k_{HGD} = 100$

$k_{DHGD} = 10$

$W = 1.2$

$\kappa_0 = 1$

$\kappa_1 = 1.1$



L4 – Real-time differentiation

$z = \sin(t)$

$\gamma_1 = \gamma_0 = 1$

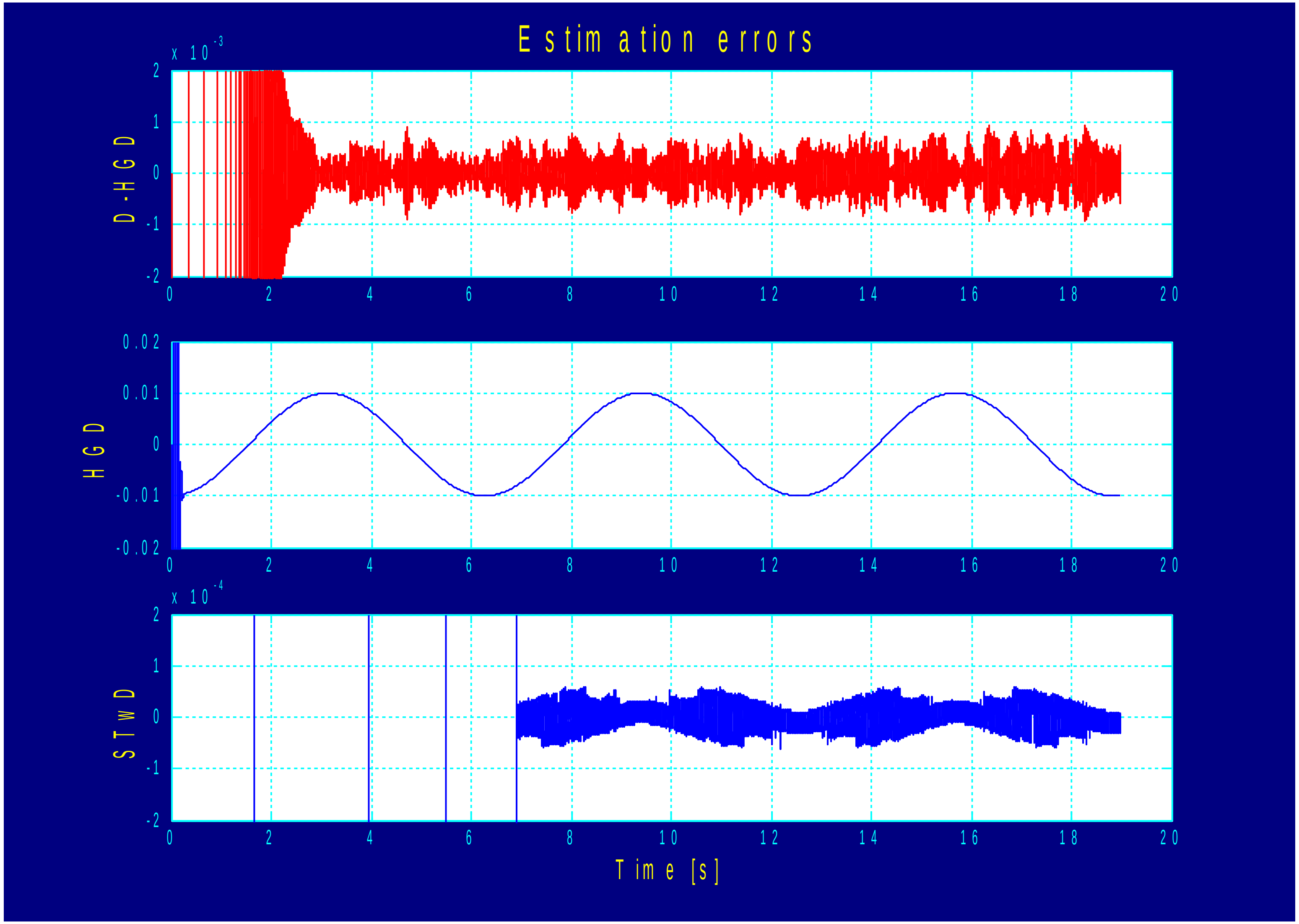
$k_{HGD} = 100$

$k_{DHGD} = 10$

$W = 1.2$

$\kappa_0 = 1$

$\kappa_1 = 1.1$



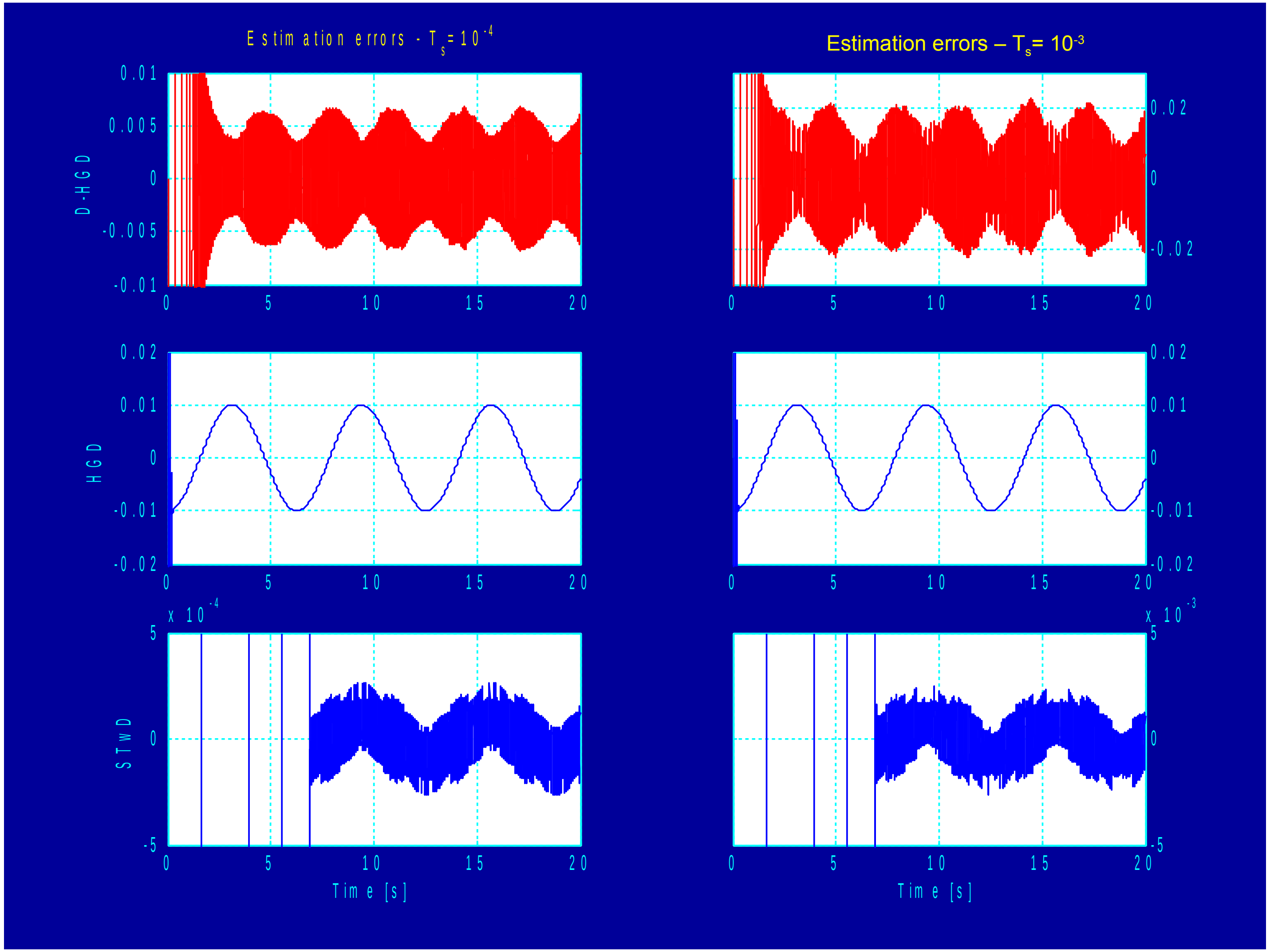
L4 – Real-time differentiation

Discrete-time implementation

$O\ (T_s^{1/2})$

$O\ (1)$

$O\ (T_s)$



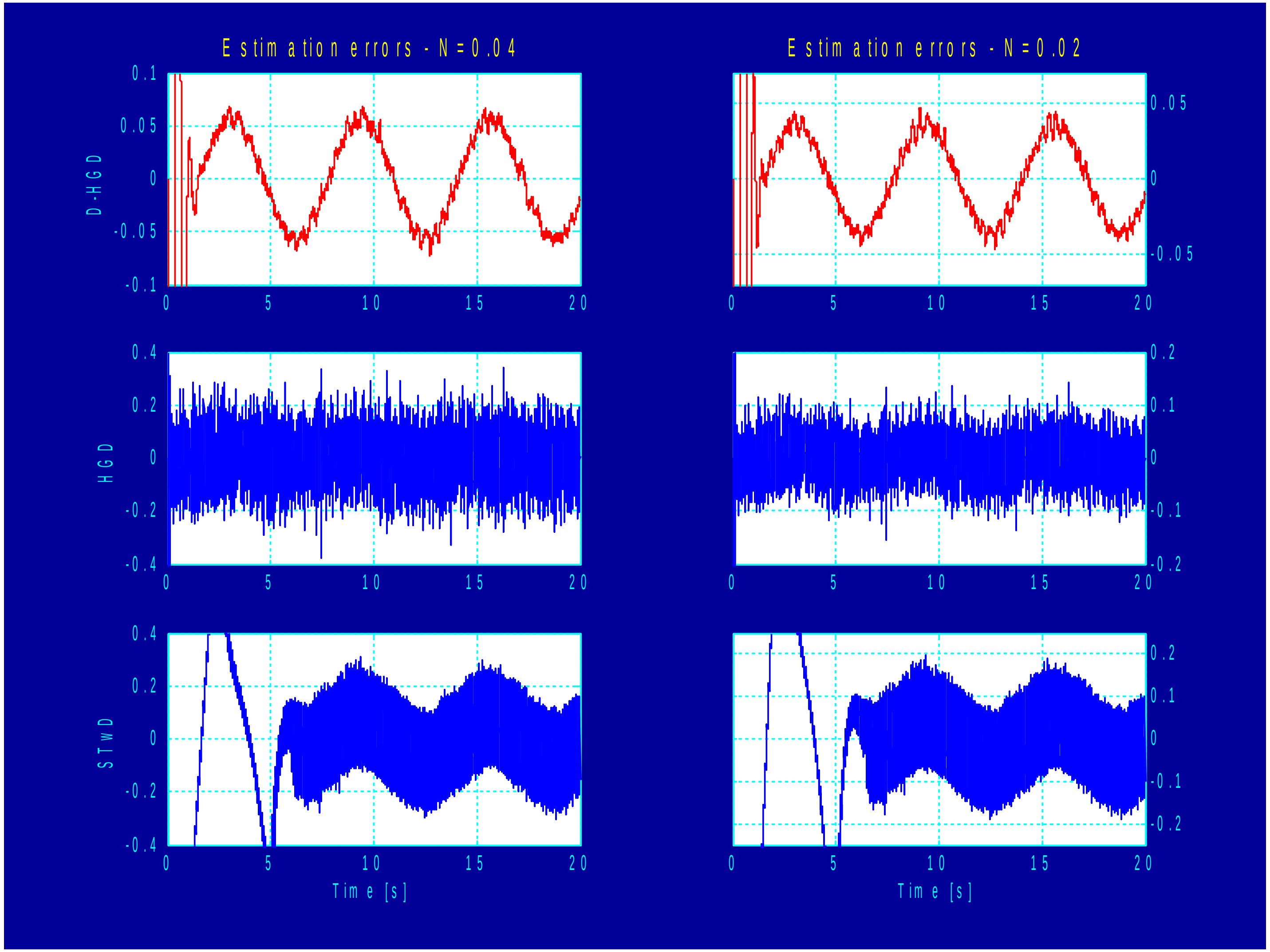
L4 – Real-time differentiation

Noise sensitivity

O (N½)

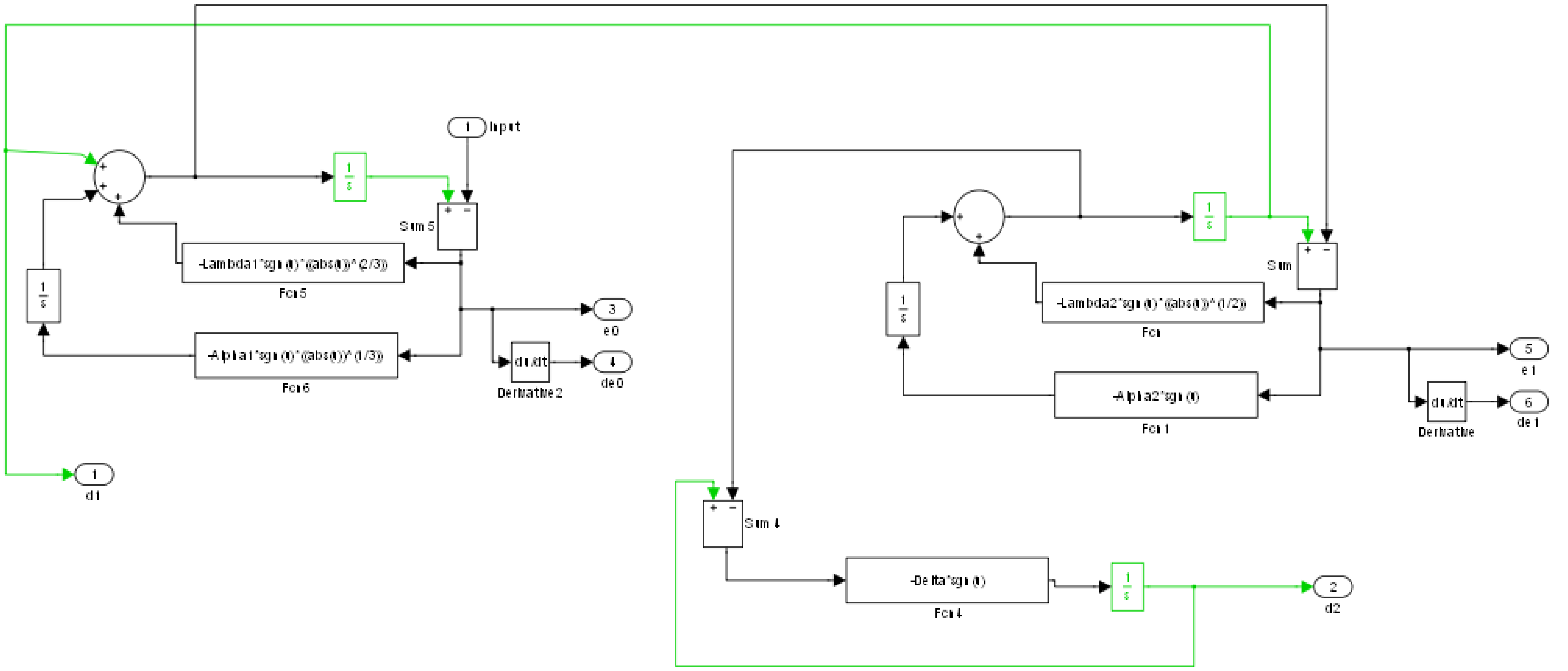
O (N)

O (N½)



L4 – Real-time differentiation

Multiple differentiation is based on nested implementation of a simple Super-Twisting robust differentiator



L4 – Real-time differentiation

More effective and precise development led to the arbitrary order sliding differentiator

$$\dot{z}_0 = v_0, \quad \dot{v}_0 = -\lambda_0 \left| z_0 - y \right|^{\frac{r}{r+1}} \operatorname{sgn}\left(z_0 - y\right) + z_1$$
$$\vdots$$
$$\dot{z}_i = v_i, \quad \dot{v}_i = -\lambda_i \left| z_i - v_{i-1} \right|^{\frac{r-i}{r-i+1}} \operatorname{sgn}\left(z_i - v_{i-1}\right) + z_{i+1}$$
$$\vdots$$
$$\dot{z}_r = -\lambda_r \operatorname{sgn}\left(z_i - v_{r-1}\right),$$

Finite time convergence
Reduced noise propagation

$$\left|\tilde{y}^{(i)}\right| \approx O\left(\Delta^{r+1-i/r+1}\right) + O\left(T_c^{\,r-i+1}\right), \qquad i = 0,1,2,\ldots$$

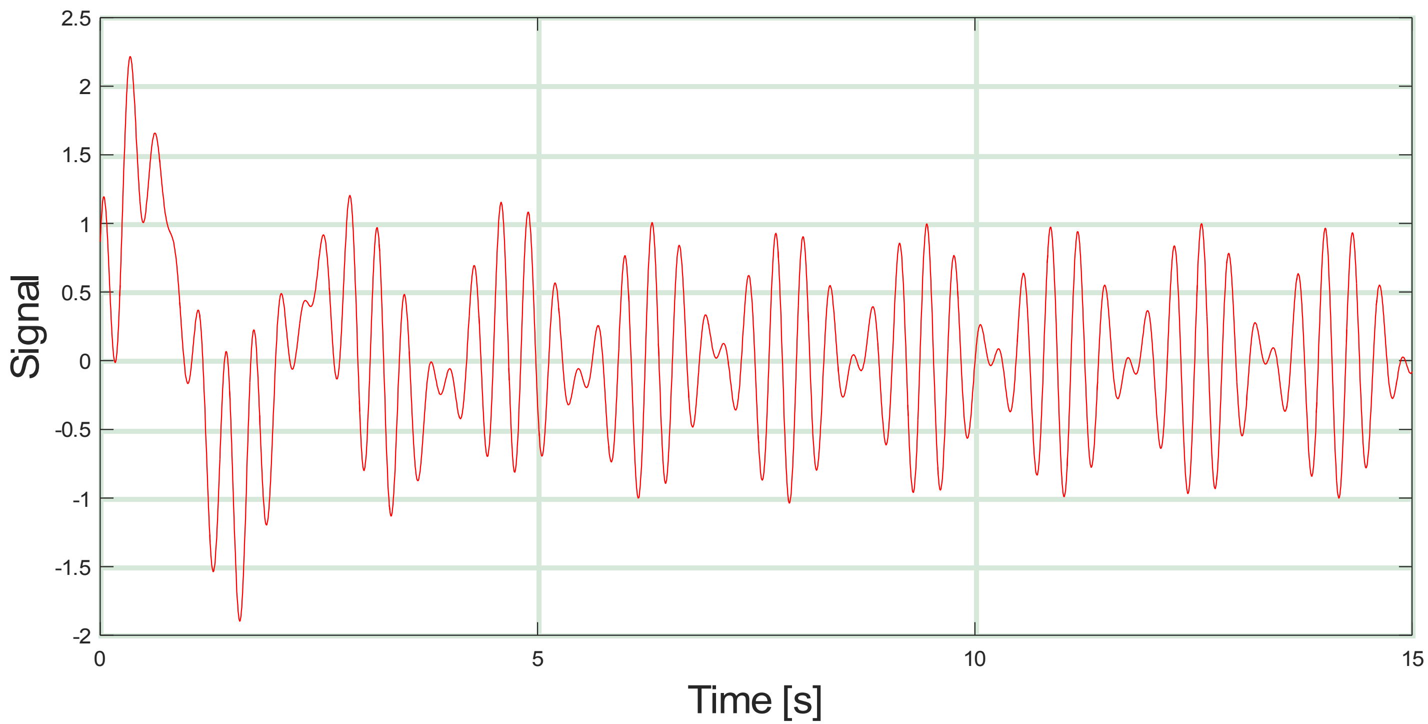
|noise|≤Δ, T_c: sampling period

An *r*-order differentiator implements a (*r+1*)-order sliding mode

L4 – Real-time differentiation

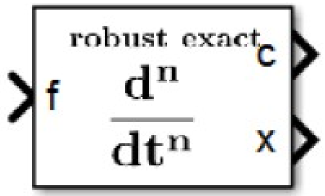
To discuss some properties of a HO-Differentiator let us consider an AM signal with a decaying transient

$$s(t) = 2e^{-\frac{t}{2}}\sin(3t) + \cos(2t)\sin(20t + \pi/3)$$



L4 – Real-time differentiation

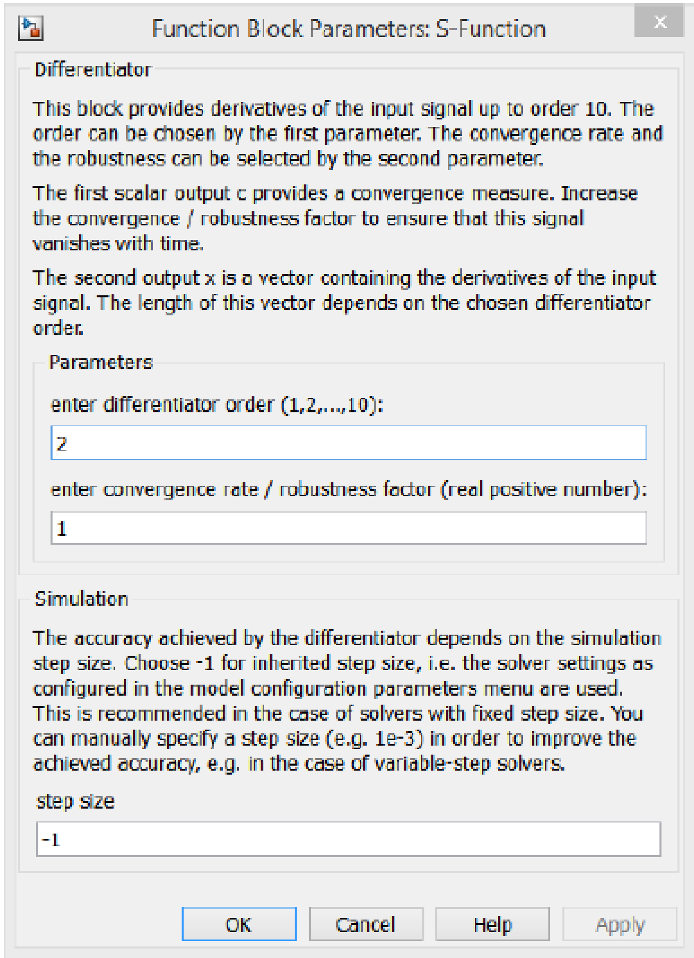
The simulink block provided by Proff. Markus Reichhartinger and Sarah Spurgeon is used to show the features of HO-Sliding Mode Differentiators



Integration method:
Variable step ODE45
(Dormand-Prince)

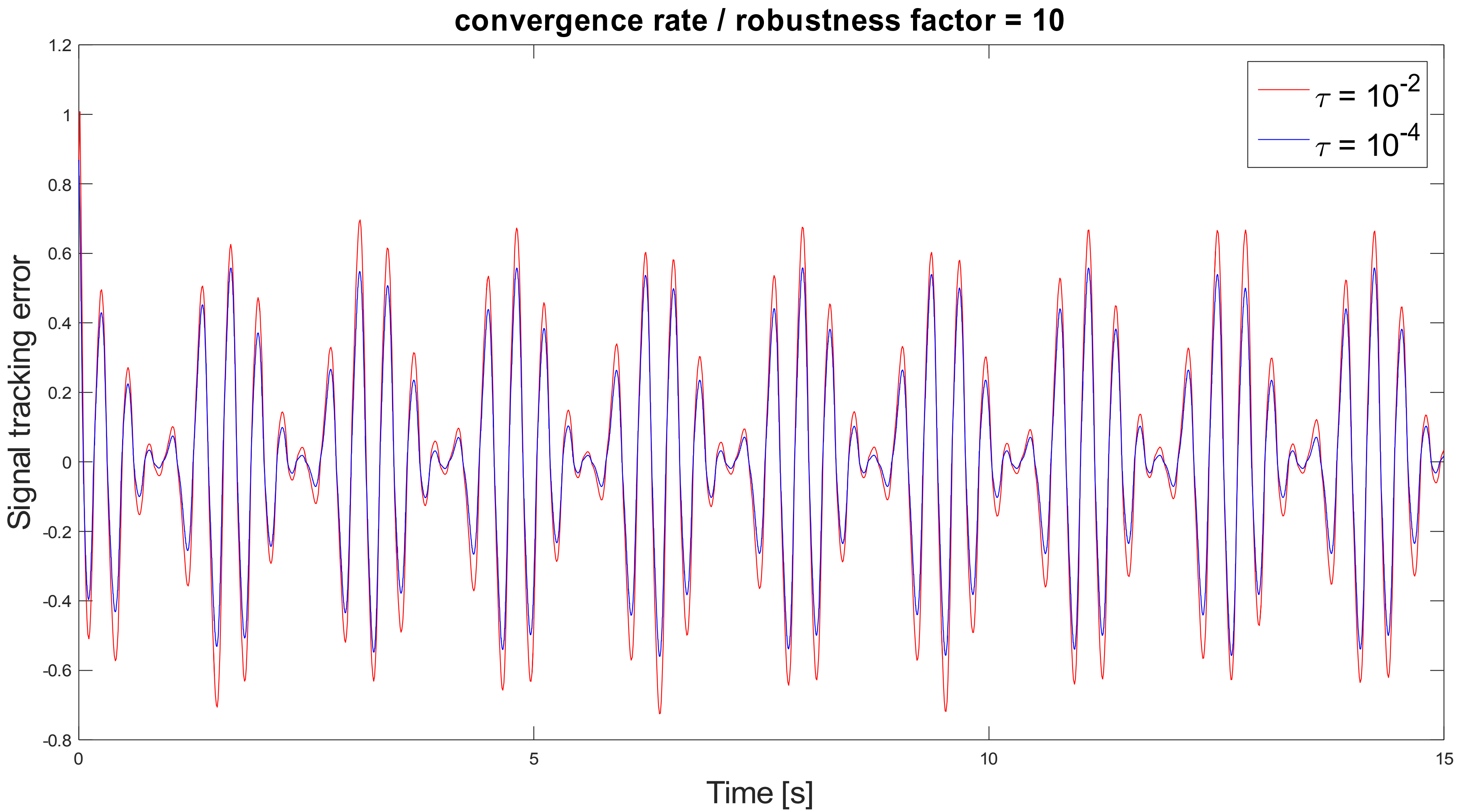
MAX step size: 10^{-5}

Min step size: 10^{-6}



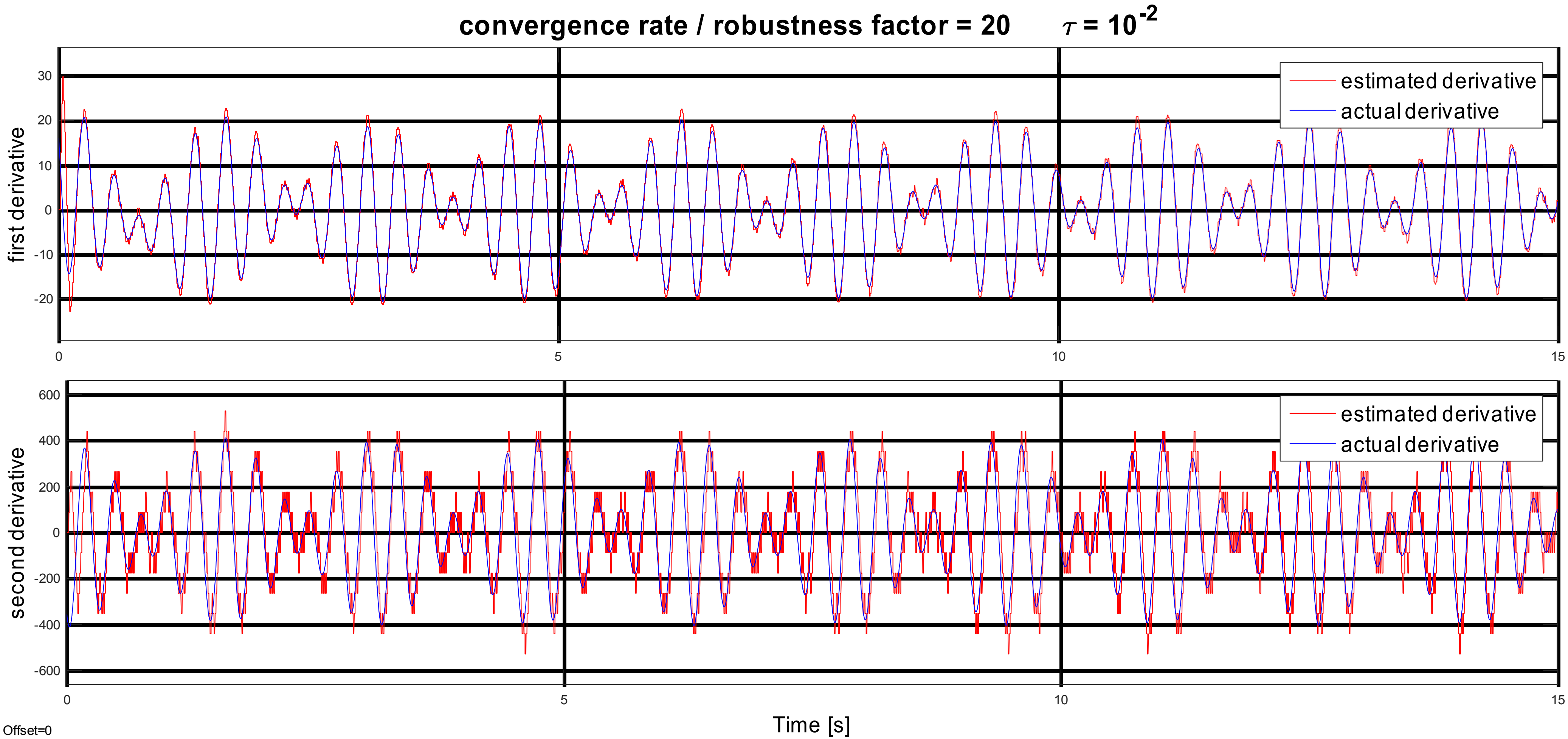
L4 – Real-time differentiation

If the "gain" parameter is not sufficiently high the SMO doesn't converge independently from the discretization period τ



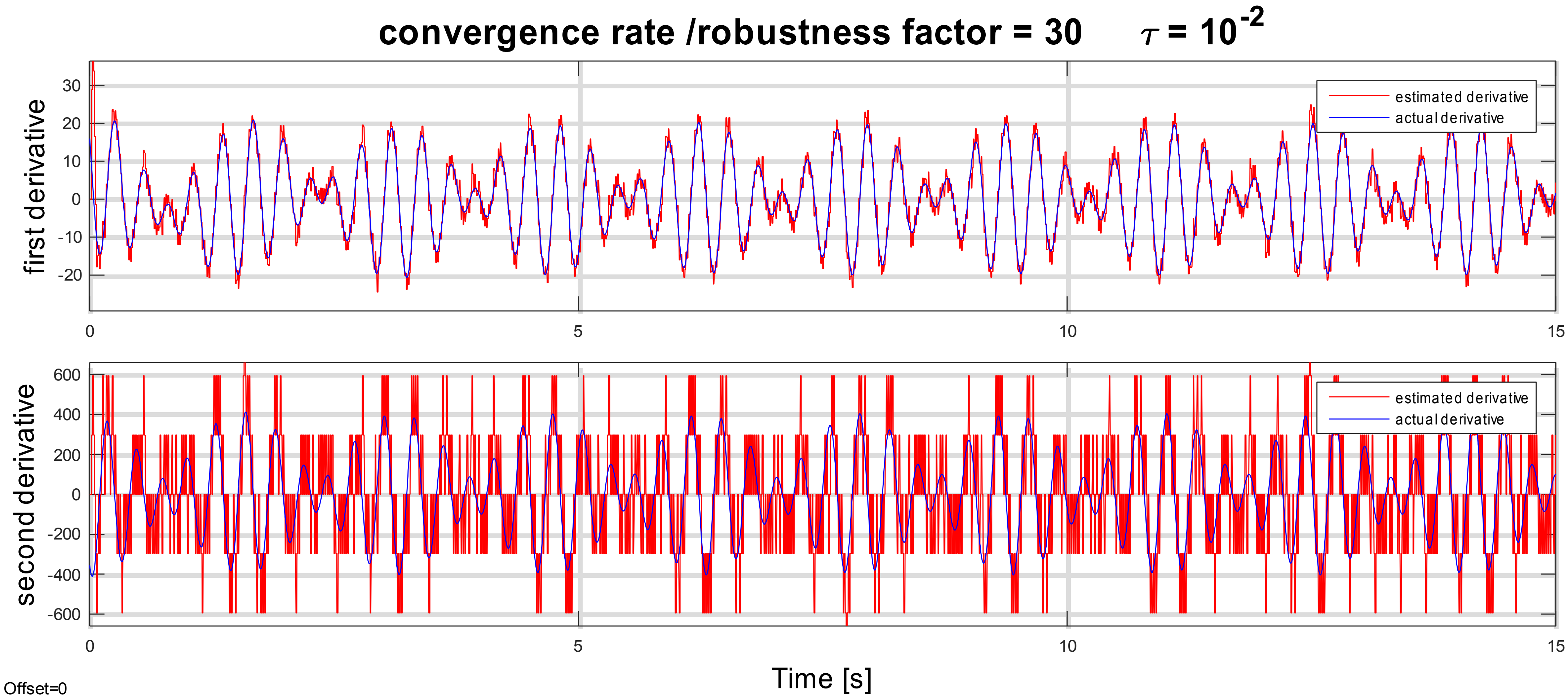
L4 – Real-time differentiation

Once the "gain" parameter is sufficiently to assure the SMO convergence its incresing worsens the accuracy at a fixed discretization period τ



L4 – Real-time differentiation

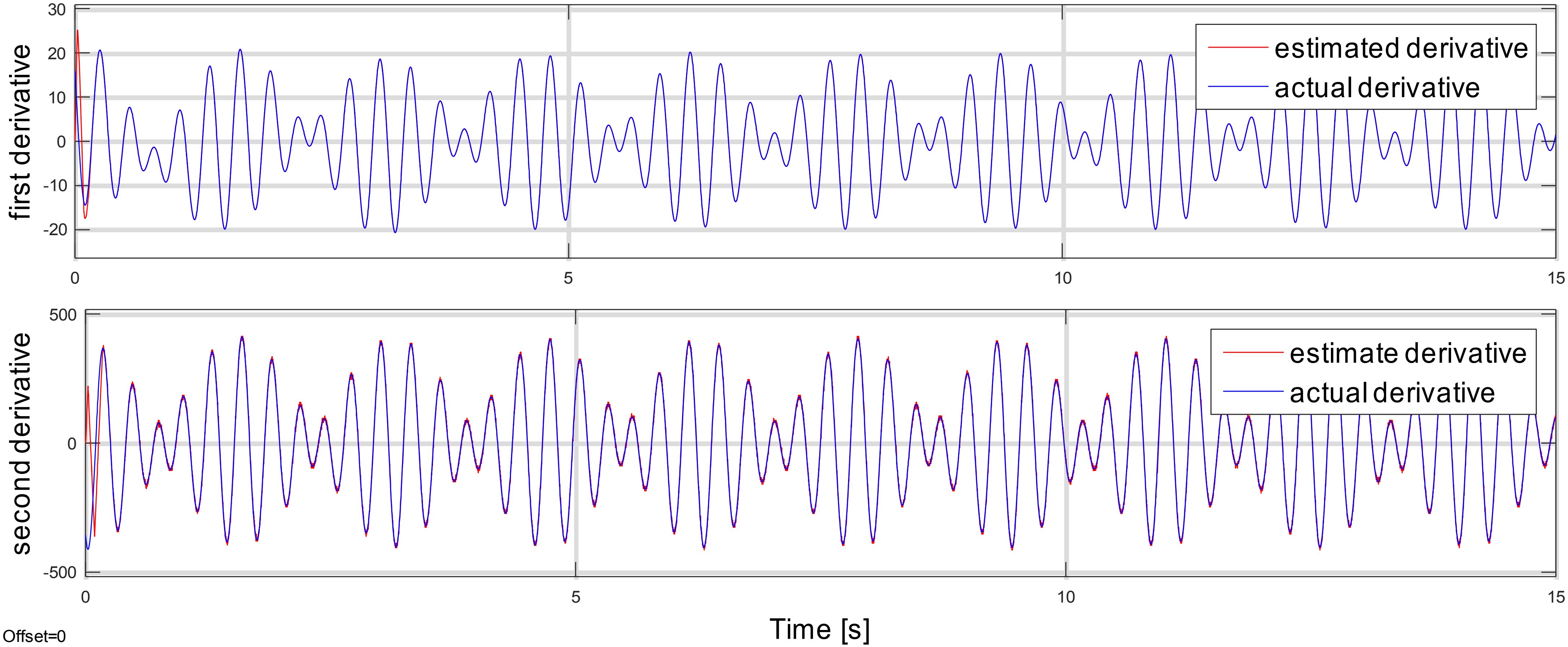
Once the “gain” parameter is sufficiently to assure the SMO convergence its incresing worsens the accuracy at a fixed discretization period τ



L4 – Real-time differentiation

Once the "gain" parameter is sufficiently to assure the SMO convergence better accuracy can be attained decreasing the discretization period τ

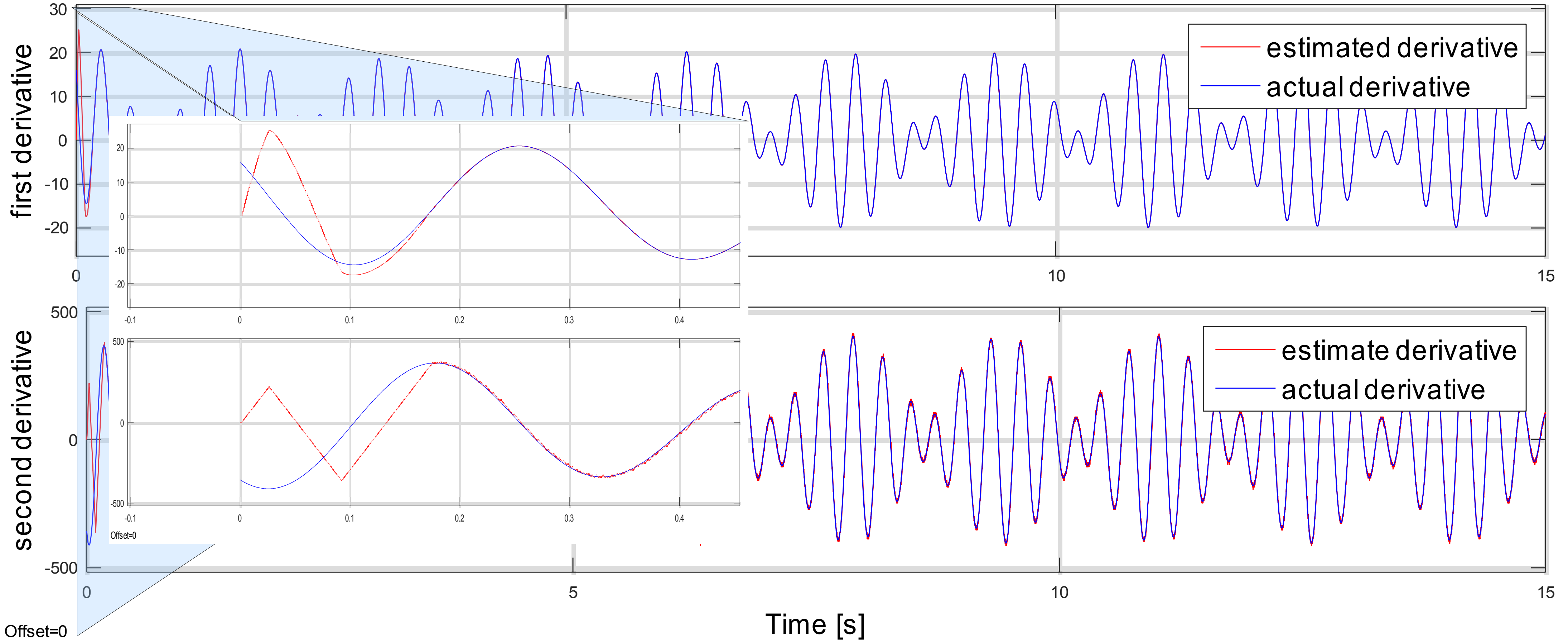
convergence rate / robustness factor = 20 $\tau = 10^{-3}$



L4 – Real-time differentiation

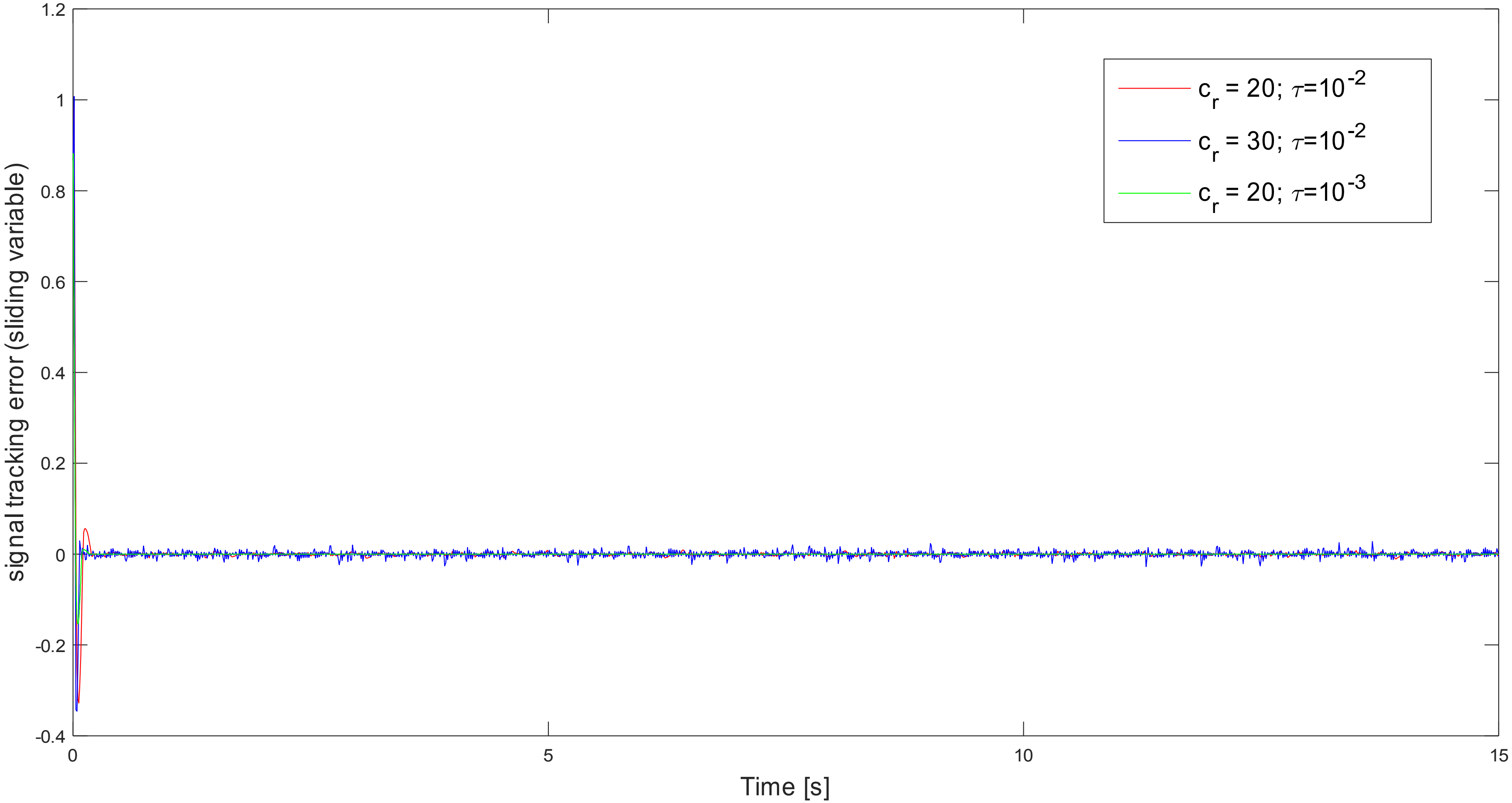
Once the "gain" parameter is sufficiently to assure the SMO convergence better accuracy can be attained decreasing the discretization period τ

convergence rate / robustness factor = 20 $\tau = 10^{-3}$



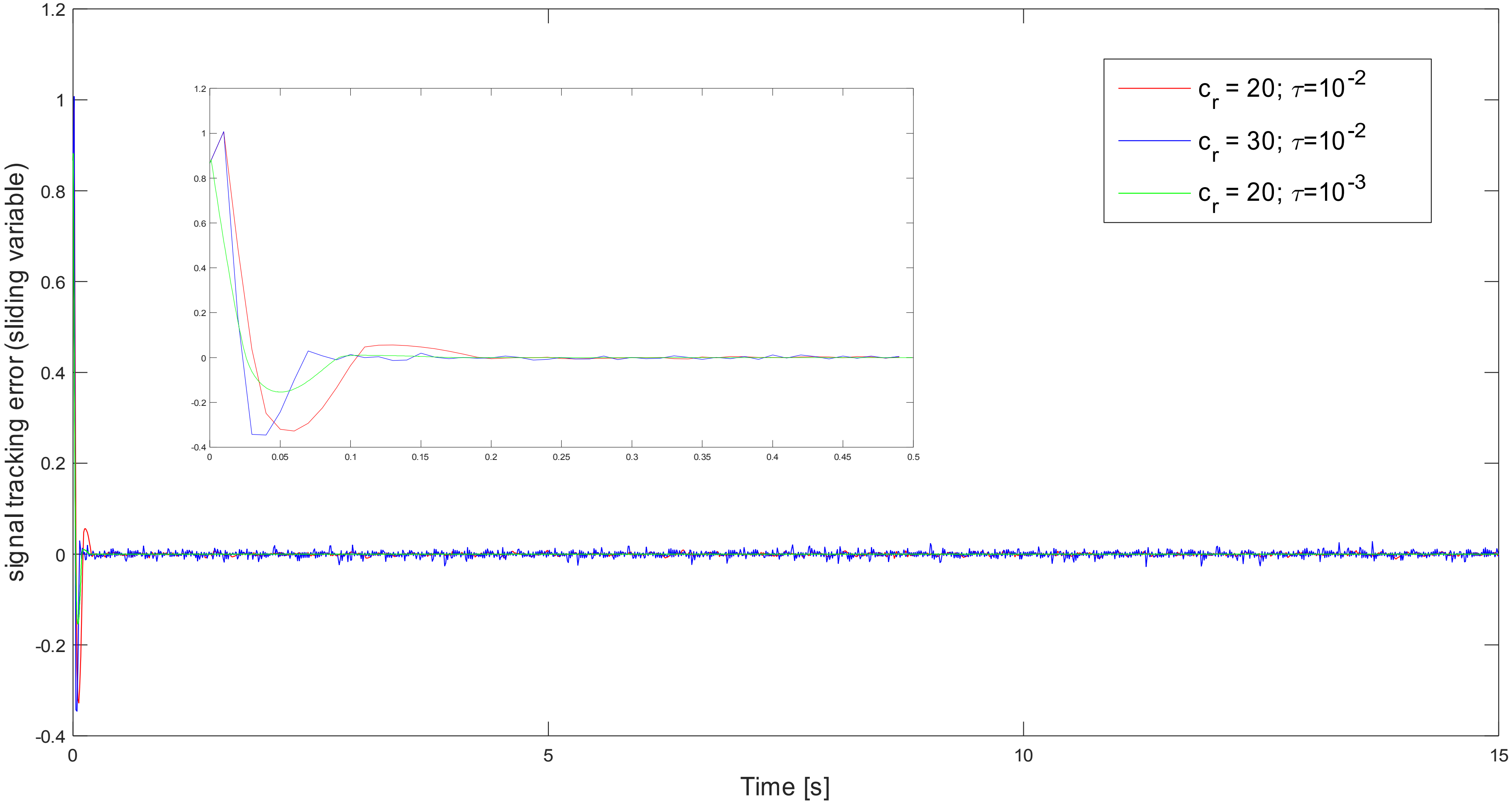
L4 – Real-time differentiation

Once the "gain" parameter is sufficiently to assure the SMO convergence better accuracy can be attained decreasing the discretization period τ



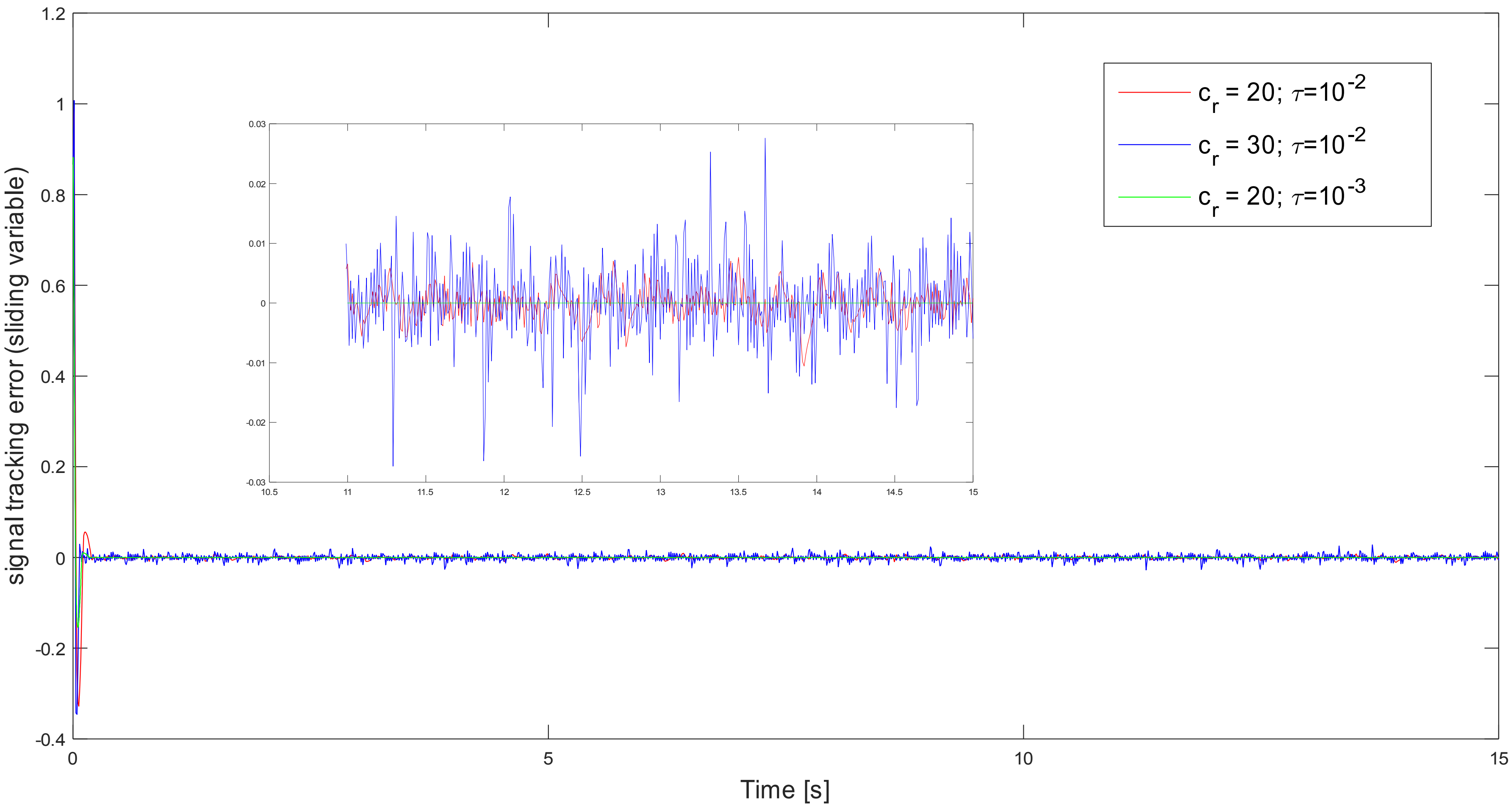
L4 – Real-time differentiation

Once the “gain” parameter is sufficiently to assure the SMO convergence better accuracy can be attained decreasing the discretization period τ



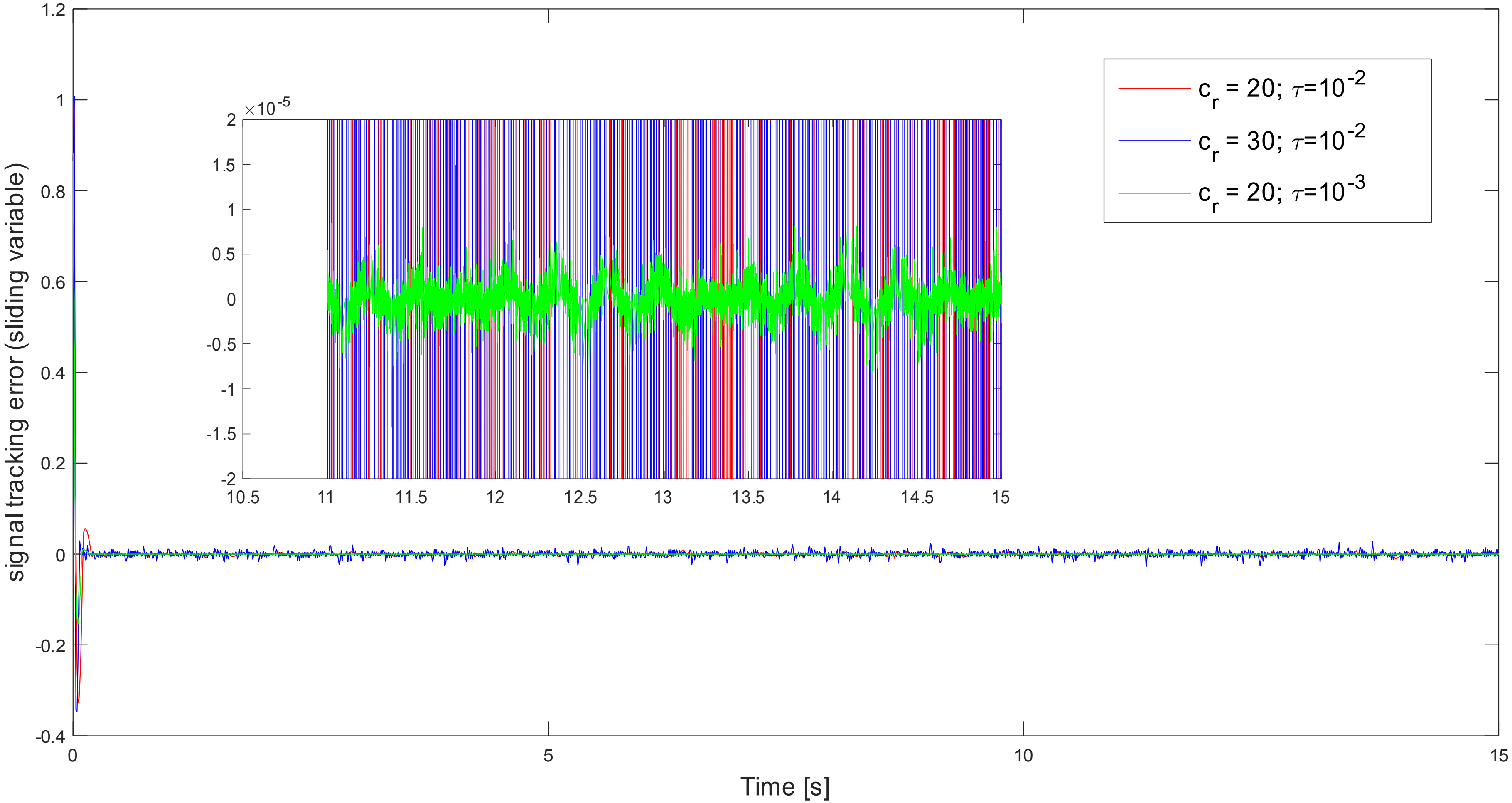
L4 – Real-time differentiation

Once the "gain" parameter is sufficiently to assure the SMO convergence better accuracy can be attained decreasing the discretization period τ



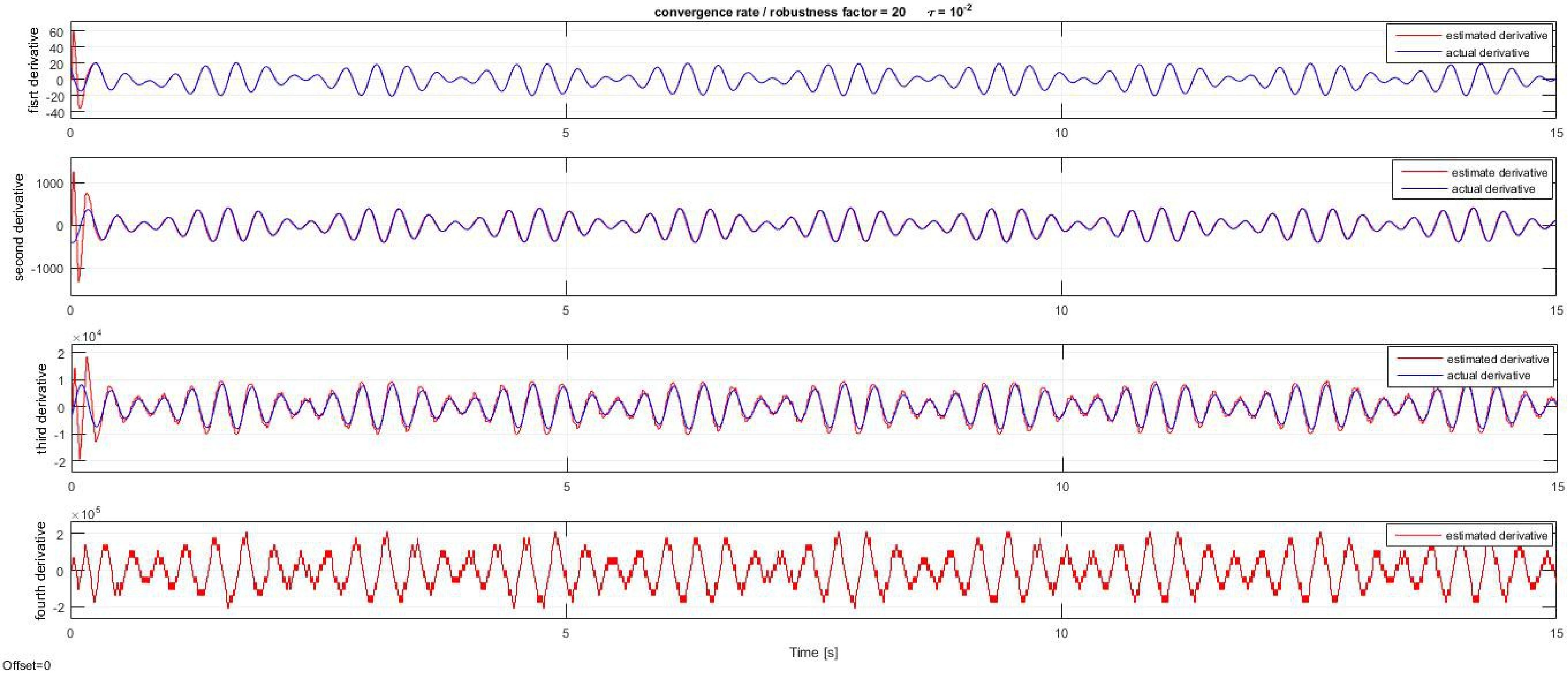
L4 – Real-time differentiation

Once the "gain" parameter is sufficiently to assure the SMO convergence better accuracy can be attained decreasing the discretization period τ



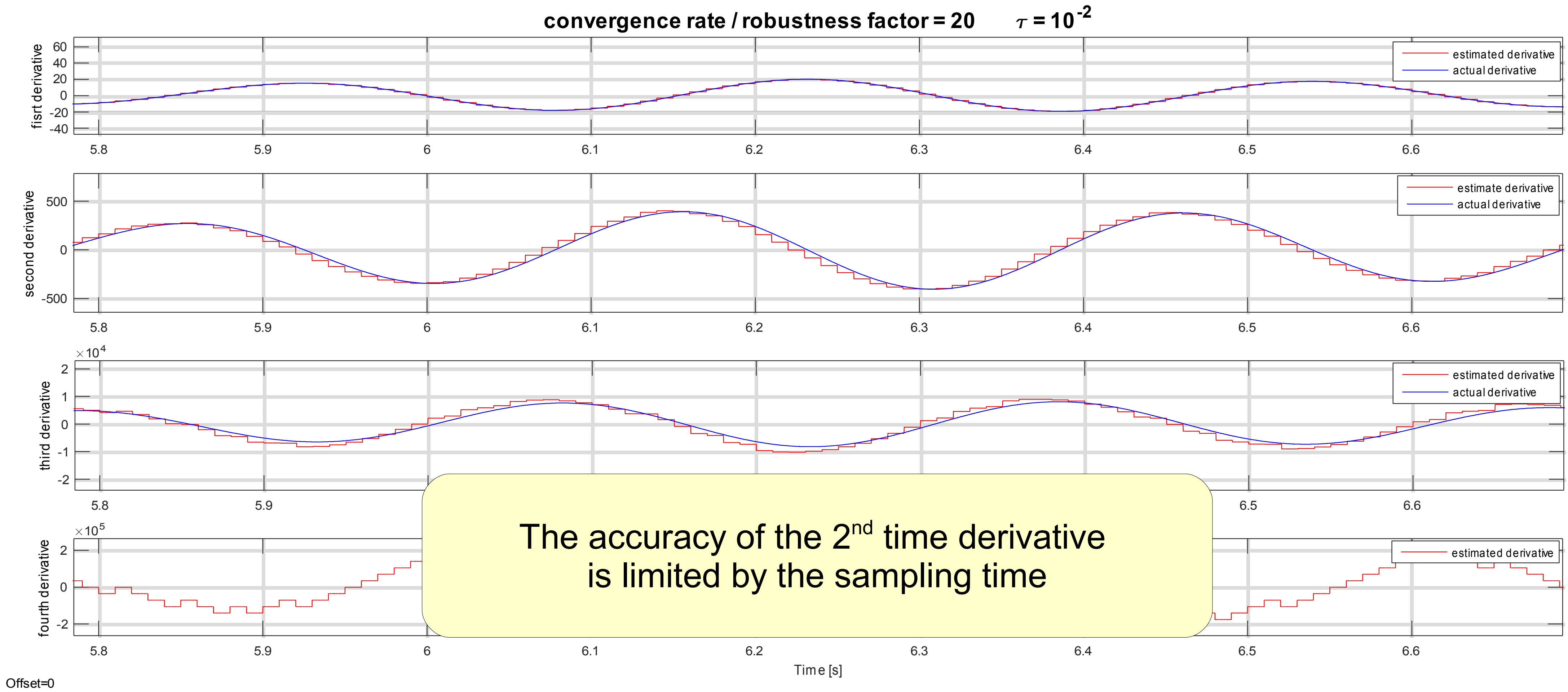
L4 – Real-time differentiation

Once the "gain" parameter is sufficiently to assure the SMO convergence better accuracy can be also attained using a differentiator of higher order



L4 – Real-time differentiation

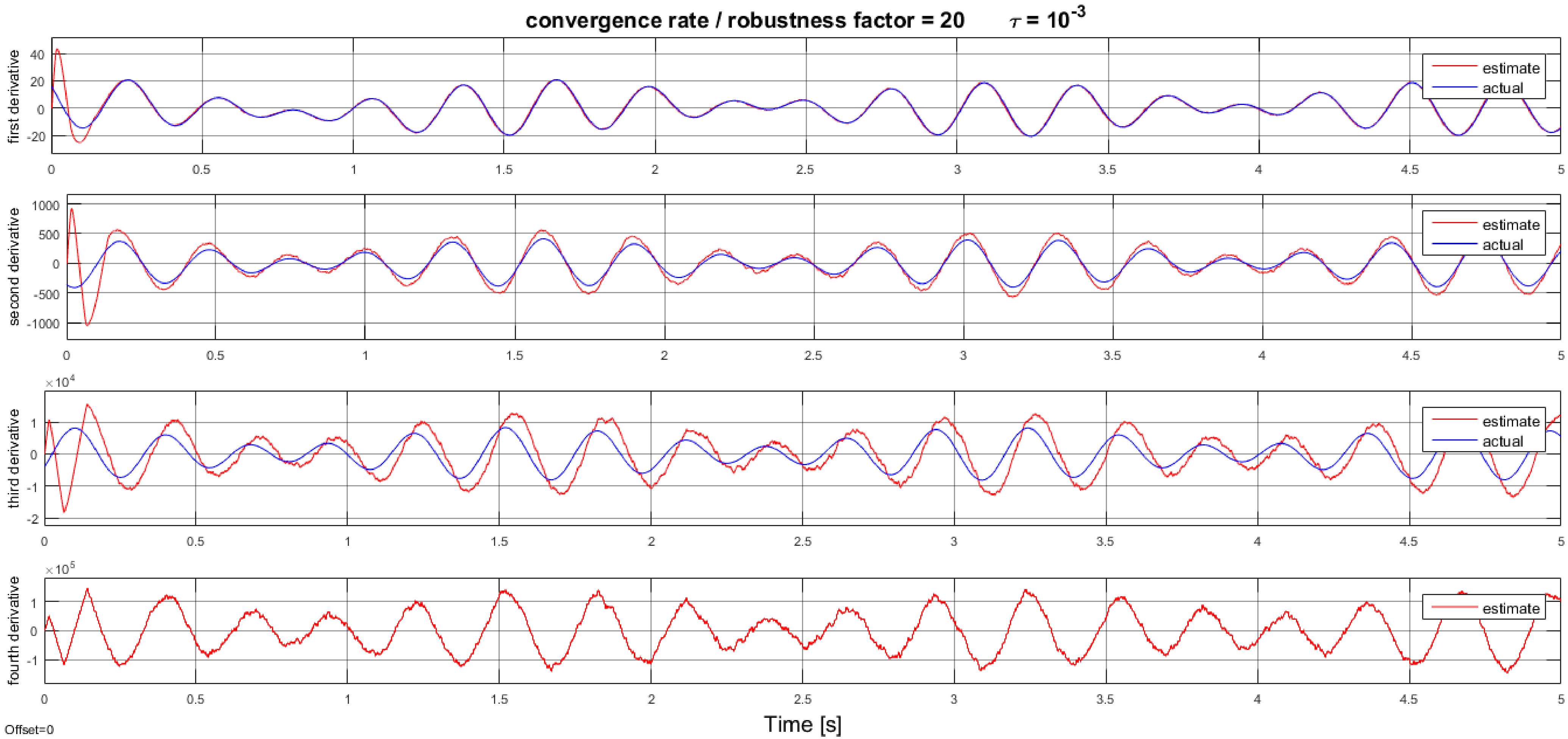
Once the “gain” parameter is sufficiently to assure the SMO convergence better accuracy can be also attained using a differentiator oh higher order



The accuracy of the 2nd time derivative is limited by the sampling time

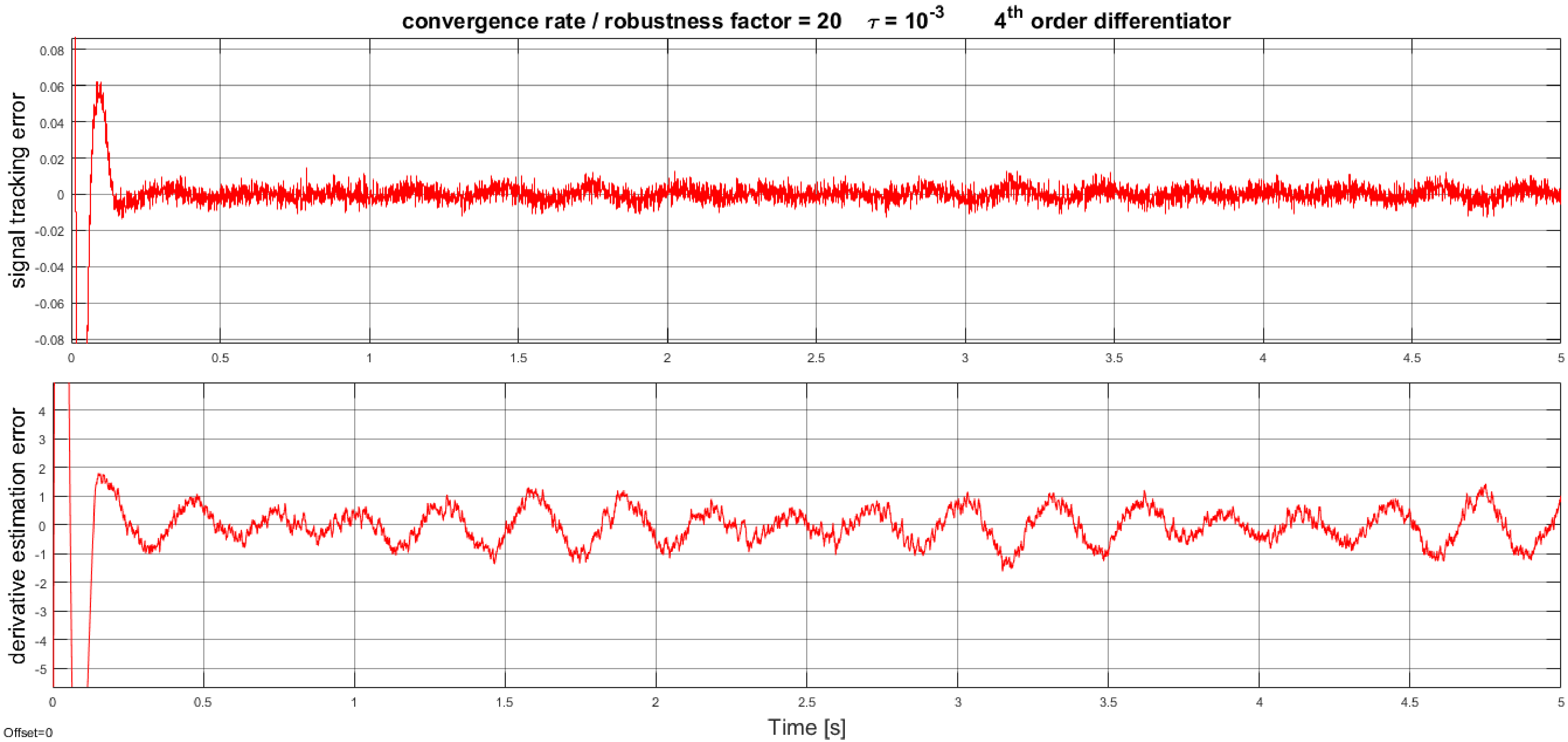
L4 – Real-time differentiation

The effect of the measurement error affects the accuracy. The following tests are performed considering a 1% accuracy (max value) in the acquired data *(band limited white noise with 10^{-3} sample time)*



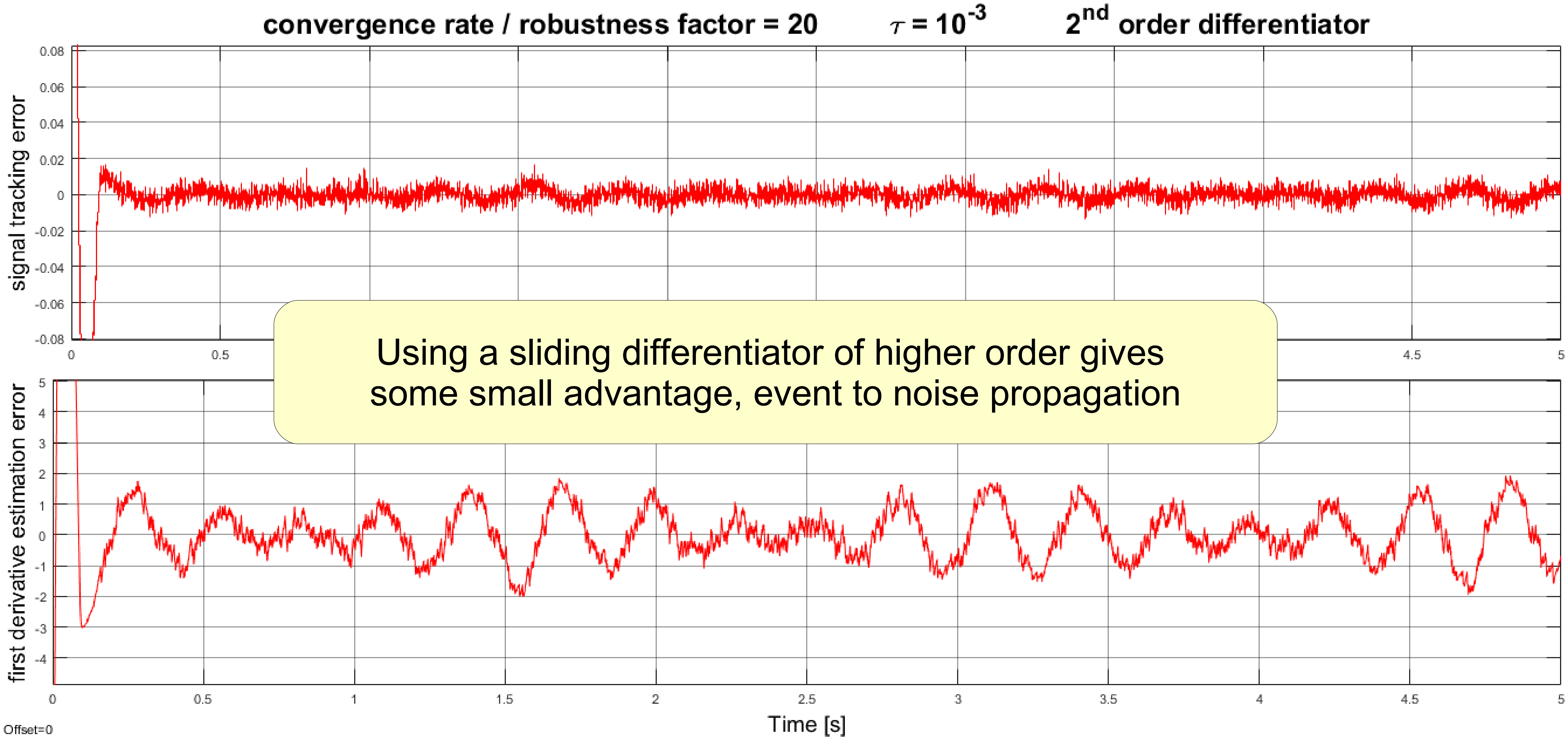
L4 – Real-time differentiation

The effect of the measurement error affects the accuracy. The following tests are performed considering a 1% accuracy (max value) in the acquired data *(band limited white noise with 10^{-3} sample time)*



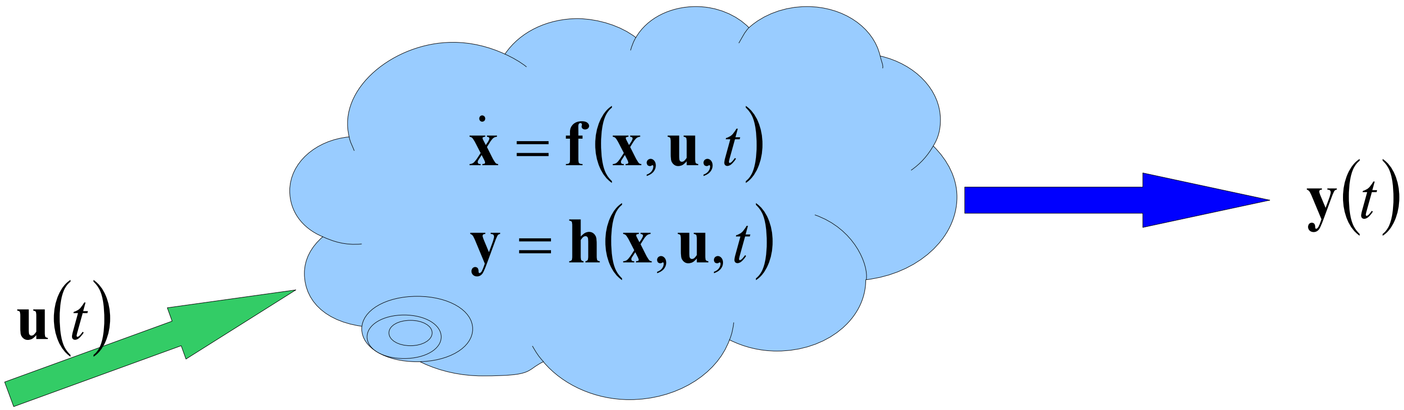
L4 – Real-time differentiation

The effect of the measurement error affects the accuracy. The following tests are performed considering a 1% accuracy (max value) in the acquired data *(band limited white noise with 10^{-3} sample time)*



Using a sliding differentiator of higher order gives some small advantage, event to noise propagation

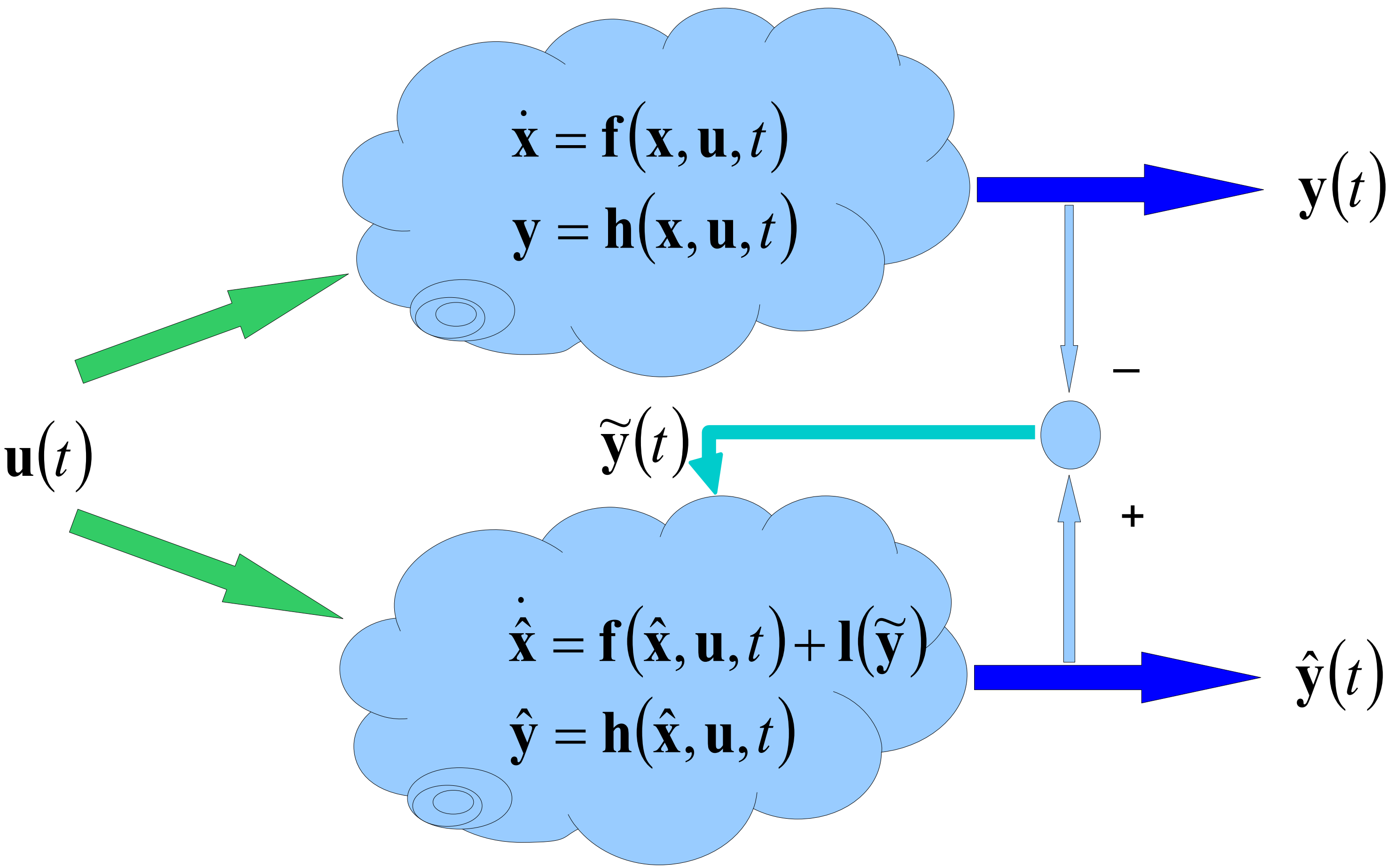
L4 – State estimation in dynamical systems



Given a dynamical system the observation problem is to estimate its internal state on the basis of the knowledge of:

- System outputs
- Systems inputs
- System dynamical model

L4 – State estimation in dynamical systems



The usual approach implies the implementation of a replica of the system with an additional input depending on the output error

L4 – State estimation in dynamical systems

The problem is relatively simple for linear, time-invariant systems

The system

$$\dot{\mathbf{x}} = \mathbf{A}\cdot\mathbf{x} + \mathbf{B}\cdot\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\cdot\mathbf{x} + \mathbf{D}\cdot\mathbf{u}$$

$\boldsymbol{x}\in\mathbb{R}^n, \quad \boldsymbol{u}\in\mathbb{R}^q, \quad \boldsymbol{y}\in\mathbb{R}^p$

$n \geq p, q$

The observer

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\cdot\hat{\mathbf{x}} + \mathbf{B}\cdot\mathbf{u} + \mathbf{L}\left(\hat{\mathbf{y}} - \mathbf{y}\right)$$
$$\hat{\mathbf{y}} = \mathbf{C}\cdot\hat{\mathbf{x}} + \mathbf{D}\cdot\mathbf{u}$$

Luemberger observer

The error

$$\dot{\tilde{\mathbf{x}}} = \left(\mathbf{A} + \mathbf{LC}\right)\cdot\tilde{\mathbf{x}}$$
$$\tilde{\mathbf{y}} = \mathbf{C}\cdot\tilde{\mathbf{x}}$$

$$\tilde{\mathbf{x}} = \mathbf{x} - \hat{\mathbf{x}}$$
$$\tilde{\mathbf{y}} = \mathbf{y} - \hat{\mathbf{y}}$$

The observability condition for linear system is that

The observability matrix

$$\mathbf{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}$$

is full rank

It is possible to arbitrarily define the eigenvalues of matrix $(\mathbf{A}+\mathbf{LC})$

Differentiating the system output

$$\begin{bmatrix} \mathbf{y} \\ \dot{\mathbf{y}} \\ \ddot{\mathbf{y}} \\ \vdots \\ \mathbf{y}^{(n-1)} \end{bmatrix} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{D} & & & & \\ \mathbf{CB} & \mathbf{D} & & & \\ \mathbf{CAB} & \mathbf{CB} & \mathbf{D} & & \\ \vdots & \vdots & & \ddots & \\ \mathbf{CA}^{n-2}\mathbf{B} & \mathbf{CA}^{n-3}\mathbf{B} & \dots & \mathbf{CB} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \dot{\mathbf{u}} \\ \ddot{\mathbf{u}} \\ \\ \mathbf{u}^{(n-1)} \end{bmatrix}$$

The system is solvable with respect to the state variables if it is possible to extract *n* independent rows from the *np* rows of the observability matrix:

i.e., the observability matrix is full rank

L4 – State estimation in dynamical systems

The general case of nonlinear systems is very much involved

$$\dot{\boldsymbol{x}}(t)=\boldsymbol{f}\left(\boldsymbol{x}(t),\boldsymbol{u}(t),t\right) \qquad \boldsymbol{x}\in\mathbb{R}^n;\ \boldsymbol{u}\in\mathbb{R}^q;\ \boldsymbol{y}\in\mathbb{R}^p$$
$$\boldsymbol{y}(t)=\boldsymbol{h}\left(\boldsymbol{x}(t),\boldsymbol{u}(t),t\right)$$

$$\begin{bmatrix}\mathbf{y}\\ \dot{\mathbf{y}}\\ \ddot{\mathbf{y}}\\ \vdots\\ \mathbf{y}^{(n-1)}\end{bmatrix}=\begin{bmatrix}L_{\mathbf{f}}^{0}\mathbf{h}(\mathbf{x},\mathbf{u},t)\\ L_{\mathbf{f}}^{1}\mathbf{h}(\mathbf{x},\mathbf{u},t)\\ L_{\mathbf{f}}^{2}\mathbf{h}(\mathbf{x},\mathbf{u},t)\\ \vdots\\ L_{\mathbf{f}}^{n-1}\mathbf{h}(\mathbf{x},\mathbf{u},t)\end{bmatrix}+\begin{bmatrix}0 & & & \\ \frac{\partial L_{\mathbf{f}}^{0}\mathbf{h}(\mathbf{x},\mathbf{u},t)}{\partial\mathbf{u}} & \ddots & & \\ \frac{\partial L_{\mathbf{f}}^{1}\mathbf{h}(\mathbf{x},\mathbf{u},t)}{\partial\mathbf{u}} & \frac{\partial L_{\mathbf{f}}^{0}\mathbf{h}(\mathbf{x},\mathbf{u},t)}{\partial\mathbf{u}} & \ddots & \\ \vdots & \vdots & \ddots & 0\\ \frac{\partial L_{\mathbf{f}}^{n-2}\mathbf{h}(\mathbf{x},\mathbf{u},t)}{\partial\mathbf{u}} & \frac{\partial L_{\mathbf{f}}^{n-3}\mathbf{h}(\mathbf{x},\mathbf{u},t)}{\partial\mathbf{u}} & \cdots & \frac{\partial L_{\mathbf{f}}^{0}\mathbf{h}(\mathbf{x},\mathbf{u},t)}{\partial\mathbf{u}}\end{bmatrix}\begin{bmatrix}\dot{\mathbf{u}}\\ \ddot{\mathbf{u}}\\ \vdots\\ \mathbf{u}^{(n-2)}\end{bmatrix}+\begin{bmatrix}0\\ \frac{\partial\mathbf{h}(\mathbf{x},\mathbf{u},t)}{\partial t}\\ \frac{\partial^{2}\mathbf{h}(\mathbf{x},\mathbf{u},t)}{\partial t^{2}}\\ \vdots\\ \frac{\partial^{n-1}\mathbf{h}(\mathbf{x},\mathbf{u},t)}{\partial t^{n-1}}\end{bmatrix}+\begin{bmatrix}0\\ 0\\ \chi_{1}(\mathbf{x},\mathbf{u},\dot{\mathbf{u}},t)\\ \vdots\\ \chi_{n-2}(\mathbf{x},\mathbf{u},\cdots,\mathbf{u}^{(n-2)},t)\end{bmatrix}$$

$$L_{\mathbf{f}}^{k+1}\mathbf{h}(\mathbf{x},\mathbf{u},t)=\begin{bmatrix}L_{\mathbf{f}}^{k+1}h_1(\mathbf{x},\mathbf{u},t)\\ L_{\mathbf{f}}^{k+1}h_2(\mathbf{x},\mathbf{u},t)\\ \vdots\\ L_{\mathbf{f}}^{k+1}h_p(\mathbf{x},\mathbf{u},t)\end{bmatrix}=\begin{bmatrix}\frac{\partial L_{\mathbf{f}}^{k}h_1(\mathbf{x},\mathbf{u},t)}{\partial x_1} & \frac{\partial L_{\mathbf{f}}^{k}h_1(\mathbf{x},\mathbf{u},t)}{\partial x_2} & \cdots & \frac{\partial L_{\mathbf{f}}^{k}h_1(\mathbf{x},\mathbf{u},t)}{\partial x_n}\\ \frac{\partial L_{\mathbf{f}}^{k}h_2(\mathbf{x},\mathbf{u},t)}{\partial x_1} & \frac{\partial L_{\mathbf{f}}^{k}h_2(\mathbf{x},\mathbf{u},t)}{\partial x_2} & \cdots & \frac{\partial L_{\mathbf{f}}^{k}h_2(\mathbf{x},\mathbf{u},t)}{\partial x_n}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial L_{\mathbf{f}}^{k}h_p(\mathbf{x},\mathbf{u},t)}{\partial x_1} & \frac{\partial L_{\mathbf{f}}^{k}h_p(\mathbf{x},\mathbf{u},t)}{\partial x_n} & \cdots & \frac{\partial L_{\mathbf{f}}^{k}h_p(\mathbf{x},\mathbf{u},t)}{\partial x_n}\end{bmatrix}\mathbf{f}(\mathbf{x},\mathbf{u},t) \qquad L_{\mathbf{f}}^{0}\mathbf{h}(\mathbf{x},\mathbf{u},t)=\mathbf{h}(\mathbf{x},\mathbf{u},t)$$

Often the simple autonomous dynamical system is referred to

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \qquad \mathbf{y} = \mathbf{h}(\mathbf{x})$$

$$\Rightarrow \quad \begin{bmatrix} \mathbf{y} \\ \dot{\mathbf{y}} \\ \ddot{\mathbf{y}} \\ \vdots \\ \mathbf{y}^{(n-1)} \end{bmatrix} = \begin{bmatrix} L_{\mathbf{f}}^0 \mathbf{h}(\mathbf{x}) \\ L_{\mathbf{f}}^1 \mathbf{h}(\mathbf{x}) \\ L_{\mathbf{f}}^2 \mathbf{h}(\mathbf{x}) \\ \vdots \\ L_{\mathbf{f}}^{n-1} \mathbf{h}(\mathbf{x}) \end{bmatrix}$$

$$d\,\boldsymbol{Y} = \begin{bmatrix} d\,\boldsymbol{y} \\ d\,\dot{\boldsymbol{y}} \\ \vdots \\ d\,\boldsymbol{y}^{(n-1)} \end{bmatrix} = \boldsymbol{O}(\boldsymbol{x})\,d\,\boldsymbol{x}$$

$$\mathbf{O}(\mathbf{x}) = \begin{bmatrix} \dfrac{\partial L_{\mathbf{f}}^0 \mathbf{h}(\mathbf{x})}{\partial \mathbf{x}} \\ \dfrac{\partial L_{\mathbf{f}}^1 \mathbf{h}(\mathbf{x})}{\partial \mathbf{x}} \\ \dfrac{\partial L_{\mathbf{f}}^2 \mathbf{h}(\mathbf{x})}{\partial \mathbf{x}} \\ \vdots \\ \dfrac{\partial L_{\mathbf{f}}^{n-1} \mathbf{h}(\mathbf{x})}{\partial \mathbf{x}} \end{bmatrix}$$

L4 – HOSM Algebraic Observers

Algebraic observability is related to the possibility to solve an algebraic system involving the system input and output and their time derivatives.

In case of linear systems, if the system is observable, there exist

$$r_i \quad (i = 1,2,\dots,p); \quad \sum_{i=1}^{p} r_i = n$$

Such that

$$\boldsymbol{O}_{sq} = \begin{bmatrix} \vdots \\ c_i \\ c_i \boldsymbol{A} \\ \vdots \\ c_i \boldsymbol{A}^{(r_i-1)} \\ \vdots \end{bmatrix}$$

is non singular

L4 – HOSM Algebraic Observers

Therefore the algebraic system can be solved by means of Higher Order Sliding Mode Differentiator of a proper order

$$r_i \quad (i = 1,2,\ldots,p); \quad \sum_{i=1}^{p} r_i = n$$

$$\boldsymbol{x} = \boldsymbol{O}_{sq}^{-1} \left[\begin{bmatrix} \vdots \\ y_i \\ \dot{y}_i \\ \vdots \\ y_i^{(r_i-1)} \\ \vdots \end{bmatrix} - \begin{bmatrix} \vdots & \vdots & \cdots & \vdots \\ d_i & 0 & \cdots & 0 \\ c_i \boldsymbol{B} & d_i & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ c_i \boldsymbol{A}^{(r_i-2)} \boldsymbol{B} & c_i \boldsymbol{A}^{(r_i-3)} \boldsymbol{B} & \cdots & d_i \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix} \begin{bmatrix} \vdots \\ \boldsymbol{u} \\ \dot{\boldsymbol{u}} \\ \vdots \\ \boldsymbol{u}^{(r_i-1)} \\ \vdots \end{bmatrix} \right]$$

L4 – HOSM Algebraic Observers

The case of nonlinear system is much more involved since the time derivative of the output depends on nonlinear functions involving both the state and the input

$$
\begin{bmatrix} \mathbf{y} \\ \dot{\mathbf{y}} \\ \ddot{\mathbf{y}} \\ \vdots \\ \mathbf{y}^{(n-1)} \end{bmatrix} = \begin{bmatrix} L_{\mathbf{f}}^{0}\mathbf{h}(\mathbf{x},\mathbf{u},t) \\ L_{\mathbf{f}}^{1}\mathbf{h}(\mathbf{x},\mathbf{u},t) \\ L_{\mathbf{f}}^{2}\mathbf{h}(\mathbf{x},\mathbf{u},t) \\ \vdots \\ L_{\mathbf{f}}^{n-1}\mathbf{h}(\mathbf{x},\mathbf{u},t) \end{bmatrix} + \begin{bmatrix} 0 & & & \\ \frac{\partial L_{\mathbf{f}}^{0}\mathbf{h}(\mathbf{x},\mathbf{u},t)}{\partial \mathbf{u}} & \ddots & & \\ \frac{\partial L_{\mathbf{f}}^{1}\mathbf{h}(\mathbf{x},\mathbf{u},t)}{\partial \mathbf{u}} & \frac{\partial L_{\mathbf{f}}^{0}\mathbf{h}(\mathbf{x},\mathbf{u},t)}{\partial \mathbf{u}} & \ddots & \\ \vdots & \vdots & \ddots & 0 \\ \frac{\partial L_{\mathbf{f}}^{n-2}\mathbf{h}(\mathbf{x},\mathbf{u},t)}{\partial \mathbf{u}} & \frac{\partial L_{\mathbf{f}}^{n-3}\mathbf{h}(\mathbf{x},\mathbf{u},t)}{\partial \mathbf{u}} & \cdots & \frac{\partial L_{\mathbf{f}}^{0}\mathbf{h}(\mathbf{x},\mathbf{u},t)}{\partial \mathbf{u}} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{u}} \\ \ddot{\mathbf{u}} \\ \vdots \\ \mathbf{u}^{(n-2)} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\partial \mathbf{h}(\mathbf{x},\mathbf{u},t)}{\partial t} \\ \frac{\partial^{2} \mathbf{h}(\mathbf{x},\mathbf{u},t)}{\partial t^{2}} \\ \vdots \\ \frac{\partial^{n-1} \mathbf{h}(\mathbf{x},\mathbf{u},t)}{\partial t^{n-1}} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \chi_{1}(\mathbf{x},\mathbf{u},\dot{\mathbf{u}},t) \\ \vdots \\ \chi_{n-2}(\mathbf{x},\mathbf{u},\cdots,\mathbf{u}^{(n-2)},t) \end{bmatrix}
$$

L4 – HOSM Algebraic Observers

Sufficient condition for existance of a solution of the algebraic observability problem for nonlinear system is that its dynamics can be described by a set of flat outputs and the corresponding squared observability matrix is non singular

$$\begin{bmatrix} \vdots \\ y_i \\ \dot{y}_i \\ \vdots \\ y_i^{(r_i-1)} \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ h_i(\boldsymbol{x}) \\ L_f h_i(\boldsymbol{x}) \\ \vdots \\ L_f^{(r_i-1)} h_i(\boldsymbol{x}) \\ \vdots \end{bmatrix}$$

$$det\left\{\begin{bmatrix} \vdots \\ \dfrac{\partial h_i(\boldsymbol{x})}{\partial \boldsymbol{x}} \\ \dfrac{\partial L_f h_i(\boldsymbol{x})}{\partial \boldsymbol{x}} \\ \vdots \\ \dfrac{\partial L_f^{(r_i-1)} h_i(\boldsymbol{x})}{\partial \boldsymbol{x}} \\ \vdots \end{bmatrix}\right\} \neq 0$$

L4 – HOSM Algebraic Observers

Consider the Chua's circuit with cubic nonlinearity

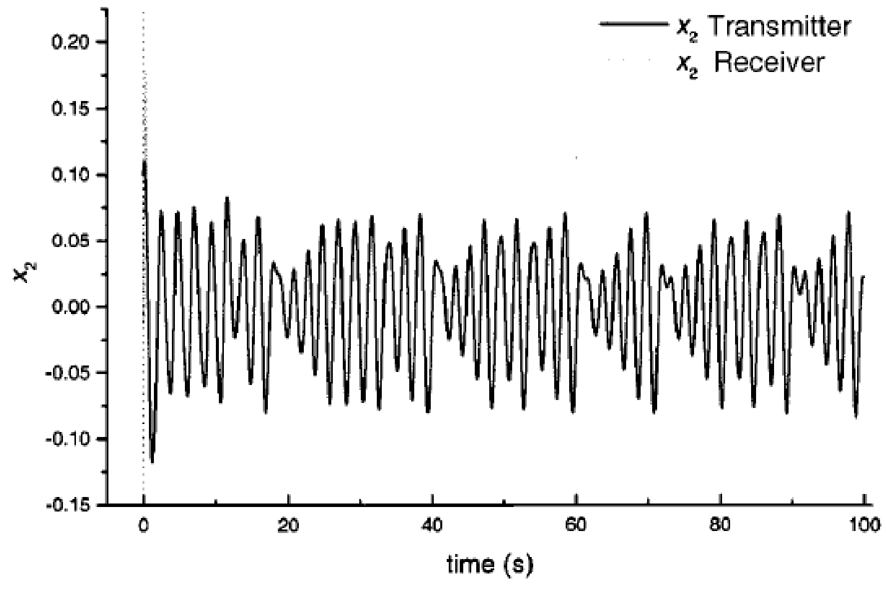
$$\begin{cases} \dfrac{dx_1}{dt} = \alpha(-cx_1 + x_2 - x_1^3) \\ \dfrac{dx_2}{dt} = x_1 - x_2 + x_3 \\ \dfrac{dx_3}{dt} = -\beta x_2. \end{cases}$$

$$\alpha = 10$$
$$\beta = 16$$
$$c = -0.143$$

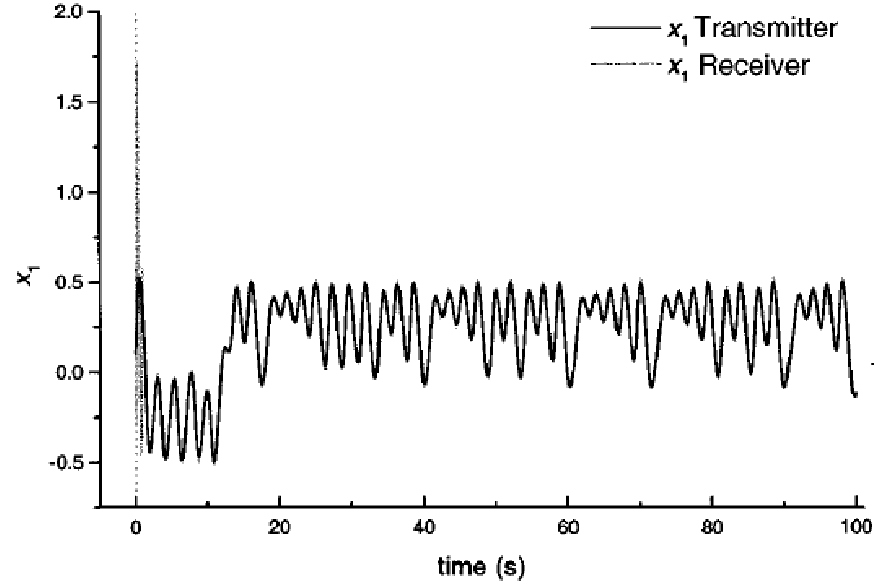
$$y = x_3$$

$$y^{(1)} + \beta x_2 = 0$$

$$y^{(2)} + y^{(1)} + \beta y + \beta x_1 = 0.$$



(a)



L4 – HOSM Dynamic Observers

Consider a nonlinear autonomous system with a sufficiently smooth dynamics

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \qquad \boldsymbol{x} \in \mathbb{R}^n, \quad \boldsymbol{y} \in \mathbb{R}^p$$

$$\mathbf{y} = \mathbf{h}(\mathbf{x}) \qquad n \geq p$$

If the system is observable, its output variables dynamics can be described as follows

$$\begin{bmatrix} \vdots \\ \dot{y}_i \\ \ddot{y}_i \\ \vdots \\ y_i^{(r_i)} \\ \vdots \end{bmatrix} = \dot{\boldsymbol{\varepsilon}} = \begin{bmatrix} \vdots \\ \dfrac{\partial h_i(\boldsymbol{x})}{\partial \boldsymbol{x}} \\ \dfrac{\partial L_{\boldsymbol{f}} h_i(\boldsymbol{x})}{\partial \boldsymbol{x}} \\ \vdots \\ \dfrac{\partial L_{\boldsymbol{f}}^{(r_i-1)} h_i(\boldsymbol{x})}{\partial \boldsymbol{x}} \\ \vdots \end{bmatrix} \dot{\boldsymbol{x}} = \boldsymbol{O}_{sq}(\boldsymbol{x}) \dot{\boldsymbol{x}} \qquad r_i \quad (i = 1,2,\ldots,p); \quad \sum_{i=1}^{p} r_i = n$$

L4 – HOSM Dynamic Observers

The observer is designed as a copy of the output dynamics plus a proper affine output injection

$$\dot{\hat{\boldsymbol{\varepsilon}}} = \begin{bmatrix} \dot{\hat{y}}_1 \\ \ddot{\hat{y}}_1 \\ \vdots \\ \hat{y}_1^{(r_1)} \\ \vdots \\ \dot{\hat{y}}_p \\ \ddot{\hat{y}}_p \\ \vdots \\ \hat{y}_p^{(r_1)} \end{bmatrix} = \mathbf{O}_{sq}(\hat{\mathbf{x}}) \cdot \dot{\hat{\mathbf{x}}} + \begin{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} & & \\ & \ddots & \\ & & \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} v_1(\hat{y}_1 - y_1) \\ \vdots \\ v_p(\hat{y}_p - y_p) \end{bmatrix}$$

$\mathbf{B}_G$

$$\begin{bmatrix} \hat{y}_1 \\ \dot{\hat{y}}_1 \\ \vdots \\ \hat{y}_1^{(r_1-1)} \\ \vdots \\ \hat{y}_p \\ \dot{\hat{y}}_p \\ \vdots \\ \hat{y}_p^{(r_p-1)} \end{bmatrix} = \begin{bmatrix} L_{\mathbf{f}}^0 h_1(\hat{\mathbf{x}}) \\ L_{\mathbf{f}}^1 h_1(\hat{\mathbf{x}}) \\ \vdots \\ L_{\mathbf{f}}^{r_1-1} h_1(\hat{\mathbf{x}}) \\ \vdots \\ L_{\mathbf{f}}^0 h_p(\hat{\mathbf{x}}) \\ L_{\mathbf{f}}^1 h_p(\hat{\mathbf{x}}) \\ \vdots \\ L_{\mathbf{f}}^{r_p-1} h_p(\hat{\mathbf{x}}) \end{bmatrix}$$

$$\hat{\mathbf{y}} = \mathbf{h}(\hat{\mathbf{x}})$$

L4 – HOSM Dynamic Observers

The observer can be implemented without any transformation

$$\dot{\hat{\mathbf{x}}} = \mathbf{f}(\hat{\mathbf{x}}) + \mathbf{O}_{sq}^{-1}(\hat{\mathbf{x}}) \cdot \mathbf{B}_G \cdot \mathbf{v}$$
$$\hat{\mathbf{y}} = \mathbf{h}(\hat{\mathbf{x}})$$

$\mathbf{B}_\mathrm{G}$ is the generalised control matrix in the control canonical form

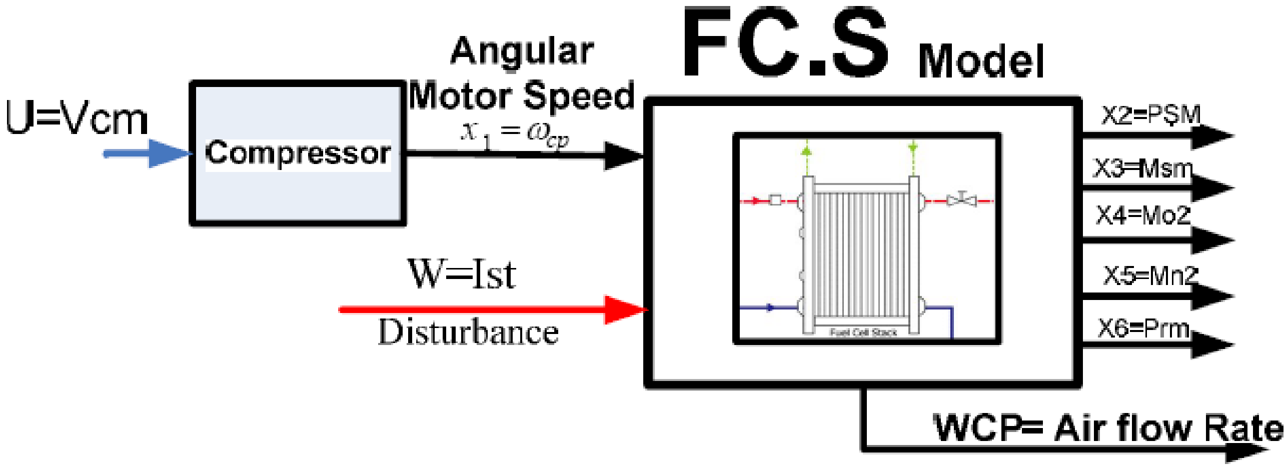
The estimation error dynamics in the output error space is defined as

$$\begin{bmatrix} \tilde{\varepsilon}_1^{(r_1)} \\ \tilde{\varepsilon}_2^{(r_2)} \\ \vdots \\ \tilde{\varepsilon}_p^{(r_p)} \end{bmatrix} = \begin{bmatrix} L_\mathbf{f}^{r_1} h_1(\hat{\mathbf{x}}) - L_\mathbf{f}^{r_1} h_1(\mathbf{x}) + v_1(\tilde{\varepsilon}_1) \\ L_\mathbf{f}^{r_2} h_2(\hat{\mathbf{x}}) - L_\mathbf{f}^{r_2} h_2(\mathbf{x}) + v_2(\tilde{\varepsilon}_2) \\ \vdots \\ L_\mathbf{f}^{r_p} h_p(\hat{\mathbf{x}}) - L_\mathbf{f}^{r_p} h_p(\mathbf{x}) + v_p(\tilde{\varepsilon}_p) \end{bmatrix} \qquad \tilde{\varepsilon}_i = \hat{y}_i - y_i \qquad (i = 1,2,\ldots,p)$$

It can be stabilised in a finite time by p HOSMC of a proper order

L4 – HOSM Dynamic Observers

Fuel cells are electro-chemical device in which hydrogen and oxygen reacts producing water and electricity



The load current depends on the reacted oxygen in the fuel cell stack

It is assumed that hydrogen pressure is constant, so that the delivered power depends on the inlet air flow

L4 – HOSM Dynamic Observers

The fuel cell model is highly nonlinear

$$\dot{x}_1 = \eta_{cm}\frac{k_t}{J_{CP}R_{cm}}V_{cm} - \eta_{cm}\frac{k_t}{J_{CP}R_{cm}}k_v x_1 - \frac{C_p}{J_{CP}}\frac{T_{atm}}{\eta_{cp}}\left(\left(\frac{x_2}{P_{atm}}\right)^{\frac{\gamma}{\gamma-1}} - 1\right)\frac{\left(B_{00} + B_{10}(x_2) + B_{20}(x_2)^2 + B_{01}(x_1) + B_{11}x_2x_1 + B_{02}(x_1)^2\right)}{x_1}$$

$$\dot{x}_2 = \frac{\gamma R_a}{v_{sm}}\left[W_{cp}\left(T_{atm} + \frac{T_{atm}}{\eta_{cp}}\left(\left(\frac{x_2}{P_{atm}}\right)^{\frac{\gamma-1}{\gamma}} - 1\right)\right) + \left(\left(-K_{sm,out}x_2 + K_{sm,out}P_{v,ca} + K_{sm,out}x_5\frac{R_{N2}T_{st}}{V_{ca}} + K_{sm,out}x_4\frac{R_{O2}T_{st}}{V_{ca}}\right)\frac{\gamma x_2}{x_3}\right)\right]$$

$$\dot{x}_3 = \left(B_{00} + B_{10}(x_2) + B_{20}(x_2)^2 + B_{01}(x_1) + B_{11}x_2x_1 + B_{02}(x_1)^2\right) + \left(-K_{sm,out}x_2 + K_{sm,out}P_{v,ca} + K_{sm,out}x_5\frac{R_{N2}T_{st}}{V_{ca}} + K_{sm,out}x_4\frac{R_{O2}T_{st}}{V_{ca}}\right)$$

$$\dot{x}_4 = \left(y_{O2,in}\cdot K_{sm,out}\cdot\left(x_2 - \frac{x_4}{M_{O2}}\frac{R_{O2}T_{st}}{V_{ca}} - \frac{x_5}{M_{N2}}\frac{R_{N2}T_{st}}{V_{ca}} - P_{v,ca}\right) - \left(\frac{x_4}{x4 + x5 + \frac{P_{v,ca}V_{ca}M_v}{R_vT_{st}}}K_{ca,out}\cdot\left(-x_6 + \frac{x_5}{M_{N2}}\frac{R_{N2}T_{st}}{V_{ca}} + \frac{x_4}{M_{O2}}\frac{R_{O2}T_{st}}{V_{ca}} + P_{v,ca}\right)\right) - \left(M_{O2}\frac{nI_{st}}{4F}\right)\right)$$

$$\dot{x}_5 = \left(\frac{(1-y_{O2,in})M_{N2}}{y_{O2,in}M_{O2} + (1-y_{O2,in})M_{N2}}\cdot\left(1 + \left(\frac{\frac{M_v\cdot\phi_{atm}\cdot P_{sat,Tatm}}{M_a^{atm}\cdot P_{atm}}}{1 - \frac{\phi_{atm}\cdot P_{sat,Tatm}}{P_{atm}}}\right)^{-1}\right)K_{sm,out}\cdot\left(x_2 - \frac{x_4}{M_{O2}}\frac{R_{O2}T_{st}}{V_{ca}} - \frac{x_5}{M_{N2}}\frac{R_{N2}T_{st}}{V_{ca}} - P_{v,ca}\right) - \frac{x_5}{x4 + x5 + \frac{P_{v,ca}V_{ca}M_v}{R_vT_{st}}}K_{ca,out}\cdot\left(-x_6 + \frac{x_4}{M_{O2}}\frac{R_{O2}T_{st}}{V_{ca}} + \frac{x_5}{M_{N2}}\frac{R_{N2}T_{st}}{V_{ca}} + P_{v,ca}\right)\right)$$

$$\dot{x}_6 = \frac{R_aT_{rm}}{V_{rm}}\left(K_{ca,out}\left(\frac{x_4}{M_{O2}}\frac{R_{O2}T_{st}}{V_{ca}} + \frac{x_5}{M_{N2}}\frac{R_{N2}T_{st}}{V_{ca}} + P_{v,ca} - x_6\right) - \left(P_{a_6}x_6^5 + P_{a_5}x_6^4 + P_{a_4}x_6^3 + P_{a_3}x_6^2 + P_{a_2}x_6 + P_{a_1}\right)\right)$$

The fuel cell model can be represented in a compact form

$$\dot{\mathbf{x}} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2, x_3, x_4, x_5) \\ f_3(x_1, x_2, x_4, x_5) \\ f_4(x_2, x_4, x_5, x_6) \\ f_5(x_2, x_4, x_5) \\ f_6(x_2, x_5, x_6) \end{bmatrix} + \begin{bmatrix} \frac{\eta_{cm} k_t}{J_{cm} R_{cm}} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} v_{cp} + \begin{bmatrix} 0 \\ 0 \\ -\frac{n M_{O_2 t}}{4F} \\ 0 \\ 0 \end{bmatrix} i_{st}$$

State variables

$$\mathbf{x} = \begin{bmatrix} \omega_{cp}\ [rad/s] \\ P_{sm}\ [Pa] \\ m_{sm}\ [kg] \\ m_{O_2}\ [kg] \\ m_{N_2}\ [kg] \\ P_{rm}\ [Pa] \end{bmatrix}$$

Compressor angular speed

Pressure in the supply manifold

Air mass in the supply manifold

Oxygen mass at cathod side

Nitrogen mass at cathod side

Pressure in the return manifold

Measured input and disturbance

$v_{cp}\ [V]$ Compressor supply voltage

$i_{st}\ [A]$ Load current from the stack

L4 – HOSM Dynamic Observers

An important parameter to be regulated in a fuel cell is the oxygen excess ratio

$$\lambda_{O_2} = \frac{W_{O_2,in}}{W_{O_2,react}} = \frac{W_{O_2,in}(x_2, x_4, x_5)}{W_{O_2,react}(d)}$$

It depends on the internal state that cannot be measured directly!

The system output is constituted by the easily measurable variables

$$\mathbf{y} = \begin{bmatrix} \omega_{cp}\,[rad/s] \\ P_{sm}\,[Pa] \\ P_{rm}\,[Pa] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \mathbf{x} = C \cdot \mathbf{x}$$

An observer is needed!

L4 – HOSM Dynamic Observers

We found out that the determinant of this ordered square observability matrix does not vanish in a sufficiently large set of working conditions

$$\mathbf{O}_{sq}^{*}(\mathbf{x}) = \begin{bmatrix} c_1 \\ c_2 \\ c_2 \nabla(f(\mathbf{x})) \\ c_3 \\ c_3 \nabla(f(\mathbf{x})) \\ c_3 \nabla(L_f^1 f(\mathbf{x})) \end{bmatrix}_{(6\times 6)}$$

The chosen outputs are sensible and quite easy and reliable to measure and the corresponding considered observation index are

$$r_1 = 1; \quad r_2 = 2; \quad r_3 = 3$$

L4 – HOSM Dynamic Observers

Since the input and disturbance matrices are constant the observer can easily implemented as a copy of the system nominal model plus the output injection

$$\dot{\hat{\mathbf{x}}} = \begin{bmatrix} f_1(\hat{x}_1, \hat{x}_2) \\ f_2(\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4, \hat{x}_5) \\ f_3(\hat{x}_1, \hat{x}_2, \hat{x}_4, \hat{x}_5) \\ f_4(\hat{x}_2, \hat{x}_4, \hat{x}_5, \hat{x}_6) \\ f_5(\hat{x}_2, \hat{x}_4, \hat{x}_5) \\ f_6(\hat{x}_2, \hat{x}_5, \hat{x}_6) \end{bmatrix} + \begin{bmatrix} \frac{\eta_{cm} k_t}{J_{cm} R_{cm}} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} v_{cp} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{n M_{O_2 t}}{4F} \\ 0 \\ 0 \end{bmatrix} i_{st} + \mathbf{O}_{sq}^{*\,-1}(\hat{x}) \left[\begin{array}{c:c:c} 1 & 0 & 0 \\ \hdashline 0 & 0 & 0 \\ 0 & 1 & 0 \\ \hdashline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right] \mathbf{v}$$

The estimation error dynamics in the output error space is defined as

$$\begin{bmatrix} \dot{\tilde{\varepsilon}}_1 \\ \ddot{\tilde{\varepsilon}}_2 \\ \dddot{\tilde{\varepsilon}}_3 \end{bmatrix} = \begin{bmatrix} f_1(\hat{\mathbf{x}}) - f_1(\mathbf{x}) + v_1(\tilde{\varepsilon}_1) \\ L_{\mathbf{f}}^1 f_2(\hat{\mathbf{x}}) - L_{\mathbf{f}}^1 f_2(\mathbf{x}) + v_2(\tilde{\varepsilon}_2) \\ L_{\mathbf{f}}^2 f_6(\hat{\mathbf{x}}) - L_{\mathbf{f}}^2 f_6(\mathbf{x}) + v_3(\tilde{\varepsilon}_3) \end{bmatrix} \qquad \tilde{\varepsilon}_i = \hat{y}_i - y_i \qquad (i = 1,2,3)$$

L4 – HOSM Dynamic Observers

The output injection terms are implemented by the omogeneous sliding mode algorithms for systems with relative degree 1, 2 and 3

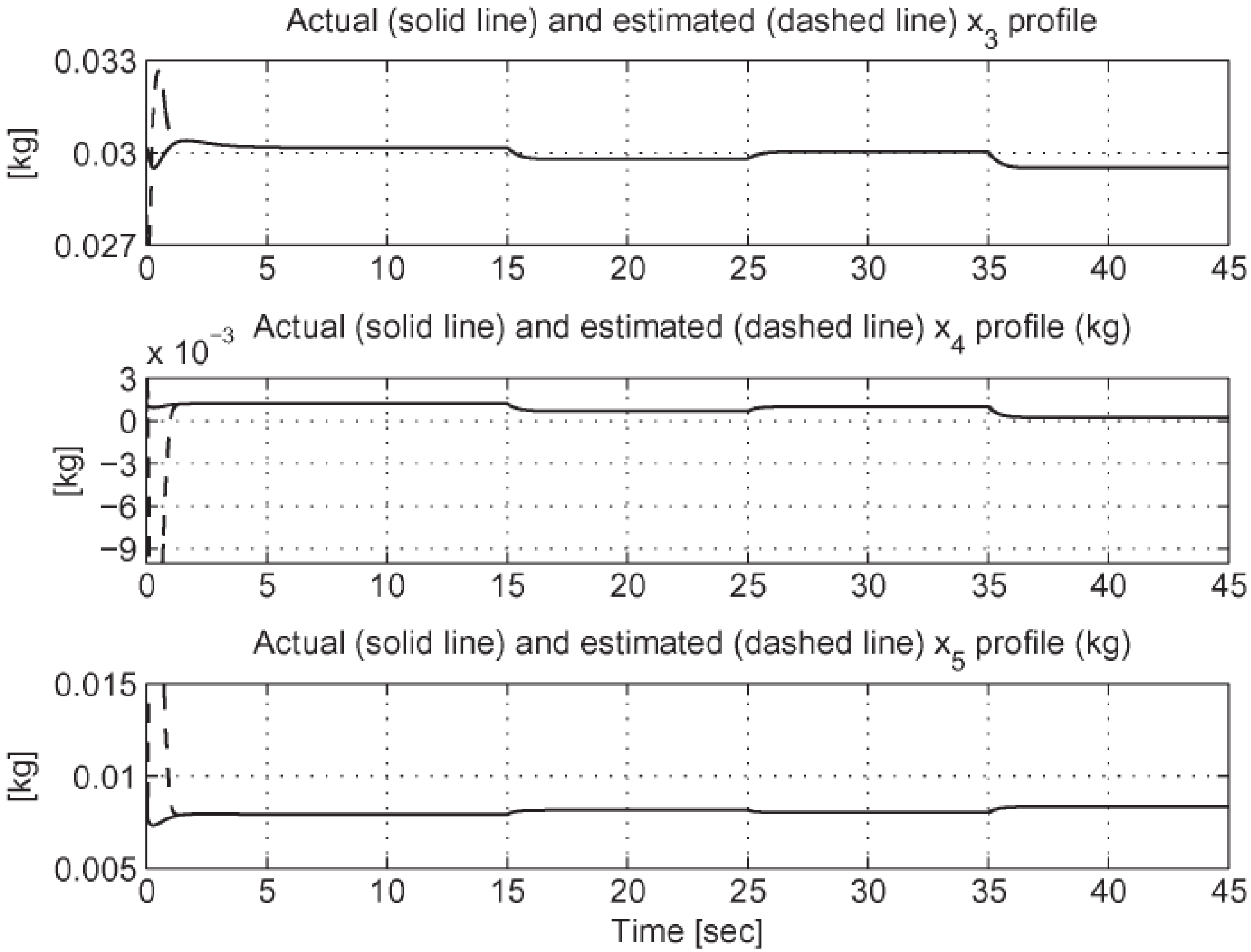
$$\nu_{1,1}(\varepsilon_1) = -\lambda_1 |\varepsilon_1|^{0.5} sign(\varepsilon_1) + \nu_{1,2}(\varepsilon_1),$$
$$\dot{\nu}_{1,2}(\varepsilon_1) = -\lambda_2 sign(\varepsilon_1).$$

$$\nu_{2,1}(\varepsilon_2) = -\alpha_1 |\varepsilon_2|^{0.5} sign(\varepsilon_2),$$
$$\nu_{2,2}(\varepsilon_2) = -\alpha_2 sign(\varepsilon_2).$$

$$\nu_{3,1}(\varepsilon_3) = -\beta_1 |\varepsilon_3|^{\frac{2}{3}} sign(\varepsilon_3),$$
$$\nu_{3,2}(\varepsilon_3) = -\beta_2 |\varepsilon_3|^{\frac{1}{3}} sign(\varepsilon_3),$$
$$\nu_{3,3}(\varepsilon_3) = -\beta_3 sign(\varepsilon_3).$$

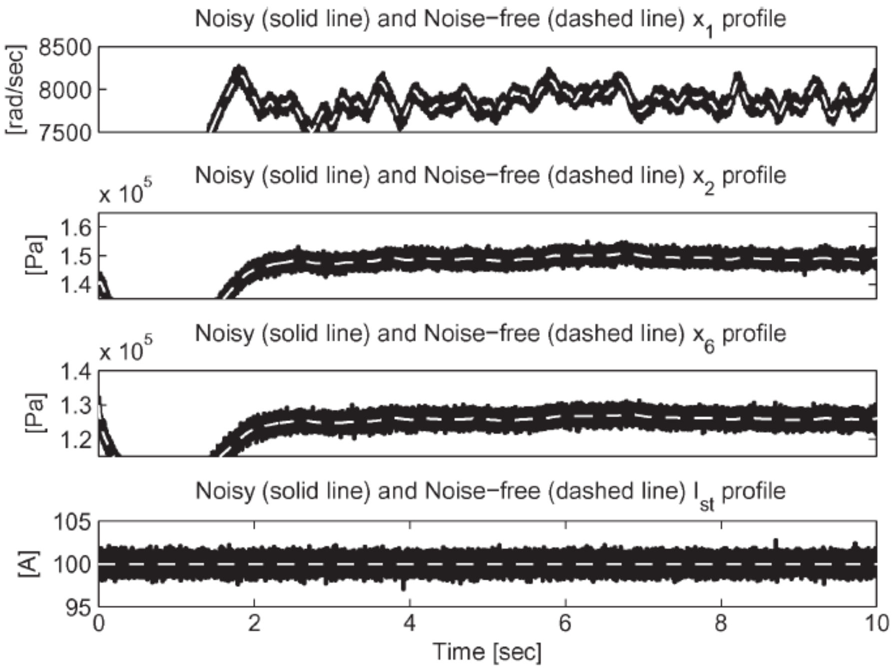
L4 – HOSM Dynamic Observers

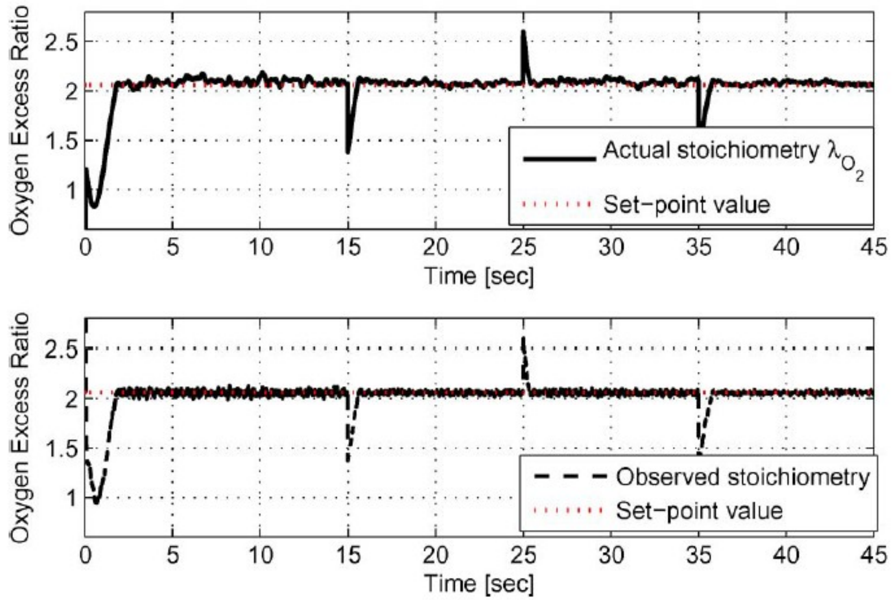
The output injection terms are implemented by the omogeneous sliding mode algorithms for systems with relative degree 1, 2 and 3



L4 – HOSM Dynamic Observers

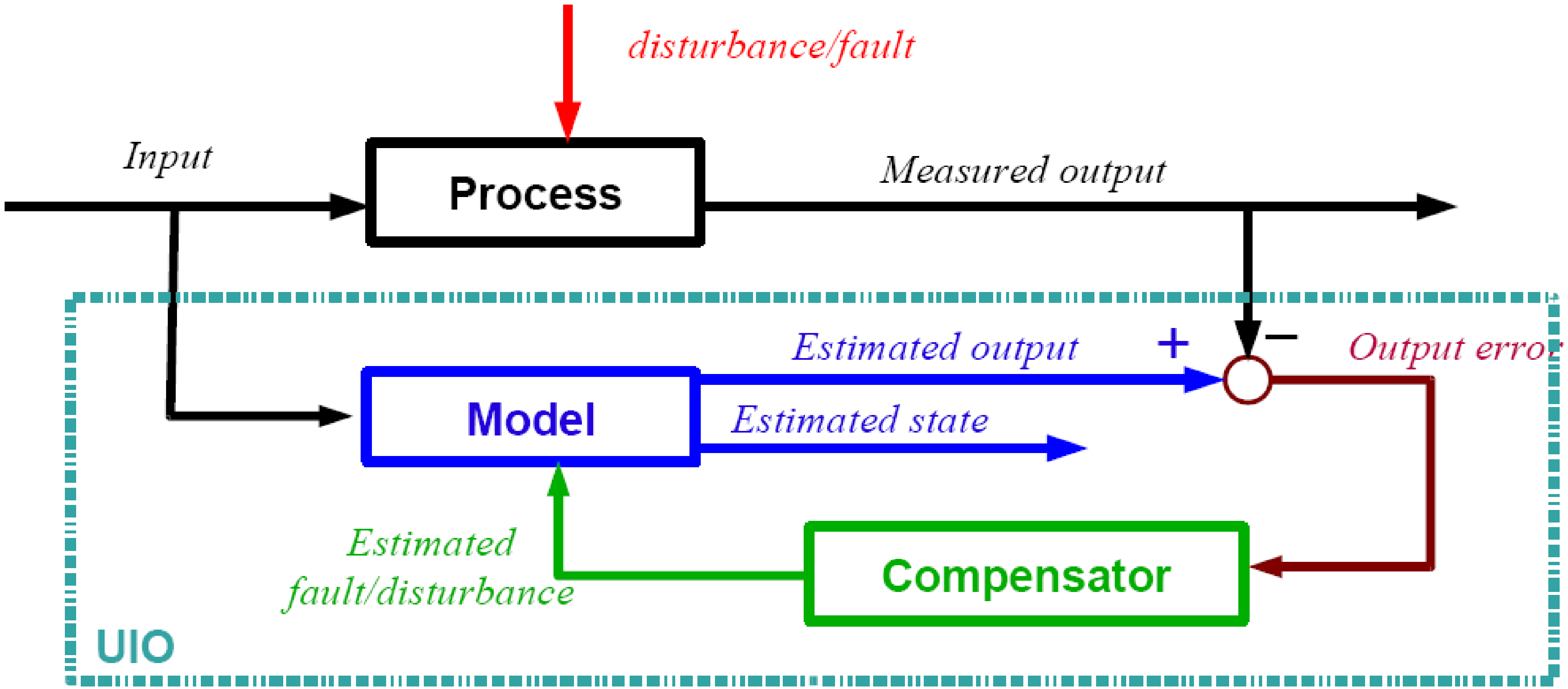
The estimates are used to regulate the oxigen excess ratio by approximate SMC even in the presence of parameters mismatching and measurement noise





L4 – Model-Based FDI via VSS

Model-based FDD and FDI via VSS resort to Sliding Mode Unknown Input Observers where the compensator is designed as a switching controller forcing a sliding mode on a properly defined sliding surface



L4 – Model-Based FDI via VSS

The system is ***strongly observable*** if $\mathbf{y} = \mathbf{0} \Rightarrow \mathbf{x} = \mathbf{0} \quad \forall \mathbf{d}$

Necessary and sufficient condition for a linear time-invariant system to be strongly observable is that the transfer matrix between the unknown input and the output has ***no invariant zeroes***

Considering time-invariant nonlinear systems with disturbances

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \Delta\mathbf{f}(\mathbf{x}, \mathbf{d})$$
$$\mathbf{y} = \mathbf{h}(\mathbf{x})$$

Condition for the system to be strongly observable is that exist

$$\rho_i : \quad L_{\Delta\mathbf{f}} L_{\mathbf{f}}^k h_i(\mathbf{x}, \mathbf{d}) = 0 \quad (k = 0, 1, \ldots, \rho_i - 2); \quad i = 1, 2, \ldots, p \quad \Rightarrow \quad \sum_{i=1}^{p} \rho_i \geq n$$

L4 – Model-Based FDI via VSS

A sufficient condition for the strong observability of a dynamical system is that a set $\mathbf{r}^*$ of r_i coefficient exists such that

$$\mathbf{r}^* = \left\{ r_1^*, r_2^*, \ldots, r_p^* \,\middle|\, \sum_{i=1}^{p} r_i^* = n; \rho_i \geq r_i^* \forall i \right\}$$

A set of *n* algebraic independent equations on the state variable only can be defined

$$\begin{bmatrix} \vdots \\ y_j \\ \dot{y}_j \\ \vdots \\ y_j^{(r_j^*-1)} \\ \vdots \end{bmatrix}_{\substack{j\in\{1,2,\ldots,p\} \\ (n\times 1)}} = \begin{bmatrix} \vdots \\ L_{\mathbf{f}}^0 h_j(\mathbf{x}) \\ L_{\mathbf{f}}^1 h_j(\mathbf{x}) \\ \vdots \\ L_{\mathbf{f}}^{r_j^*-1} h_j(\mathbf{x}) \\ \vdots \end{bmatrix}_{\substack{j\in\{1,2,\ldots,p\} \\ (n\times 1)}} \Rightarrow \begin{array}{l} \bar{\mathbf{y}}^* = \Phi(\mathbf{x}) \\ \mathbf{x} = \Phi^{-1}(\bar{\mathbf{y}}^*) \end{array}$$

Defines a diffeomorphism

L4 – Model-Based FDI via VSS

The system dynamics in the *n*-dimensional output space can be reduced into a canonical form

$$\dot{\bar{\mathbf{y}}}^* = \mathbf{A}_G \cdot \bar{\mathbf{y}}^* + \mathbf{B}_G \cdot \begin{bmatrix} \vdots \\ L_{\mathbf{f}}^{r_j^*} h_j(\mathbf{x}) + L_{\Delta\mathbf{f}} L_{\mathbf{f}}^{r_j^*-1} h_j(\mathbf{x}, \mathbf{d}) \\ \vdots \end{bmatrix}$$

$$\mathbf{A}_G = \left[\begin{array}{c:c:c} \ddots & & \\ \hdashline & \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix}_{(r_j^* \times r_j^*)} & \\ \hdashline & & \ddots \end{array}\right] \qquad \mathbf{B}_G = \left[\begin{array}{c:c:c} \ddots & & \\ \hdashline & \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{(r_j^* \times 1)} & \\ \hdashline & & \ddots \end{array}\right]$$

L4 – Model-Based FDI via VSS

The system dynamics in the *n*-dimensional output space can be reduced into a canonical form

$$\dot{\bar{\mathbf{y}}}^* = \mathbf{A}_G \cdot \bar{\mathbf{y}}^* + \mathbf{B}_G \cdot \begin{bmatrix} \vdots \\ L_{\mathbf{f}}^{r_j^*} h_j(\mathbf{x}) + L_{\Delta\mathbf{f}} L_{\mathbf{f}}^{r_j^*-1} h_j(\mathbf{x}, \mathbf{d}) \\ \vdots \end{bmatrix}$$

$$\mathbf{A}_G = \left[\begin{array}{c:c:c} \ddots & & \\ \hdashline & \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix}_{\left(r_j^* \times r_j^*\right)} & \\ \hdashline & & \ddots \end{array}\right] \qquad \mathbf{B}_G = \left[\begin{array}{c:c:c} \ddots & & \\ \hdashline & \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{\left(r_j^* \times 1\right)} & \\ \hdashline & & \ddots \end{array}\right]$$

L4 – Model-Based FDI via VSS

The system dynamics in the *n*-dimensional output space can be reduced into a canonical form

$$\dot{\bar{\mathbf{y}}}^* = \mathbf{A}_G \cdot \bar{\mathbf{y}}^* + \mathbf{B}_G \cdot \begin{bmatrix} \vdots \\ L_{\mathbf{f}}^{r_j^*} h_j(\mathbf{x}) + L_{\Delta\mathbf{f}} L_{\mathbf{f}}^{r_j^*-1} h_j(\mathbf{x}, \mathbf{d}) \\ \vdots \end{bmatrix}$$

Matrices $\mathbf{A}_G$ and $\mathbf{B}_G$ are block diagonal matrices in the so called

Generalized Controllable Form

Matrix $\mathbf{A}_G$ is a square nxn matrix while $\mathbf{B}_G$ is a nxn^*

$$n^* = \mathrm{card}\left\{ r_j^* \middle| r_j^* \neq 0 \right\}$$

L4 – Model-Based FDI via VSS

The observer can be designed as a replica of the nominal model plus an additional input term: the output injection

$$\dot{\hat{\mathbf{x}}} = \mathbf{f}(\hat{\mathbf{x}}) + \mathbf{O}_{sq}^{*\,-1}(\hat{\mathbf{x}}) \cdot \mathbf{B}_G \cdot \mathbf{v}\left(\hat{\overline{\mathbf{y}}}^* - \overline{\mathbf{y}}^*\right) \qquad \hat{\mathbf{y}} = \mathbf{h}(\hat{\mathbf{x}})$$

The same diffeomorphism applies for both the observer and the plant

$$\hat{\mathbf{x}} = \Phi^{-1}\left(\hat{\overline{\mathbf{y}}}^*\right) \qquad \hat{\overline{\mathbf{y}}}^* = \Phi(\hat{\mathbf{x}}) \qquad \mathbf{x} = \Phi^{-1}\left(\overline{\mathbf{y}}^*\right) \qquad \overline{\mathbf{y}}^* = \Phi(\mathbf{x})$$

The estimation error dynamics in the output state space is represented by n^* independent uncertain differential equations each having order $r_i{}^*$

$$\dot{\tilde{\overline{\mathbf{y}}}}^* = \mathbf{A}_G \cdot \tilde{\overline{\mathbf{y}}}^* + \mathbf{B}_G \cdot \begin{bmatrix} \vdots \\ \tilde{\varphi}_j^*(\hat{\mathbf{x}}, \mathbf{x}, \mathbf{d}) \\ \vdots \end{bmatrix} + \mathbf{B}_G \cdot \mathbf{v}\left(\hat{\overline{\mathbf{y}}}^* - \overline{\mathbf{y}}^*\right)$$

$$\tilde{\varphi}_j^*(\hat{\mathbf{x}}, \mathbf{x}, \mathbf{d}) = L_{\mathbf{f}}^{r_j^*} h_j(\hat{\mathbf{x}}) - L_{\mathbf{f}}^{r_j^*} h_j(\mathbf{x}) - L_{\Delta\mathbf{f}} L_{\mathbf{f}}^{r_j^*-1} h_j(\mathbf{x}, \mathbf{d})$$

L4 – Model-Based FDI via VSS

Once the sliding mode is attained in the output error state space, the state estimated perfectly coincide with the actual state and the dynamics simplify into

$$\dot{\tilde{\mathbf{y}}}^* = \mathbf{A}_G \cdot \tilde{\mathbf{y}}^* + \mathbf{B}_G \cdot \begin{bmatrix} \vdots \\ \tilde{\varphi}_j^*(\mathbf{x},\mathbf{x},\mathbf{d}) \\ \vdots \end{bmatrix} + \mathbf{B}_G \cdot \mathbf{v}\left(\tilde{\mathbf{y}}^*\right) = \mathbf{B}_G \cdot \left(\begin{bmatrix} \vdots \\ \tilde{\varphi}_j^*(\mathbf{x},\mathbf{x},\mathbf{d}) \\ \vdots \end{bmatrix} + \mathbf{v}\left(\tilde{\mathbf{y}}^*\right) \right)$$

$$\tilde{\varphi}_j^*(\mathbf{x},\mathbf{x},\mathbf{d}) = L_{\mathbf{f}}^{r_j^*} h_j(\mathbf{x}) - L_{\mathbf{f}}^{r_j^*} h_j(\mathbf{x}) - L_{\Delta\mathbf{f}} L_{\mathbf{f}}^{r_j^*-1} h_j(\mathbf{x},\mathbf{d}) = -L_{\Delta\mathbf{f}} L_{\mathbf{f}}^{r_j^*-1} h_j(\mathbf{x},\mathbf{d})$$

In the sliding mode the switching output injection can be substituted by its equivalent counterpart

$$v_{j_{eq}}\left(\tilde{y}_j^*\right) = -\tilde{\varphi}_j^*(\mathbf{x},\mathbf{x},\mathbf{d}) = L_{\Delta\mathbf{f}} L_{\mathbf{f}}^{r_j^*-1} h_j(\hat{\mathbf{x}},\mathbf{d})$$

The equivalent control contains information about the fault that can be recovered on appropriate conditions

L4 – Model-Based FDI via VSS

Receding horizon analysis can be used to avoid local zero crossing

$$\varepsilon_j^*(\hat{\mathbf{x}},\mathbf{d},t)=\int_{t-\Delta T}^{t}\psi\left(\hat{v}_{j_{eq}}^*(\hat{\mathbf{x}},\mathbf{d},\tau)\right)d\tau \quad j=1,2,\ldots \quad \psi(\cdot)>0$$

Residual can be used to detect the faulty condition

Under the assumption $\boldsymbol{\varepsilon}^*(\hat{\mathbf{x}},\mathbf{0},t)=\mathbf{0}$

Faulty condition can be detected by threshold checking any positive definite function can be used as the argument of $\delta(\mathbf{d},t)$ provided

$$\delta(\mathbf{d},t)=\begin{cases}1 & \mathbf{d}\neq\mathbf{0}\\ 0 & \mathbf{d}=\mathbf{0}\end{cases}$$

$$\delta(\mathbf{d},t)=\mathrm{bool}\left\{\left\|\boldsymbol{\varepsilon}^*(\hat{\mathbf{x}},\mathbf{d},t)\right\|_\infty>\varepsilon_{MAX}^*\right\}$$

$$\delta(\mathbf{d},t)=\mathrm{bool}\left\{\left\|\boldsymbol{\varepsilon}^*(\hat{\mathbf{x}},\mathbf{d},t)\right\|_2>\varepsilon_{MAX}^*\right\}$$

L4 – Model-Based FDI via VSS

Once the faulty condition is detected some information from the estimated output injection can be extracted

Location Look at $\arg\left\{\varepsilon_j^*(\hat{\mathbf{x}},\mathbf{d},t) > \varepsilon_{j\,MAX}^* ; j=1,2,\ldots\right\}$

Identification Evaluate $\mathbf{d} = -\tilde{\varphi}_j^{*-1}\left(\hat{\mathbf{x}},\hat{\mathbf{x}},\hat{\mathbf{v}}_{eq}^*\right) = \left(L_{\Delta\mathbf{f}} L_{\mathbf{f}}^{r_j^*-1} h_j\left(\hat{\mathbf{x}},\hat{\mathbf{v}}_{eq}^*\right)\right)^{-1}$

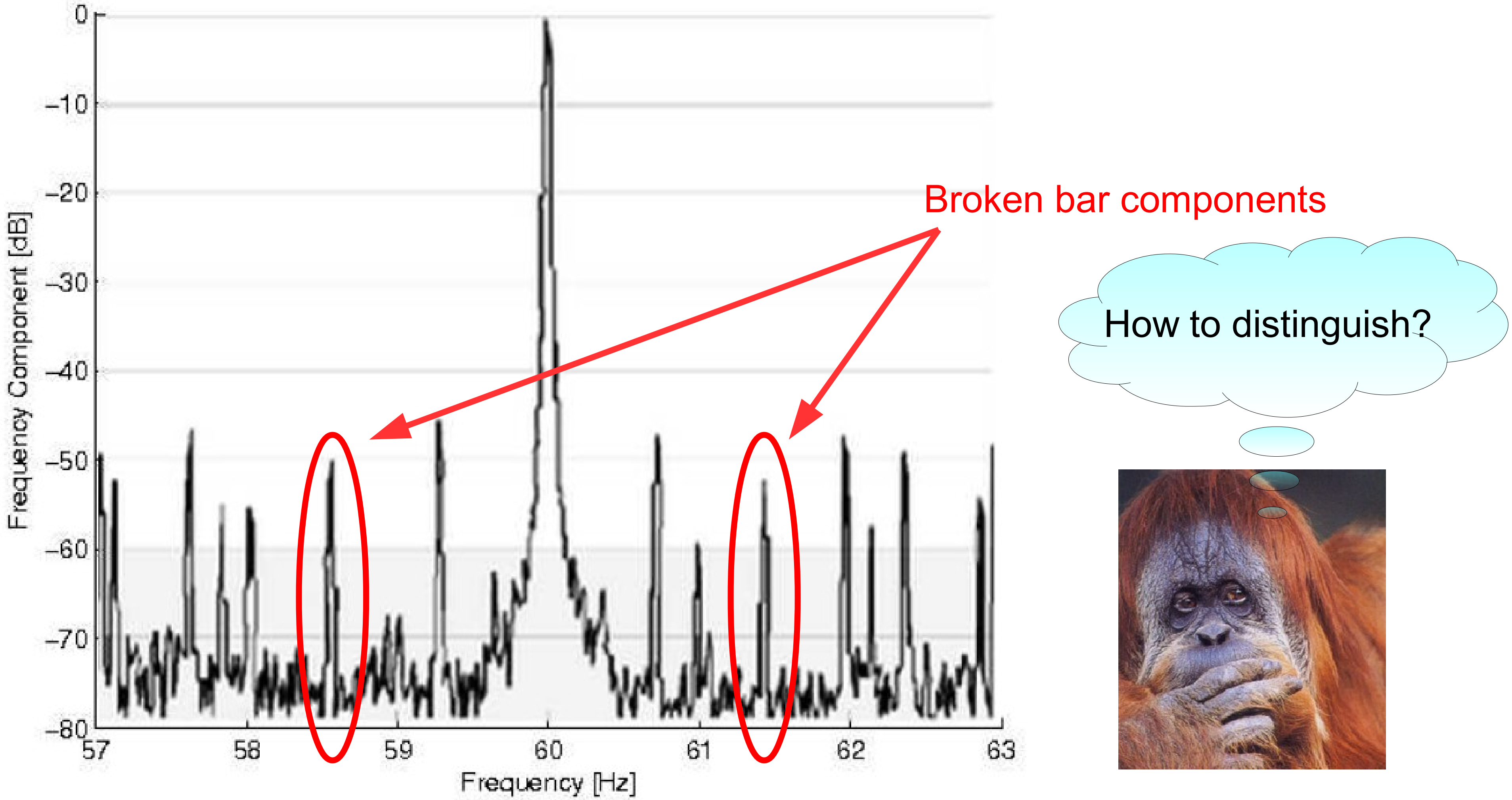
Diagnosis Evaluate and recognise the pattern of some analytic functions

Pattern of the spectrum of the Fourier transform of the output injection or of the identified fault/disturbance

$$V_k(j\omega) = \mathcal{F}\left\{\hat{v}_{k_{eq}}^*\left(\hat{\mathbf{x}},\mathbf{d},t\right)\right\} \qquad D_k(j\omega) = \mathcal{F}\left\{d_k(t)\right\}$$

L4 – Model-Based FDI via VSS

Stator currents power spectrum of an IM with broken bars and transmission gears



L4 – Model-Based FDI via VSS

The dynamics of the Induction Motor in the stator frame can be represented by the model

$$\begin{cases} \dot{x}_1 = a_1\left(x_3x_4 - x_2x_5\right) - a_2x_1 + a_3T_L \\ \dot{x}_2 = b_1x_4 - b_2x_2 + b_3x_1x_3 + b_4u_{s_\alpha} \\ \dot{x}_3 = b_1x_5 - b_2x_3 - b_3x_1x_2 + b_4u_{s_\beta} \\ \dot{x}_4 = c_1x_2 - c_2x_4 - n_px_1x_5 \\ \dot{x}_5 = c_1x_3 - c_2x_5 + n_px_1x_4 \end{cases}$$

$$a_1 = \frac{n_pL_m}{JL_r} \qquad a_2 = \frac{f_v}{J} \qquad a_3 = -\frac{1}{J}$$

$$b_1 = \frac{L_mR_r}{\sigma L_sL_r^2} \qquad b_2 = \frac{L_m^2R_r + L_r^2R_s}{\sigma L_sL_r^2} \qquad \sigma = 1 - \frac{L_m^2}{L_rL_s}$$

$$b_3 = \frac{n_pL_m}{\sigma L_sL_r} \qquad b_4 = \frac{1}{\sigma L_s}$$

$$c_1 = \frac{R_r}{L_r}L_m \qquad c_2 = \frac{R_r}{L_r}$$

State Variables:	
Shaft speed	x_1
$\alpha\ \beta$ Stator Currents	$x_{2,3}$
$\alpha\ \beta$ Rotor Fluxes	$x_{4,5}$
Input variables:	
Load Torque	T_L
$\alpha\ \beta$ Stator Voltage Supply	$u_{\alpha,\beta}$
Model Parameters:	
Pole pairs number	n_p
Viscous friction coefficient	f_v
Rotor inertia	J
Rotor and stator resistance	$R_{r,s}$
Rotor, stator and mutual inductance	$L_{r,s,m}$

L4 – Model-Based FDI via VSS

We assume that both the fault in the IM and the load torque are unknown

$$\begin{cases} \dot{x}_1 = a_1\left(x_3x_4 - x_2x_5\right) - a_2x_1 + a_3T_L \\ \dot{x}_2 = b_1x_4 - b_2x_2 + b_3x_1x_3 + b_4\left(u_{s_\alpha} + f_{BB_\alpha}\right) \\ \dot{x}_3 = b_1x_5 - b_2x_3 - b_3x_1x_2 + b_4\left(u_{s_\beta} + f_{BB_\beta}\right) \\ \dot{x}_4 = c_1x_2 - c_2x_4 - n_px_1x_5 \\ \dot{x}_5 = c_1x_3 - c_2x_5 + n_px_1x_4 \end{cases}$$

We supply the IM with a known voltage three phase line, and we measure the shaft angular velocity and the three phase currents.

We want to estimate the unknown inputs, assuming to have ***bounded total time derivative***, on the basis of the IM model and of the available inputs and outputs

L4 – Model-Based FDI via VSS

Consider a replica of the nominal plant plus some additional voltage and torque inputs to be defined

$$
\begin{cases}
\dot{\hat{x}}_1 = a_1\left(\underline{x_3}\hat{x}_4 - \underline{x_2}\hat{x}_5\right) - a_2\underline{x_1} + a_3 v_1 \\
\dot{\hat{x}}_2 = b_1\hat{x}_4 - b_2\underline{x_2} + b_3\underline{x_1}\hat{x}_5 + b_4(u_{s\alpha} + v_2) \\
\dot{\hat{x}}_3 = b_1\hat{x}_5 - b_2\underline{x_3} - b_3\underline{x_1}\hat{x}_4 + b_4(u_{s\beta} + v_3) \\
\dot{\hat{x}}_4 = c_1\underline{x_2} - c_2\hat{x}_4 - n_p\underline{x_1}\hat{x}_5 \\
\dot{\hat{x}}_5 = c_1\underline{x_3} - c_2\hat{x}_5 + n_p\underline{x_1}\hat{x}_4
\end{cases}
$$

Defining the state error $e_i(t) = \hat{x}_i(t) - x_i(t)$ $\quad i = 1,\ldots,5$;
the estimation error dynamics derives

$$
\begin{aligned}
\dot{e}_1 &= a_1x_3e_4 - a_1x_2e_5 + a_3(v_1 - T_L) \\
\dot{e}_2 &= b_1e_4 + b_3x_1e_5 + b_4(v_2 - f_{BB_\alpha}) \\
\dot{e}_3 &= b_1e_5 - b_3x_1e_4 + b_4(v_3 - f_{BB_\beta}) \\
\dot{e}_4 &= -c_2e_4 - n_px_1e_5 \\
\dot{e}_5 &= -c_2e_5 + n_px_1e_4
\end{aligned}
$$

L4 – Model-Based FDI via VSS

The last two rows of the error dynamics are decoupled prom the previous ones

$$
\begin{aligned}
\dot{e}_1 &= a_1 x_3 e_4 - a_1 x_2 e_5 + a_3 (v_1 - T_L) \\
\dot{e}_2 &= b_1 e_4 + b_3 x_1 e_5 + b_4 (v_2 - f_{BB_\alpha}) \\
\dot{e}_3 &= b_1 e_5 - b_3 x_1 e_4 + b_4 (v_3 - f_{BB_\beta}) \\
\dot{e}_4 &= -c_2 e_4 - n_p x_1 e_5 \\
\dot{e}_5 &= -c_2 e_5 + n_p x_1 e_4
\end{aligned}
$$

and they are exponentially stable

$$V = \frac{1}{2}\left(e_4^2 + e_5^2\right) \qquad \dot{V} = e_4 \dot{e}_4 + e_5 \dot{e}_5 = - \ c_2 \ V$$

L4 – Model-Based FDI via VSS

The first three rows of the error dynamics are a first order dynamics that can be stabilised in a finite time

$$
\begin{aligned}
\dot{e}_1 &= a_1 x_3 e_4 - a_1 x_2 e_5 + a_3 (v_1 - T_L) \\
\dot{e}_2 &= b_1 e_4 + b_3 x_1 e_5 + b_4 (v_2 - f_{BB_\alpha}) \\
\dot{e}_3 &= b_1 e_5 - b_3 x_1 e_4 + b_4 (v_3 - f_{BB_\beta}) \\
\dot{e}_4 &= -c_2 e_4 - n_p x_1 e_5 \\
\dot{e}_5 &= -c_2 e_5 + n_p x_1 e_4
\end{aligned}
$$

$$\dot{z} = K(\phi(t) + u(t) - f(t))$$

$$\left| \frac{d}{dt} f(t) \right| \le F_D$$

$\phi(t)$ is a vanishing term

K is a constant term

$$u(t) = u_1(t) + u_2(t)$$

$$
\begin{cases}
u_1(t) = -k \sqrt{|z(t)|}\ \text{sign}(z(t)) \\
\dot{u}_2(t) = -w\ \text{sign}(z(t)), \quad u_2(0) = 0
\end{cases}
$$

$$w > F_D \quad , \quad k^2 > 4F_D \cdot \frac{w + F_D}{w - F_D}$$

L4 – Model-Based FDI via VSS

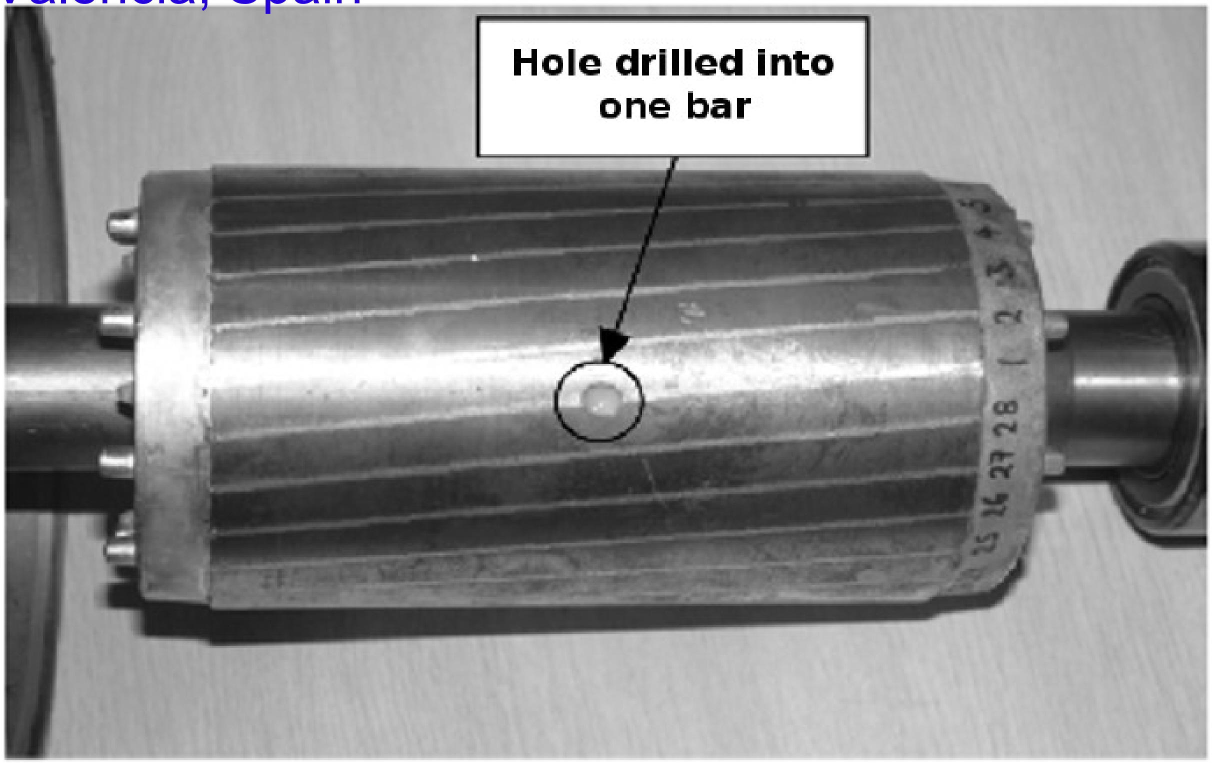
The system replica with the three output injections defines an unknown input observer having the following characteristics

- The output estimation errors tend to zero in a finite time
- The flux estimate tend to the actual flux exponentially
- The system is **strongly detectable**
- The output injections tends to the unknown inputs exponentially

We designed an unknown input observer whose output injections can recover the unknown input as well.

L4 – Model-Based FDI via VSS

The test was performed by means of MATLAB-Simulink implementations using the experimental data on a test run given by the Electrical Machines and Drives research group at the Department of Electrical Engineering, UPV, Valencia, Spain

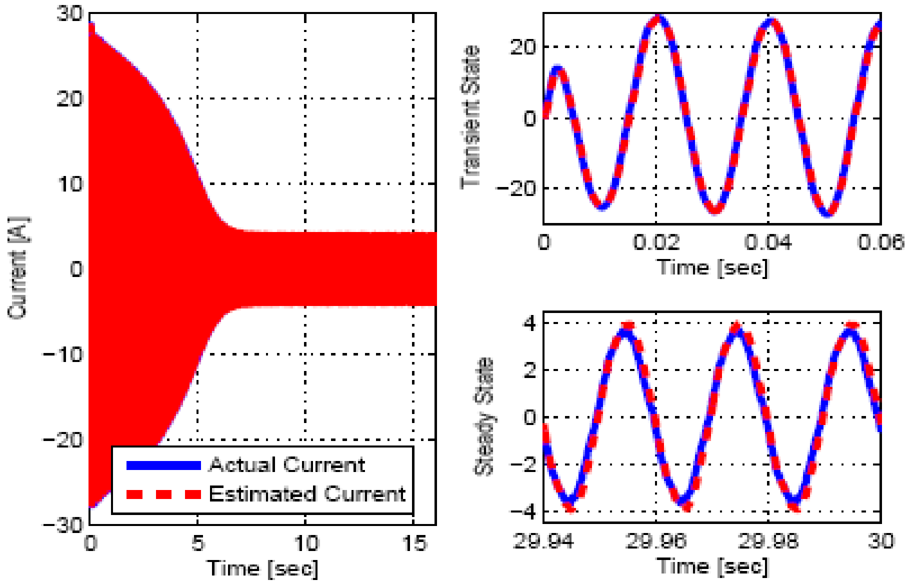


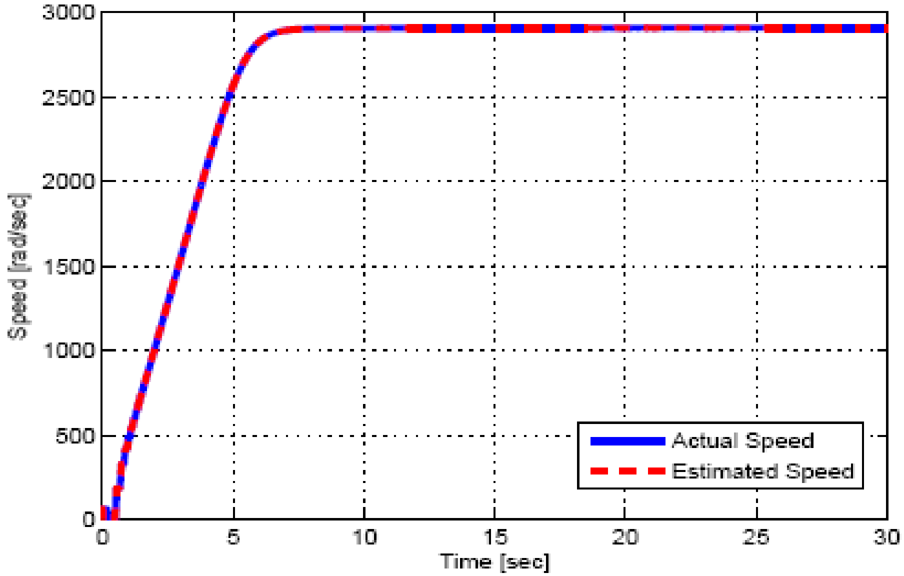
The electrical motors has been properly modified to reproduce the fault

L4 – Model-Based FDI via VSS

α stator current

Motor speed

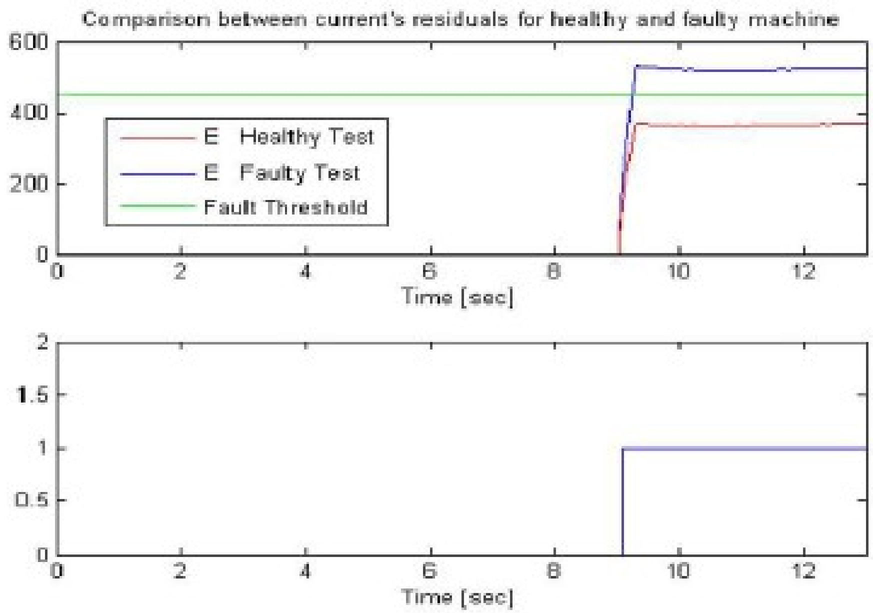




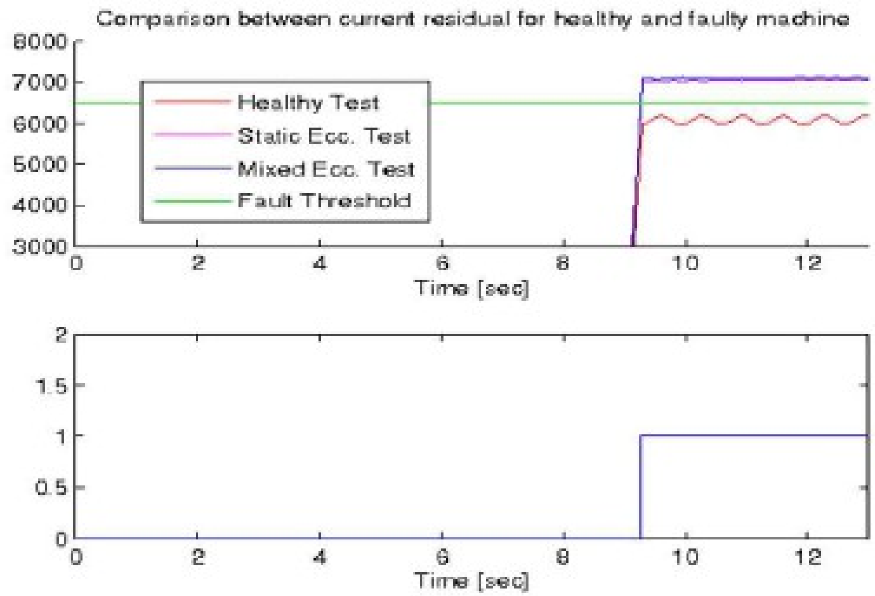
Observer convergence

L4 – Model-Based FDI via VSS

Broken bar test



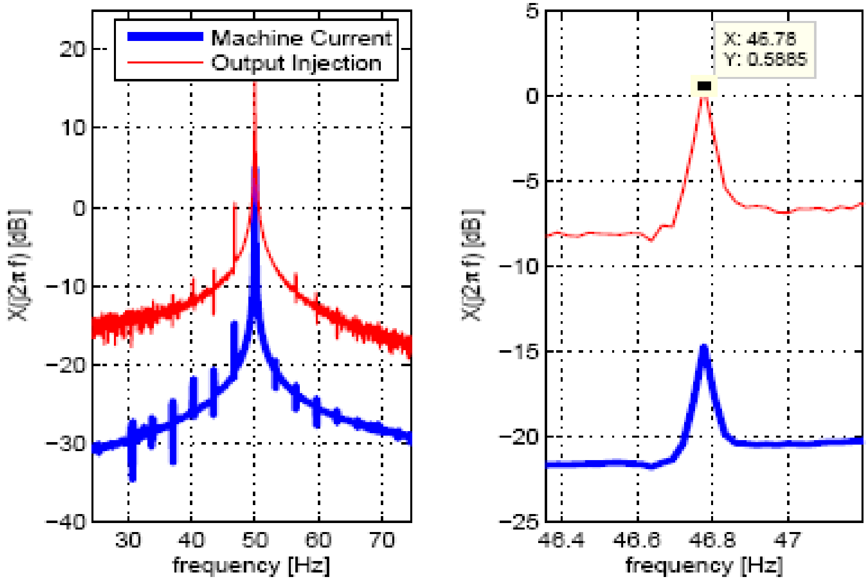
Eccentricity test



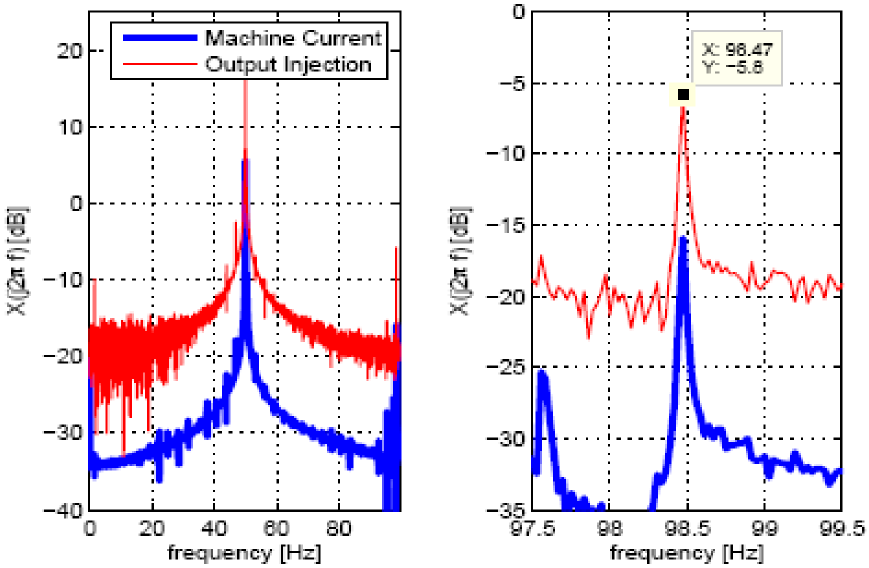
Residual evaluation

L4 – Model-Based FDI via VSS

Broken bar test



Eccentricity test



L4 – References

- Bartolini G., Ferrara A., Usai E., "Real-Time Output Derivatives Estimation by Means of Higher Order Sliding Modes", Proceedings of the IMACS Multiconference (CESA'98)– Symposium on Modeling, Analysis and Control, vol.1, pp. 782-787, Nebeul-Hammamet, Tunisia, April 1-4 1998.
- Cannas B., Cincotti S., Usai E., “Chaos Synchronisation via Sliding Modes”, in Computing Anticipatory System: CASYS 2000-Fourth Int. Conf. (D.M. Dubois ed.), AIP Conference Proceedings, 573,229-241, 2001.
- Cannas B., Cincotti S., Usai E., “An algebraic observability approach to chaos synchronisation by sliding differentiators”, IEEE Trans. Circuit & Systems-I: Fundamental Theory and Applications, 49, 1000-1006, 2002.
- Cannas B., Cincotti S., Usai E., “A chaotic modulation scheme based on algebraic observability and sliding mode differentiators”, Chaos, Solitons and Fractals, 26, 363-377, 2005.
- Davila J, Fridman L, Pisano A., Usai E., “Finite-time state observation for nonlinear uncertain systems via higher order sliding modes”, Int. J. of Control, 82, 1564–1574, 2009.

L4 – References

- Davila J., Fridman L., Levant A., "Second-order SM observer for mechanical systems", IEEE Trans. Automatic Control, 50, pp. 1785–1789, 2005.
- Edwards C., Spurgeon S.K., Patton R.J., "Sliding mode observers for fault detection and isolation", *Automatica*, 39, pp. 541–553, 2000.
- Floquet T., Edwards C., Spurgeon S. K., On sliding mode observers for systems with unknown inputs, Int. J. Adaptive Control and Signal Processing, 21, 638–656, 2007.
- Fridman, L., Shtessel, Y., Edwards, C., Yan, X.,'Higher-order Sliding-mode Observer for State Estimation and Input Reconstruction in Nonlinear Systems', Int. J. Robust Nonlinear Control, 18, 399–413, 2008.
- Levant A., "Robust exact differentiation via sliding mode technique", Automatica, 34, pp. 379–384, 1998.
- Levant A., "Higher-order sliding modes, differentiation and output-feedback control", *Int. J. Control*, 76, pp. 924–941, 2003.
- Levant A., "Quasi–continuous high–order sliding mode controllers", *IEEE Trans. Automatic Control*, 50, pp. 1812-1816, 2005.

L4 – References

- Orani, N.; Pisano, A.; Usai, E., “On a new sliding-mode differentiation scheme”; Proceedings of the IEEE International Conference on Industrial Technology (ICIT 2006), pp. 2652 – 2657, Mumbay, India, December 2006.
- Pilloni A., Pisano A., Usai E., Puche-Panadero R., “Detection of rotor broken bar and eccentricity faults in induction motors via second order sliding mode observer”, Proceedings of the 51th IEEE Conference on Decision and Control (CDC 2012), pp. 7614 - 7619 , Maui, Hawai (USA), 10-13 Dec. 2012.
- Pilloni A., Pisano A., Usai E., “Observer Based Air Excess Ratio Control of a PEMFuel Cell System via High Order Sliding Mode”, IEEE Trans. on Industrial Electronics, 62, 5236-5246, 2015.

  Reichhartinger M., Spurgeon S.K., A Robust Exact Dierentiator Block for MATLAB®-Simulink®, DOI: 10.13140/RG.2.1.3243.4803, Technical University of Graz, 2016.
- Spurgeon S.K., “Sliding Mode Observers - A Survey”, Int. J. Syst. Sci., 39, pp. 751–768, 2008.
- Shtessel Y.B., Baev S., Edwards C., Spurgeon S., “HOSM observer for a class of non-minimum phase causal nonlinear MIMO systems,” IEEE Trans. Autom. Control, 55, 543–548, 2010.