Institut für Regelungs- und Automatisierungstechnik
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Sliding Mode Control: Basic Theory, Advances and Applications

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Lecture 3

Implementation Issues
of
Sliding Mode Control Systems

- Approximate Sliding Modes
- Discrete Time implementation
- Effect of Parasitic Dynamics
- Effect of Measurement Noise
- Chattering Attenuation
  - Control magnitude adaptation
  - Parameter tuning in 2-SMC
  - System dynamics shaping

L3 – Approximate SM

Ideal Sliding Modes in Control Systems can be established if infinite frequency switching in the closed loop dynamics appears

Real devices has low-pass characteristics and therefore cannot perform infinite frequency switching

Switching delay appears

The system state is no more constrained on the sliding surface

WHAT is the EFFECT of Switching Delays?

L3 – Approximate SM

Assume that the sliding surface is an attractive set of the closed-loop dynamics

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At certain time instant t_0 the system state is within a vicinity of the sliding surface and the system dynamics is represented by the input-output and internal dynamics

$$\begin{cases} \dot{\mathbf{y}}(t) = \varphi(\mathbf{y}(t), \mathbf{w}(t), \mathbf{u}(t), t) \\ \dot{\mathbf{w}}(t) = \psi(\mathbf{y}(t), \mathbf{w}(t), t) \end{cases} \qquad \mathbf{y} = \sigma(\mathbf{x}) \in \mathrm{R}^q, \quad \mathbf{w} \in \mathrm{R}^{n-q}, \quad \mathbf{u} \in \mathrm{R}^q$$

$$\frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}} \cdot f(\mathbf{x}(t), \mathbf{u}(t), t) = \varphi(\mathbf{y}(t), \mathbf{w}(t), \mathbf{u}(t), t)$$

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{w} \end{bmatrix} = \Phi(\mathbf{x}) \qquad \mathbf{x} \in \mathrm{R}^n, \quad \mathbf{y} \in \mathrm{R}^q, \quad \mathbf{w} \in \mathrm{R}^{n-q}$$

L3 – Approximate SM

General treatment of the analysis of the system behaviour nearby the sliding surface is quite complex and could need a Poicaré analysis

Complete results can be quite easily obtained in the linear case for the classic first order sliding mode control systems

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\cdot\mathbf{x}(t) + \mathbf{B}(t)\cdot\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\cdot\mathbf{x}(t)$$

$$\mathbf{x}\in\mathrm{R}^{n},\quad \mathbf{u}\in\mathrm{R}^{q},\quad \mathbf{y}\in\mathrm{R}^{q},\quad q<n$$

$$\begin{bmatrix}\mathbf{y}\\ \mathbf{w}\end{bmatrix} = \begin{bmatrix}\mathbf{C}_{11} & \mathbf{C}_{12}\\ \mathbf{0} & \mathbf{C}_{22}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix} = \hat{\mathbf{C}}\cdot\mathbf{x} \qquad \mathbf{x}_1\in\mathrm{R}^{q}\quad \mathbf{x}_2\in\mathrm{R}^{n-q},\quad \mathbf{y}\in\mathrm{R}^{q},\quad \mathbf{w}\in\mathrm{R}^{n-q}$$

$$\Downarrow$$

$$\begin{cases}\dot{\mathbf{y}}(t) = \mathbf{\Phi}_1(t)\cdot\mathbf{y}(t) + \mathbf{\Phi}_2(t)\cdot\mathbf{w}(t) + \mathbf{C}\cdot\mathbf{B}(t)\cdot\mathbf{u}(t)\\ \dot{\mathbf{w}}(t) = \mathbf{\Psi}_1(t)\cdot\mathbf{y}(t) + \mathbf{\Psi}_2(t)\cdot\mathbf{w}(t)\end{cases}$$

L3 – Approximate SM

Assuming that the control is designed taking into account for the nominal dynamics

$$\mathbf{u}(t) = -[\mathbf{C}\cdot\mathbf{B}(t)]^{-1}\left(\overline{\mathbf{\Phi}}_1(t)\cdot\mathbf{y}(t) + \overline{\mathbf{\Phi}}_2(t)\cdot\mathbf{w}(t) + U\,\mathrm{sgn}(\mathbf{y}(t))\right)$$

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$$\begin{cases} \dot{\mathbf{y}}(t) = \tilde{\mathbf{\Phi}}_1(t)\cdot\mathbf{y}(t) + \tilde{\mathbf{\Phi}}_2(t)\cdot\mathbf{w}(t) - U\,\mathrm{sgn}(\mathbf{y}(t)) \\ \dot{\mathbf{w}}(t) = \mathbf{\Psi}_1(t)\cdot\mathbf{y}(t) + \mathbf{\Psi}_2(t)\cdot\mathbf{w}(t) \end{cases}$$

If the system were in ideal sliding mode the system dynamics will be characterized by its zero dynamics

$$\begin{cases} \dot{\overline{\mathbf{y}}}(t) = \tilde{\mathbf{\Phi}}_2(t)\cdot\overline{\mathbf{w}}(t) - U\,\mathrm{sgn}(\overline{\mathbf{y}}(t)) \\ \dot{\overline{\mathbf{w}}}(t) = \mathbf{\Psi}_2(t)\cdot\overline{\mathbf{w}}(t) \end{cases} \qquad \frac{U}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon}\mathrm{sgn}(\overline{\mathbf{y}}(\tau))\,d\tau \xrightarrow[\varepsilon\to 0]{} \tilde{\mathbf{\Phi}}_2(t)\cdot\overline{\mathbf{w}}(t)$$

L3 – Approximate SM

The dynamics of the error between the ideal and real sliding behaviour

$$\begin{cases} \dot{\tilde{\mathbf{y}}}(t) = \tilde{\mathbf{\Phi}}_1(t)\cdot\tilde{\mathbf{y}}(t) + \tilde{\mathbf{\Phi}}_2(t)\cdot\mathbf{w}(t) - U\,\mathrm{sgn}(\mathbf{y}(t)) \\ \dot{\tilde{\mathbf{w}}}(t) = \mathbf{\Psi}_1(t)\cdot\tilde{\mathbf{y}}(t) + \mathbf{\Psi}_2(t)\cdot\tilde{\mathbf{w}}(t) \end{cases} \qquad \begin{array}{l} \tilde{\mathbf{y}}(t) = \mathbf{y}(t) \\ \tilde{\mathbf{w}}(t) = \mathbf{w}(t) - \overline{\mathbf{w}}(t) \end{array}$$

Assume that T is the switching delay and that the sliding dynamics can be upper bounded by a constant D

$$\left\|\tilde{\mathbf{y}}(t)\right\| \le \left\|\tilde{\mathbf{y}}_0\right\| + DT$$

Assume also that the matrices in the error dynamics can be upper bound by proper constants during the switching delay

$$\left\|\mathbf{\Psi}_1(t)\right\| < Q \qquad \left\|\mathbf{\Psi}_2(t)\right\| < P$$

L3 – Approximate SM

$$\begin{cases} \left\|\tilde{\mathbf{w}}(t)\right\| \leq \left\|\tilde{\mathbf{w}}_0\right\| + Q\left\|\mathbf{y}_0\right\|T + \frac{1}{2}QDT^2 + P\int_{t_0}^{t_0+T}\left\|\tilde{\mathbf{w}}(\tau)\right\|d\tau \\ \left\|\tilde{\mathbf{y}}(t)\right\| \leq \left\|\tilde{\mathbf{y}}_0\right\| + DT \end{cases}$$

Taking into account the ISS assumption for the internal dynamics and that T is the switching delay

$$\left\|\tilde{\mathbf{x}}(t)\right\| \leq \Delta \qquad \Delta = \upsilon T \qquad \forall t \in [t_0, t_0 + T]$$

The system trajectory remains confined within a $O(T)$ vicinity of the ideal sliding trajectory

L3 – Approximate SM

Theorem
Consider system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u \qquad \sigma : \mathrm{R}^n \to \mathrm{R}$$

Assume that conditions for convergence and stability of a r-SM are fulfilled by a homogeneous r–order sliding mode controller

$$\begin{cases} \left\| L_{\mathbf{f}}^{r}\sigma \right\| \le \Phi \\ L_{\mathbf{g}}L_{\mathbf{f}}^{k}\sigma = 0, \quad k = 0,1,\ldots,r-2 \\ 0 < \Gamma_m \le L_{\mathbf{g}}L_{\mathbf{f}}^{r-1}\sigma \le \Gamma_M \end{cases}$$

$$u = -\alpha\,\mathrm{sgn}\left(\varPhi_{r-1,r}\left(\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)}\right)\right)$$

Then, if the switching device has a switching delay T, the real r-SM has the following finite time accuracy

$$\left|\sigma^{(k)}\right| \le \upsilon_k T^{r-k} \qquad k = 0,1,\ldots,r-1$$

L3 – Approximate SM

Proof

Under the considered assumption the system trajectories are infinitely extendible in time for any Lebesgue-measurable bounded feedback control and at the ideal switching time the sliding variable and its time derivatives are bounded

$$\left|\sigma^{(k)}\right| \le \Sigma_k \qquad k = 0,1,\ldots,r$$

Applying the Lagrange theorem

$$\left|\frac{d^p\sigma^{(k)}}{dt^p}\right|_{t^*\in[t,t+T]} \le K_k \left|\sup_{t\in[t,t+T]} \sigma^{(k)}\right| T^{-p} \qquad \forall p = 0,1,\ldots,r-1-k$$

Integrating $\sigma^{(k)}$ k times and taking into account above inequalities

$$\left|\sigma^{(k)}\right| \le \upsilon_k\left(\Sigma_{k+1}\right)T^{r-k} \qquad k = 0,1,\ldots,r-1 \qquad \upsilon_k \in \mathcal{K}_\infty$$

L3 – Approximate SM

Real actuation devices cannot implement infinite switching and therefore the system trajectory cannot be constrained on the sliding surface

The *real* sliding is a motion confined into a vicinity of the sliding surface

The thickness of the *real* sliding vicinity depends on the the switching delay T and on the control magnitude (Σ_r)

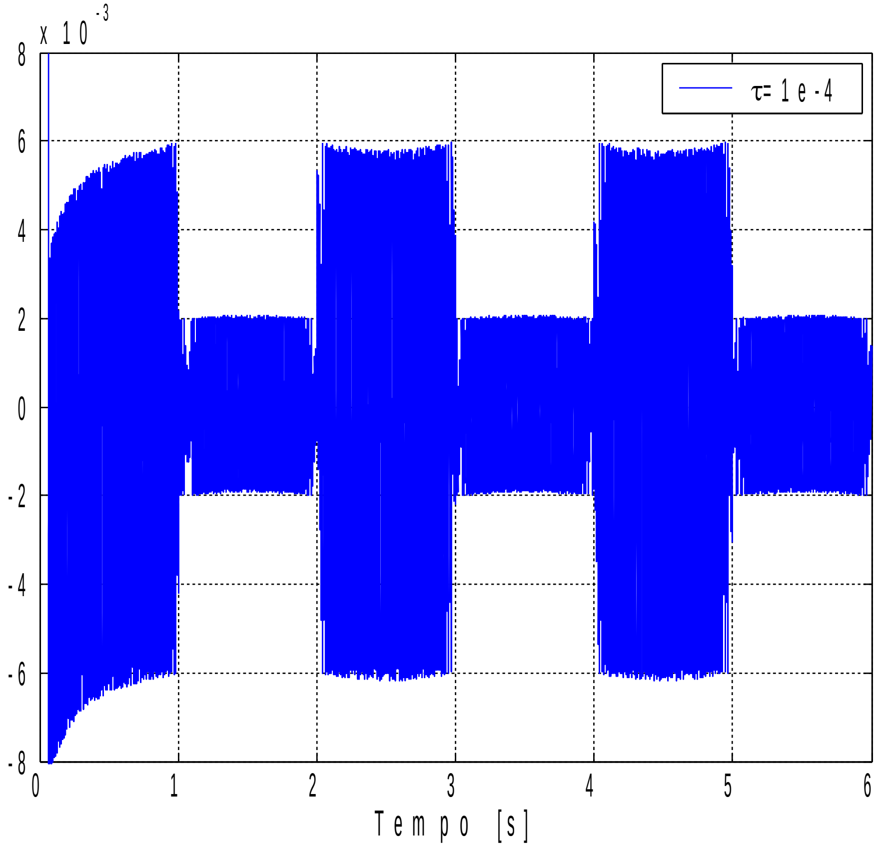
The *real* sliding accuracy can be improved by means of HOSM, if the switching delay is $T<1$

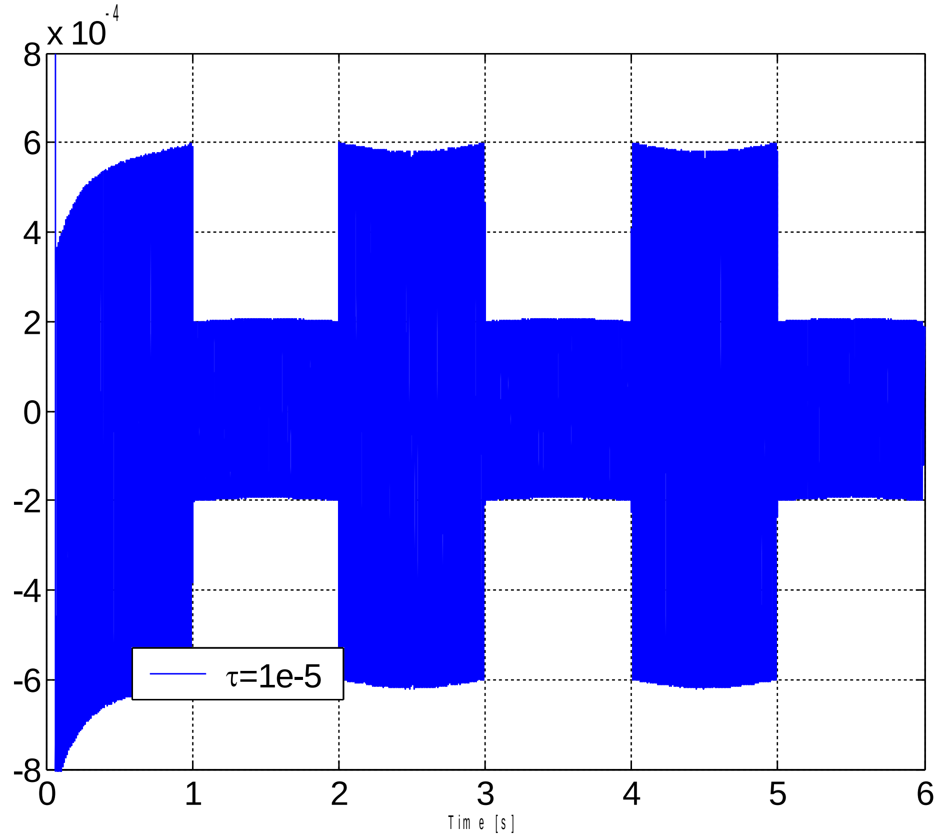
The accuracy can be also marginally improved by avoiding unnecessary large magnitude controls

L3 – Approximate SM

Example

$$m(t)\ddot{y} + (b_1 + b_2|\dot{y}|)y + (k_1 + k_3 y^2)y = u - \sin(\pi t)$$
$$\sigma = \dot{y} + cy$$





L3 – Discrete Time Implementation

The most common cause of delay switching is the digital implementation of the controller

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Discrete-time sliding mode control

$$
\begin{aligned}
\bar{\mathbf{x}}[k+1] &= \mathbf{A}_d \bar{\mathbf{x}}[k] + \mathbf{B}_d \boldsymbol{\sigma}[k] \\
\boldsymbol{\sigma}[k+1] &= \boldsymbol{\Phi}_d \left(\bar{\mathbf{x}}[k], \boldsymbol{\sigma}[k], k\right) + \boldsymbol{\Gamma}_d \mathbf{u}[k] \\
\mathbf{u}[k] &= -\alpha \operatorname{sgn}(\boldsymbol{\sigma}[k])
\end{aligned}
$$

Discrete-time sliding mode control can appear also in systems with continuous right-hand-side (e.g., deadbeat control)

Discrete time sliding mode control is sensible only in the presence of uncertainties or disturbances

L3 – Discrete Time Implementation

What is a discrete time sliding mode?

$$\boldsymbol{\sigma}[k+1] = \mathbf{0} \qquad k = 0,1,2,\ldots$$

?

$$\boldsymbol{\sigma}[k+1] - \boldsymbol{\sigma}[k] = \mathbf{0} \qquad k = 0,1,2,\ldots$$

The second is not convincing and does not imply the first

Effective approach is constituted by continuous time design and subsequent discretization analysis

The system behaviour within a sampling period is almost unpredictable, apart from the maximum deviation from the sliding surface

In some conditions chaotic behavior within the boundary layer has been recognized

L3 – Discrete Time Implementation

The usual implementation of the control law has two parts
 * the nominal part
 * the discontinuous part to cope with uncertainties

This allows for implementing learning and adaptive methods that can improve the accuracy by one order, i.e., $O(T) \rightarrow O(T^2)$

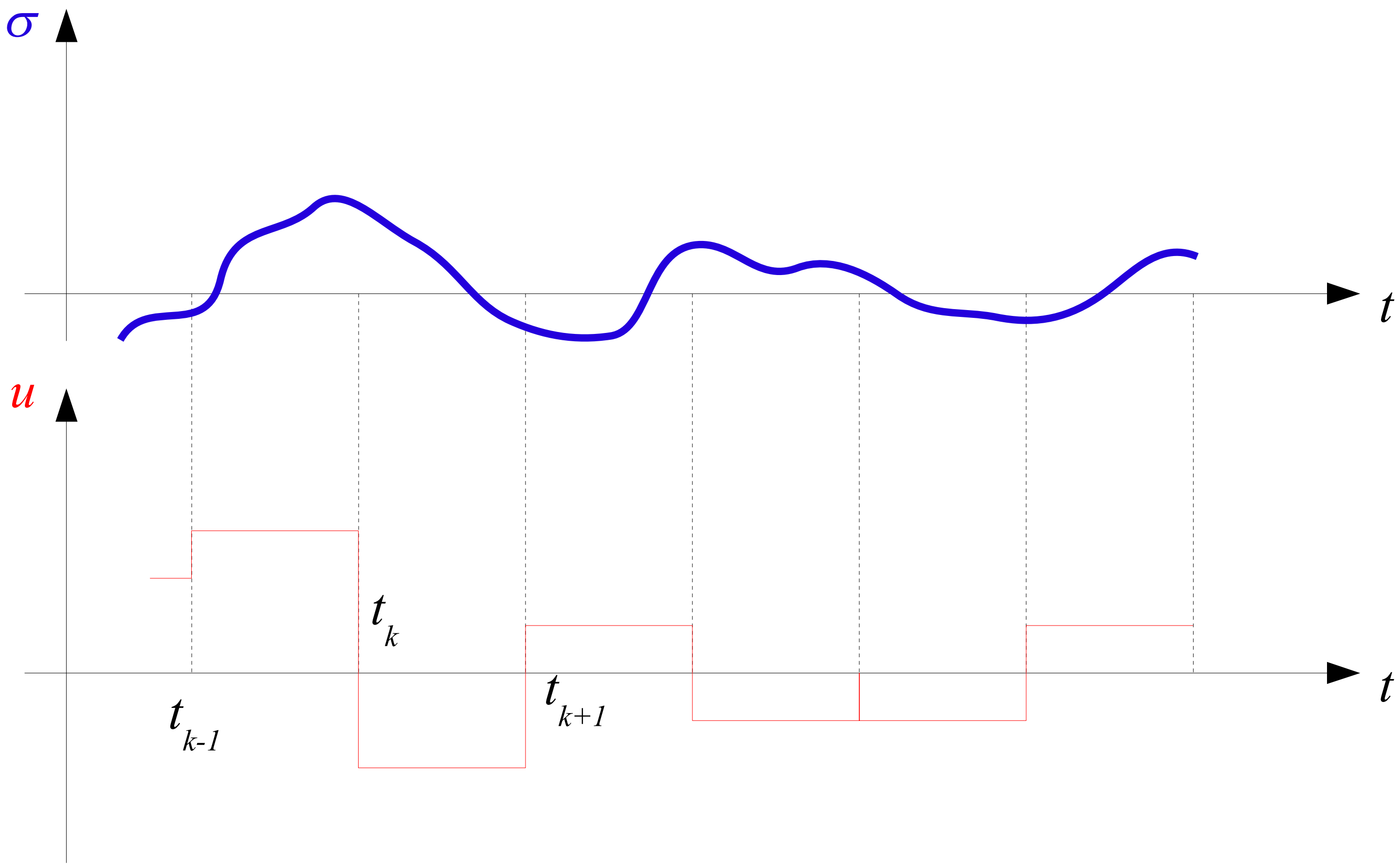
$$\begin{cases} \dot{x}_i = x_{i+1} & i = 1,2,\dots n-1 \\ \dot{x}_n = f(\mathbf{x},t) + b(\mathbf{x},t)u \end{cases} \qquad |f(\mathbf{x},t)| \le F(\|\mathbf{x}\|), \quad 0 < b_m(\|\mathbf{x}\|) \le b(\mathbf{x},t) \le b_M(\|\mathbf{x}\|)$$

$$\sigma = x_n + \sum_{i=1}^{n-1} c_i x_i$$

$$\dot{\sigma} = f(\mathbf{x},t) + b(\mathbf{x},t)u + \sum_{i=1}^{n-1} c_i x_{i+1}$$

$$u(t) = -\frac{F(\|\mathbf{x}(t_k)\| + \kappa T) + \left|\sum_{i=1}^{n-1} c_i x_{i+1}(t_k)\right| + \|\mathbf{c}\|\kappa T + \eta^2}{b_m(\|\mathbf{x}(t_k)\| + \kappa T)} \operatorname{sgn}(\sigma(t_k)) \qquad t \in (t_k, t_k + T]$$

L3 – Discrete Time Implementation



The switching delay due to sampling causes an approximate sliding motion in a $O\ (T)$ boundary layer of the ideal sliding

L3 – Discrete Time Implementation

In the ideal case the equivalent control can be estimated by a low-pass filter

$$\tau \dot{u}_{av}(t) + u_{av}(t) = u(t)$$

Since the sliding variable in constrained in a $O(T)$ boundary layer of the ideal sliding and the equivalent control remains bounded

$$\left| u_{av}(t) - u_{eq}(t) \right| \leq \kappa_1 \tau + \kappa_2 T + \kappa_3 \frac{T}{\tau}$$

The estimation error can be minimized and the actual value of the average control computed exactly at each sampling time

$$\tau = \sqrt{\frac{\kappa_1}{\kappa_3}} \sqrt{T} \quad \Rightarrow \quad \left| u_{av}(t) - u_{eq}(t) \right| \leq \kappa_4 \sqrt{T}$$

$$u_{av}[k+1] = e^{-T/\tau} u_{av}[k] + \left(1 - e^{-T/\tau} \right) u[k]$$

L3 – Discrete Time Implementation

The control input can be implemented as a combination of two components

$$u[k] = u_{av}[k] - \kappa_5 \sqrt{T}\, \frac{F\left(\|\mathbf{x}(t_k)\| + \kappa T\right) + \left|\sum_{i=1}^{n-1} c_i x_{i+1}(t_k)\right| + \|\mathbf{c}\|\kappa T + \eta^2}{b_m\left(\|\mathbf{x}(t_k)\| + \kappa T\right)}\, \mathrm{sgn}(\sigma(t_k)) \qquad t \in (t_k, t_k + T]$$

$$\Downarrow$$

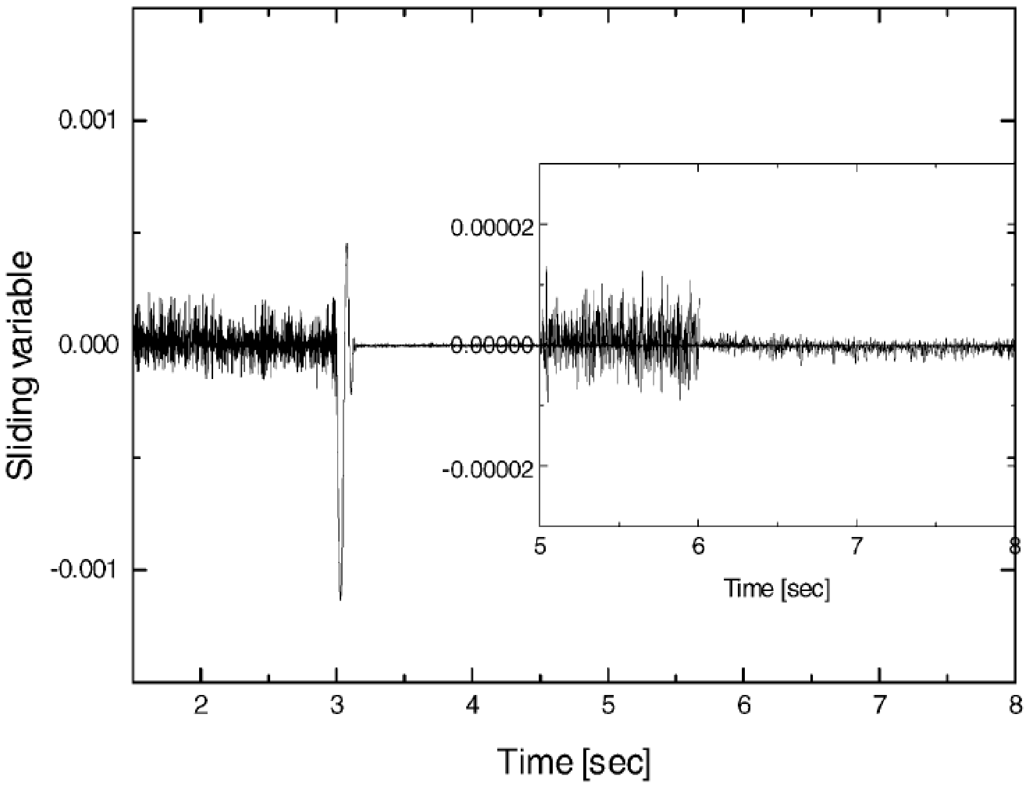
$$|\sigma[k]| \xrightarrow[k \to K_1]{} O\left(T^{3/2}\right)$$

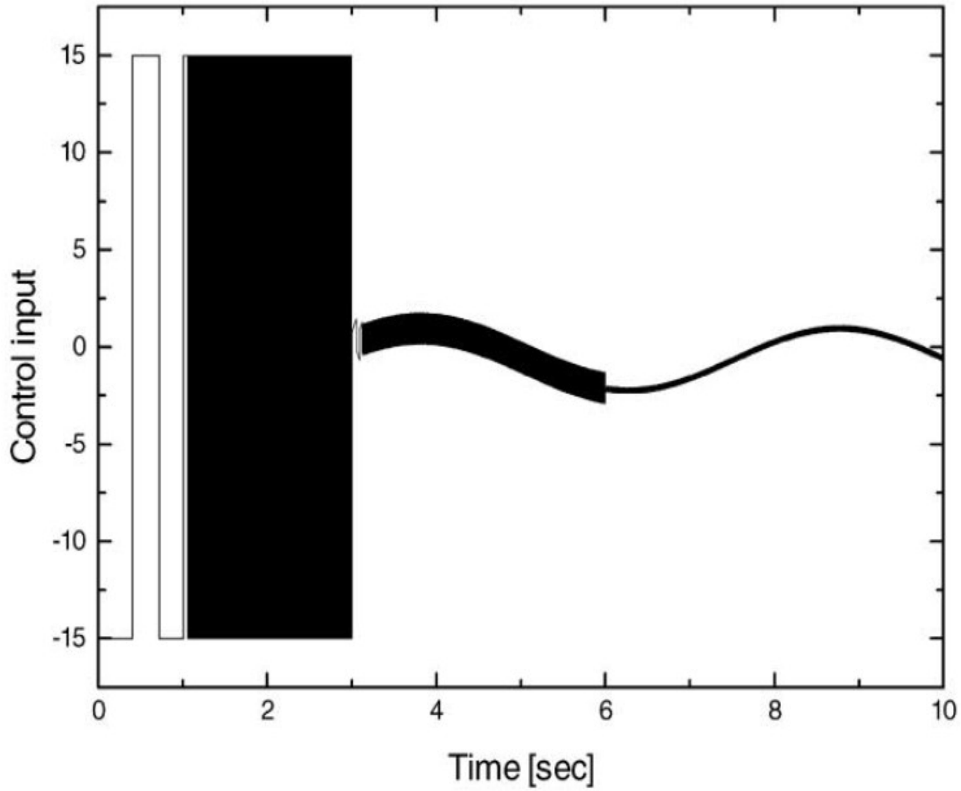
Adapting step by step the time constant of the filter and the magnitude of the discontinuous control

$$\begin{aligned} \tau_j &= \sqrt{\frac{\kappa_{1_j}}{\kappa_{3_j}} T^{1-2^{-j}}} \\ U_j &= \kappa_{5_j} T^{1-2^{-j}} \end{aligned} \quad \Rightarrow \quad |\sigma| \le O\left(T^{2-2^{-j}}\right) \quad \forall t > K_j T \quad j\ldots = 1,2,$$

L3 – Discrete Time Implementation

Example of discrete time sliding mode control with recursive estimation and adaptation of the control





L3 – Effect of the parasitic dynamics

If the switching control is applied to the plant by means of a dynamic actuator the relative degree between the sliding variable and the switching control increases and the ideal sliding cannot be achieved

$$
\begin{aligned}
&\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x}) z_1 && \mathbf{f}, \mathbf{g} : \mathrm{R}^n \to \mathrm{R}^n \\
&\mu \dot{\mathbf{z}} = \mathbf{h}(\mathbf{z}, u) && \mathbf{h} : \mathrm{R}^m \times \mathrm{R} \to \mathrm{R}^m \\
&s = \sigma(\mathbf{x}) && \sigma : \mathrm{R}^n \to \mathrm{R} \\
&u = switch(\sigma)
\end{aligned}
$$

$$
\begin{aligned}
&s^{(r)} = L_{\mathbf{f}}^{r} \sigma\left(s, \dot{s}, \ldots, s^{r-1}, \mathbf{w}\right) + L_{\mathbf{g}} L_{\mathbf{f}}^{r-1} \sigma\left(s, \dot{s}, \ldots, s^{r-1}, \mathbf{w}\right) \cdot z_1 && L_{\mathbf{f}}^{r} \sigma, L_{\mathbf{g}} L_{\mathbf{f}}^{r-1} : \mathrm{R}^r \times \mathrm{R}^{n-r} \to \mathrm{R} \\
&\dot{\mathbf{w}} = \boldsymbol{\psi}\left(s, \dot{s}, \ldots, s^{r-1}, \mathbf{w}\right) && \boldsymbol{\psi} : \mathrm{R}^r \times \mathrm{R}^{n-r} \to \mathrm{R}^{n-r} \\
&\mu \dot{\mathbf{z}} = \mathbf{h}(\mathbf{z}, u) && \mathbf{h} : \mathrm{R}^m \times \mathrm{R} \to \mathrm{R}^m \\
&s = \sigma\left(s, \dot{s}, \ldots, s^{r-1}, \mathbf{w}\right) && \sigma : \mathrm{R}^r \times \mathrm{R}^{n-r} \to \mathrm{R} \\
&u = switch(\sigma)
\end{aligned}
$$

L3 – Effect of the parasitic dynamics

If the parameter μ is sufficiently small the actuator dynamics is a singular perturbation of the nominal dynamics

a) Poincaré analysis of the fast dynamics, freezing the slow dynamics

b) Phase trajectory analysis considering differential inclusions with switching delays

$$s^{(r)} \in [-\Lambda_r, \Lambda_r] + [\Gamma_m, \Gamma_M] z_1$$

c) Homogeneity of the differential inclusion

Method a) is very much involved and hard to implement for nonlinear uncertain systems;

Method b) is relative simple only for relative degree 2 sliding dynamics

Method c) is general but require the homogeneity of the controller

L3 – Effect of the parasitic dynamics

All methods confirmed that the accuracy of the sliding mode depends on the singular parameter μ only, no matter the relative degree m of the parasitic dynamics is

$$\left|\sigma^{(k)}\right| = O\left(\mu^{r-k}\right) \qquad k = 0,1,\ldots,r-1$$

In general information about the system behavior within the boundary layer are not available apart for linear system

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Approximate method

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Describing Function

Exact methods

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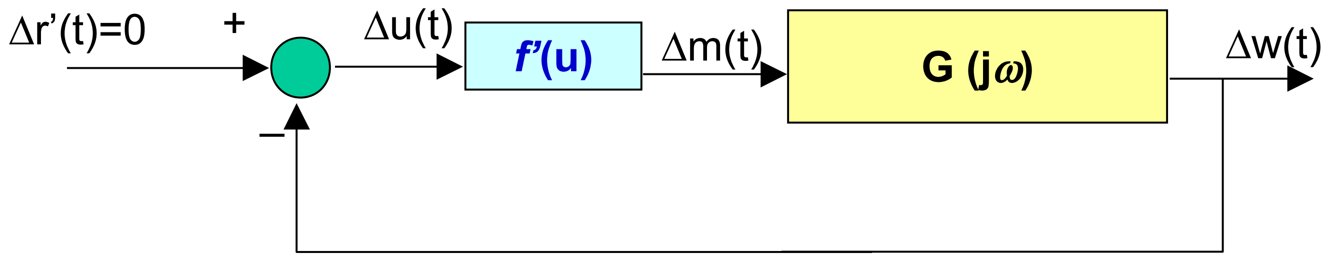
Tzipkin locus

LPRS

L3 – Effect of the parasitic dynamics

This methods refer to linear systems with nonlinear static feedback



Give only necessary conditions for the stability of limit cycles because they consider the steady state behavior only, and are based on the harmonic balance of the feedback loop

$$Ue^{j\bar{\omega}t} = -\sum_{k=1}^{\infty} G_k e^{j\varphi_k} M_k e^{j\vartheta_k} e^{jk\bar{\omega}t}$$

If the linear system has low-pass characteristics the Describing Function method can be applied

$$1 + G(j\omega)N(U,\omega) = 0 \qquad\qquad N(U,\omega) = \frac{1}{U}(b_1 + ja_1)$$

L3 – Effect of the parasitic dynamics

Example

$$G(j\omega) = \frac{k_t}{j\omega(j\omega L_a + R_a)(j\omega J + B) + k_t k_e}$$

R=0.4; % rotor resistance

L=0.001; % rotor inductance

ke=0.3; % voltage feedback constant

kt=0.3; % torque constant

Jm=0.01; % motor inertia

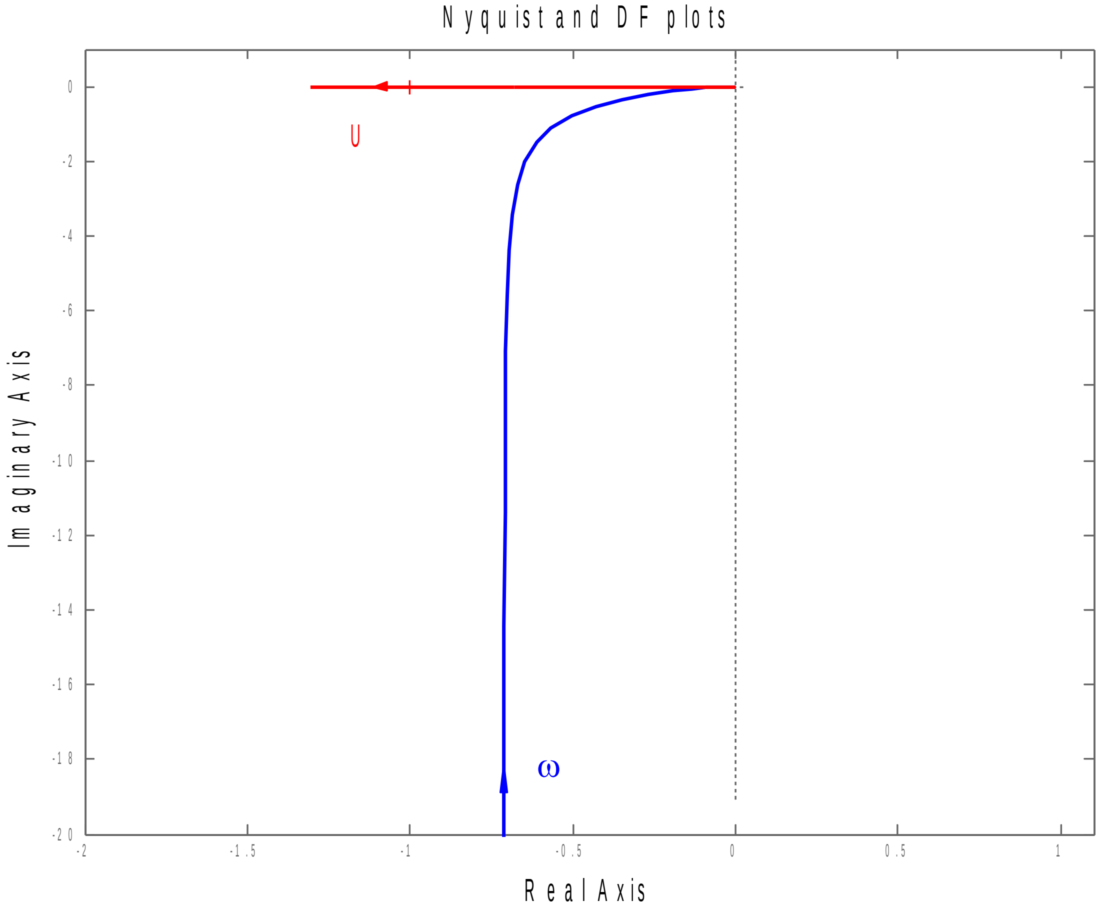
Jl=0.09 % load inertia

Bm=0.05; % motor friction coefficient

Bl=0.05; %load friction coefficient

J=Jm+Jl;

B=Bm+Bl;



$$\omega\big|_{\Im(W_p(j\omega))=0} = \omega_{cr} = \sqrt{\frac{R_a B + k_t k_e}{L_a J}} = 36.056 \text{ rad/s} \qquad \overline{U} = -\frac{4M}{\pi} \cdot \Re(W_p(j\omega_{cr})) = 0.1759 \text{ rad}$$

L3 – Effect of the parasitic dynamics

Example

$$G(j\omega)=\frac{k_t}{j\omega(j\omega L_a+R_a)(j\omega J+B)+k_t k_e}$$

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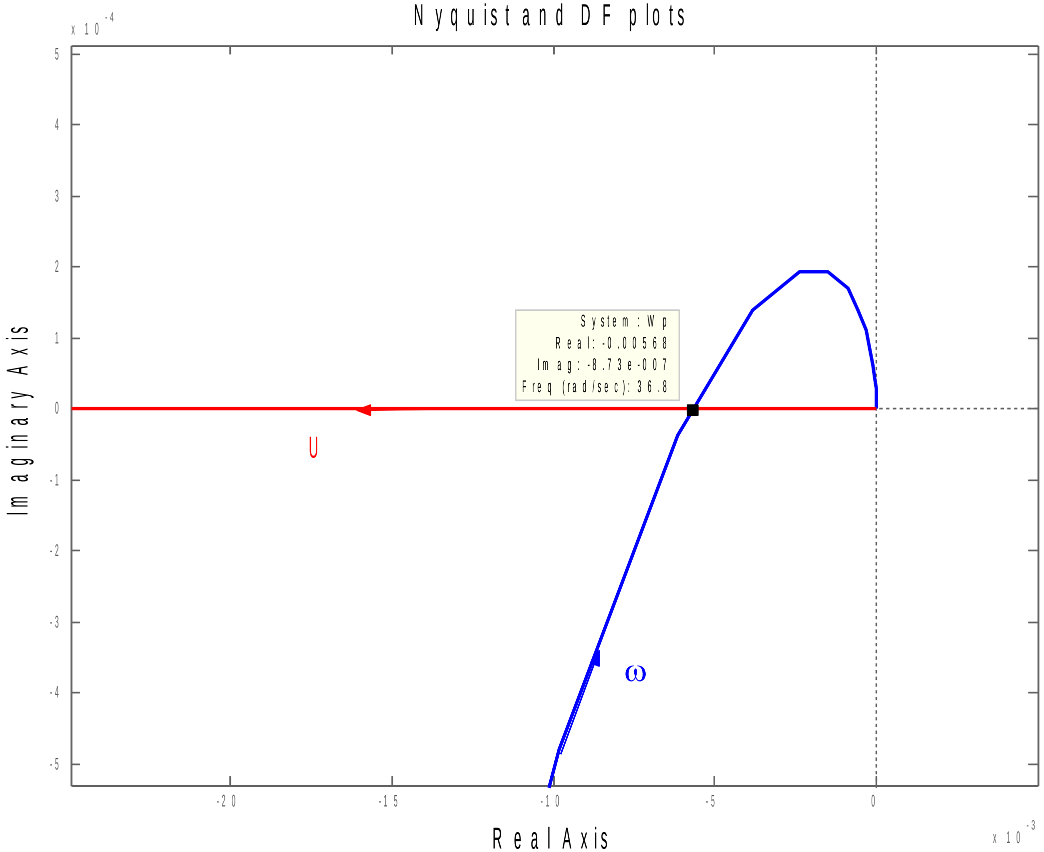
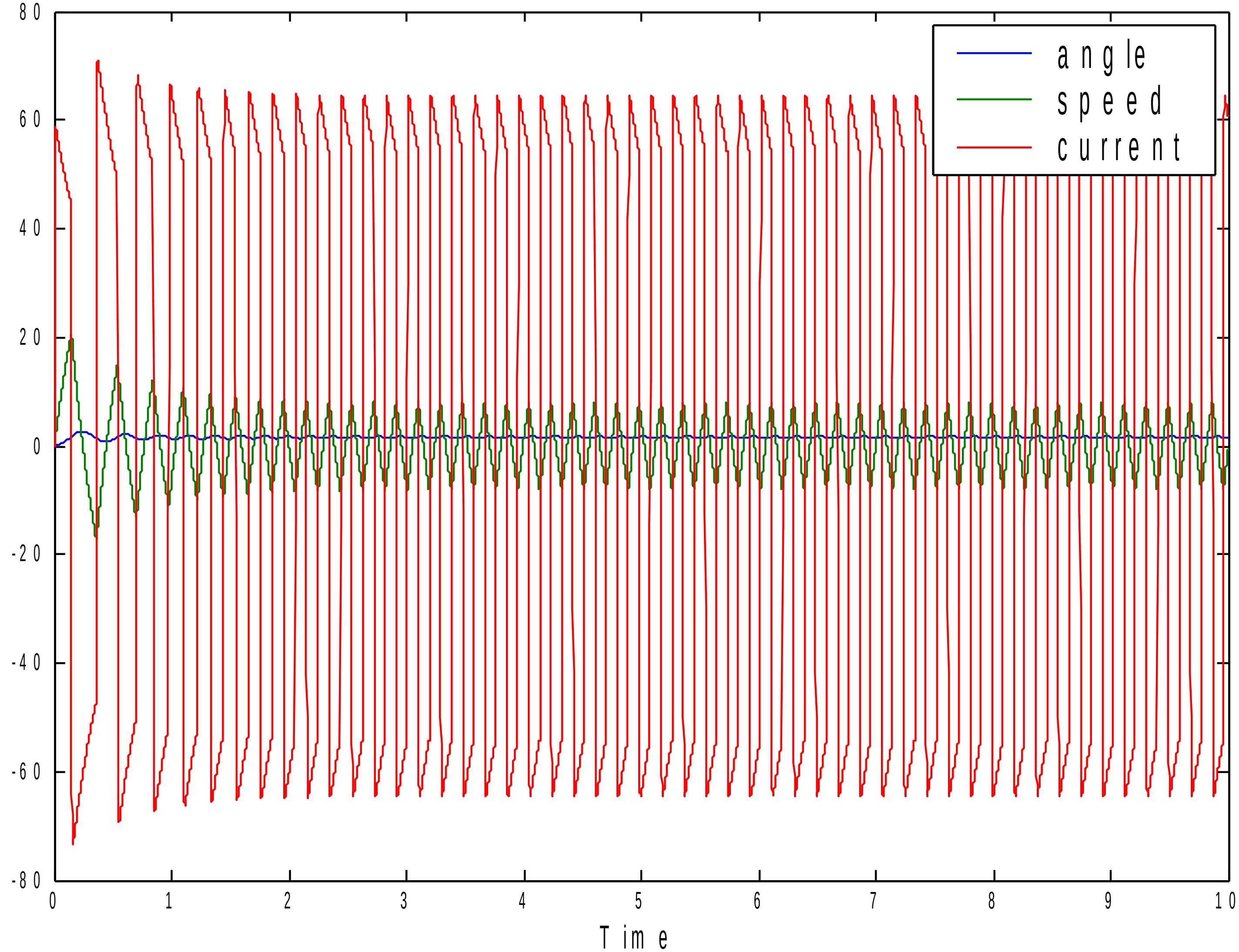


$$\omega\big|_{\Im(W_p(j\omega))=0}=\omega_{cr}=\sqrt{\frac{R_a B+k_t k_e}{L_a J}}=36.056\ \text{rad/s}$$

$$\overline{U}=-\frac{4M}{\pi}\cdot\Re\left(W_p(j\omega_{cr})\right)=0.1759\ \text{rad}$$

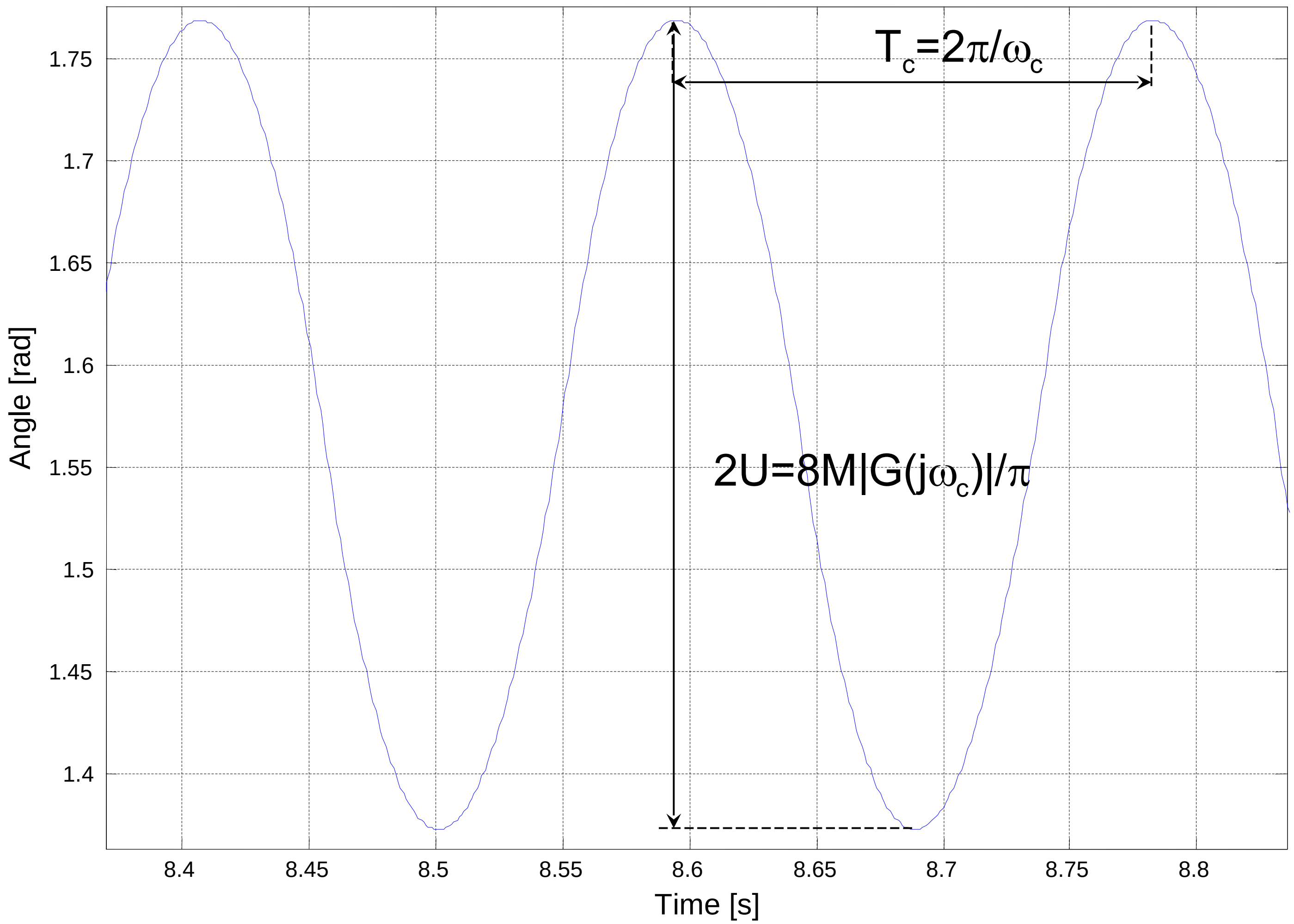
L3 – Effect of the parasitic dynamics

The system presents a periodic steady-state oscillation



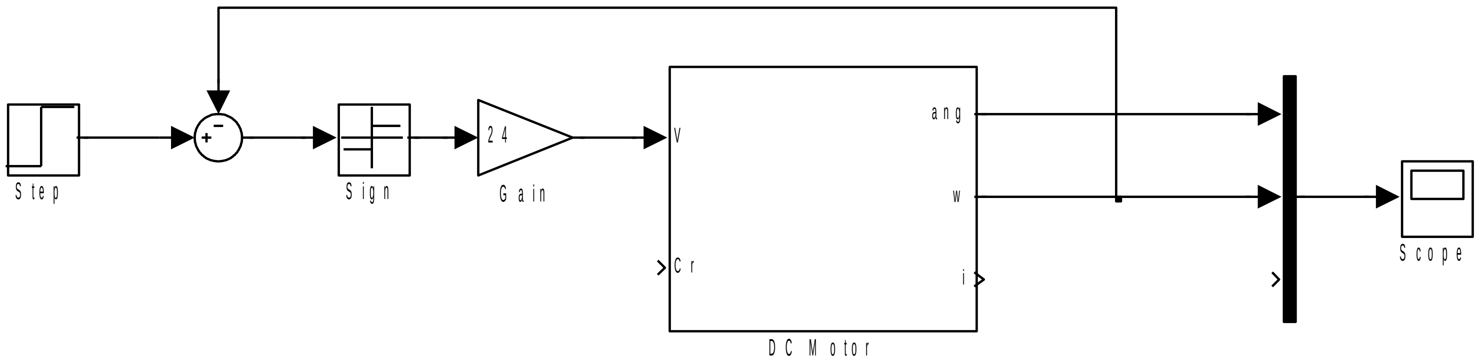
L3 – Effect of the parasitic dynamics

The system presents a periodic steady-state oscillation



L3 – Effect of the parasitic dynamics

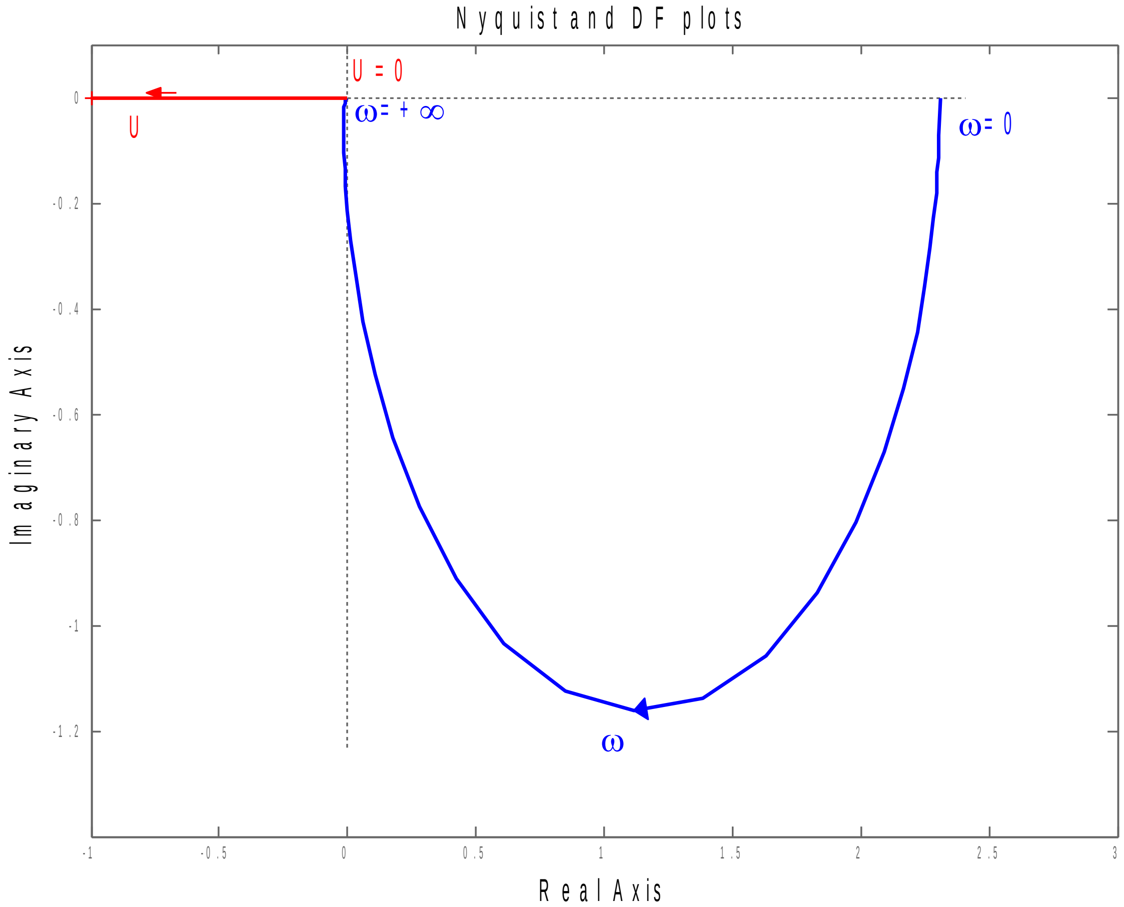
What does it happen if the same control law is applied to the speed control problem?



$$W_\omega(s) = \frac{k_t}{(sL_a + R_a)(sJ + B) + k_t k_e}$$

The linear plant is characterised by an all-pole transfer function with relative degree two, therefore there is no crossing of the Nyquist plot with the real negative axis, but the origin at $\omega=+\infty$ (corresponding to U=0 in the DF plot)

L3 – Effect of the parasitic dynamics



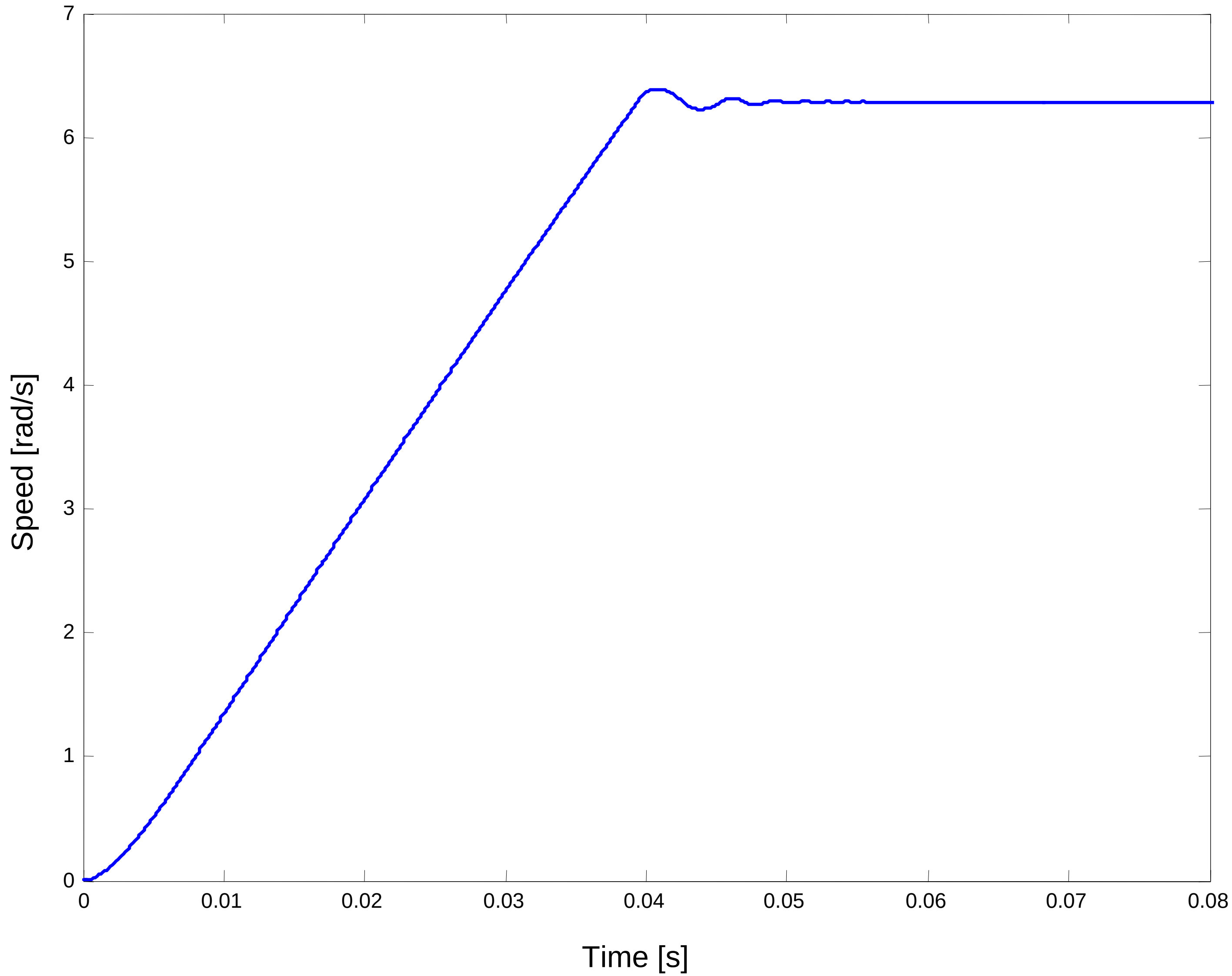
The Nyquist plot of the linear system is tangent to the negative reciprocal of the DF at the origin, i.e., U=0 and ω=+∞

A sliding mode behaviour is established asymptotically

L3 – Effect of the parasitic dynamics

Looking at the step response, it is apparent that the stabilisation is asymptotic

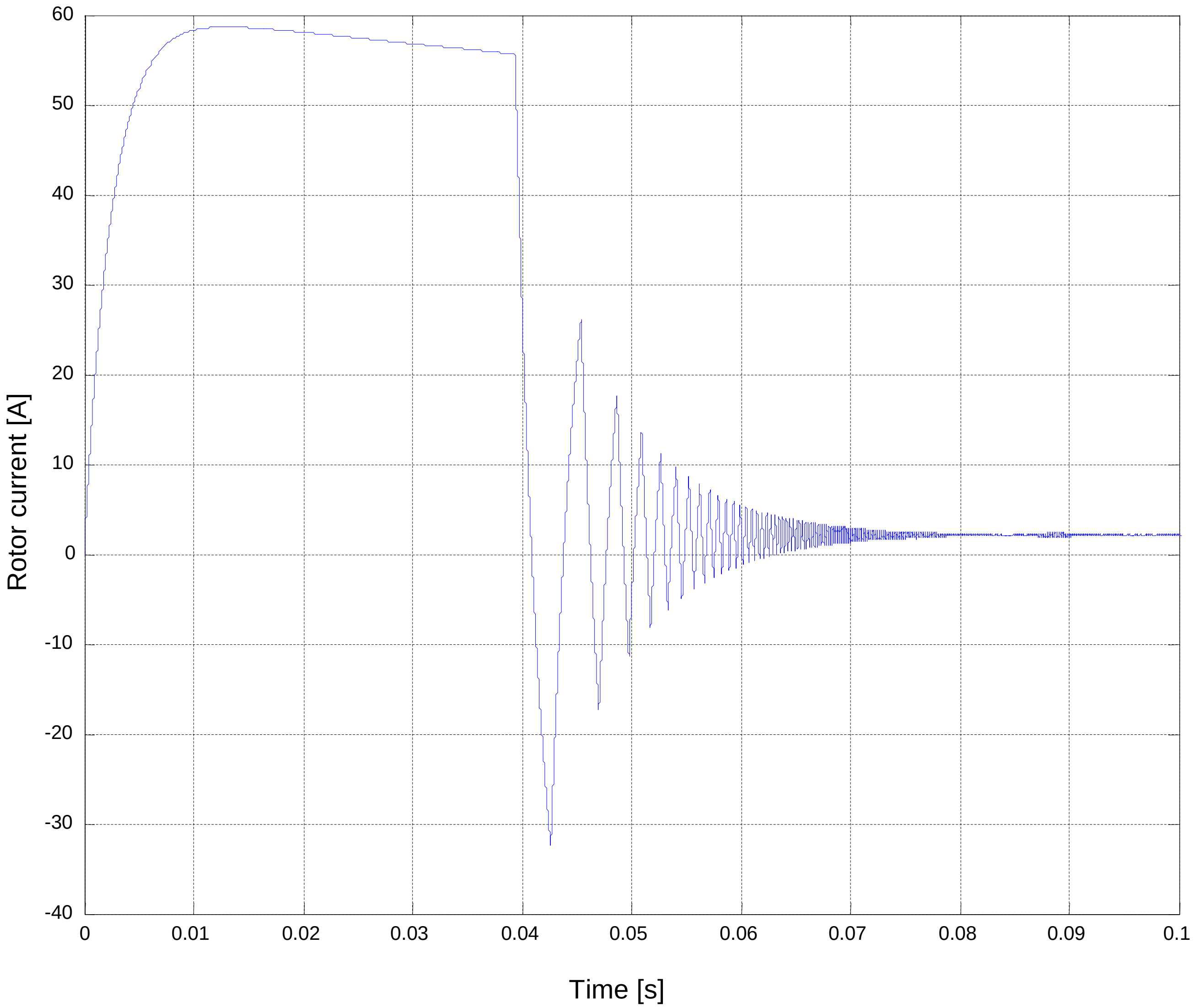
Can it be a sliding mode behaviour?



L3 – Effect of the parasitic dynamics

Looking at the step response, it is apparent that the stabilisation is asymptotic

The rotor current tends to a constant value, i.e. an asymptotic (*2nd order*) sliding sliding mode appears



L3 – Effect of the parasitic dynamics

Most effective use of the Describing Function approach to sliding modes….

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Analysis of the characteristics of a chattering behaviour due to unmodelled dynamics of sensors and/or actuators

What is chattering? *It appears as oscillations of the system variables, whose magnitude is related to the influence of the neglected dynamics on the system bandwidth*

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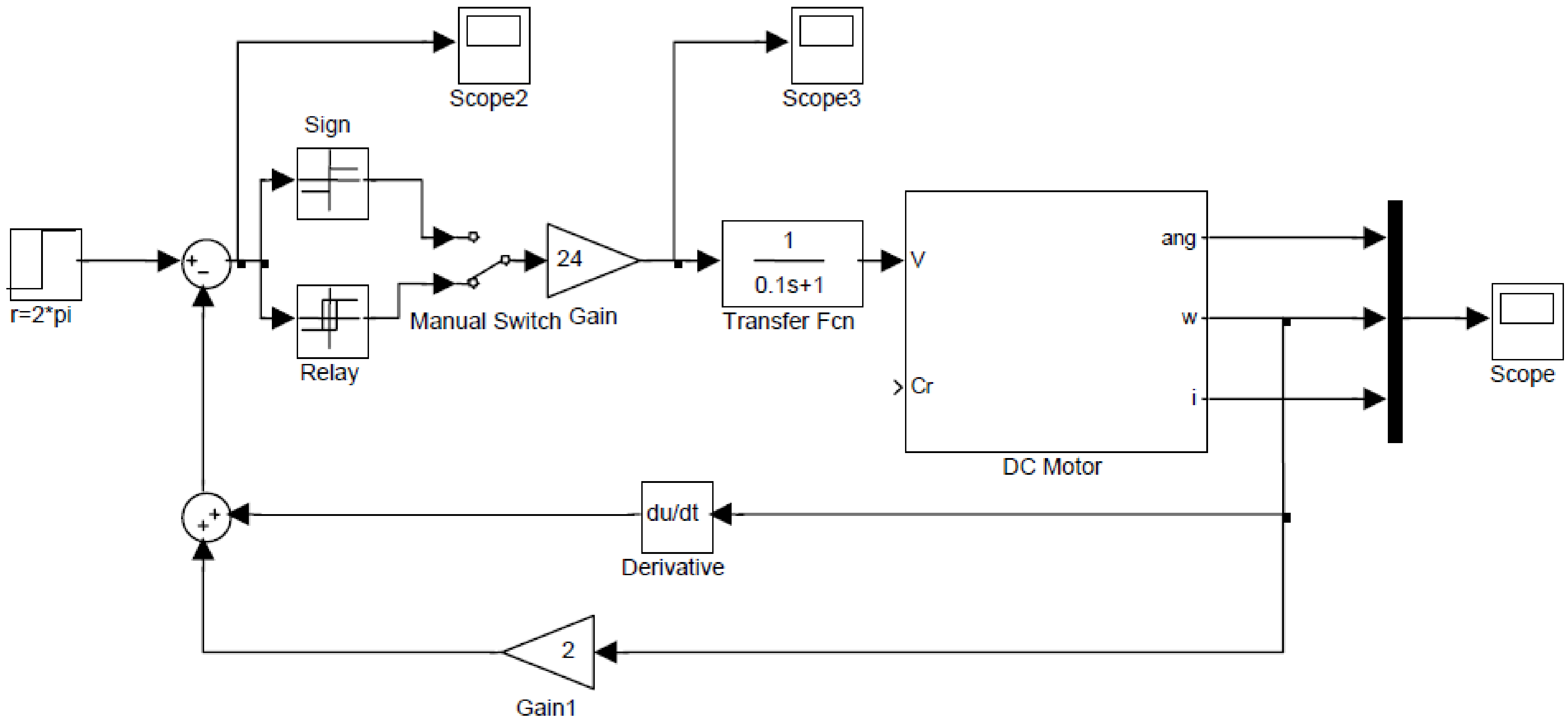
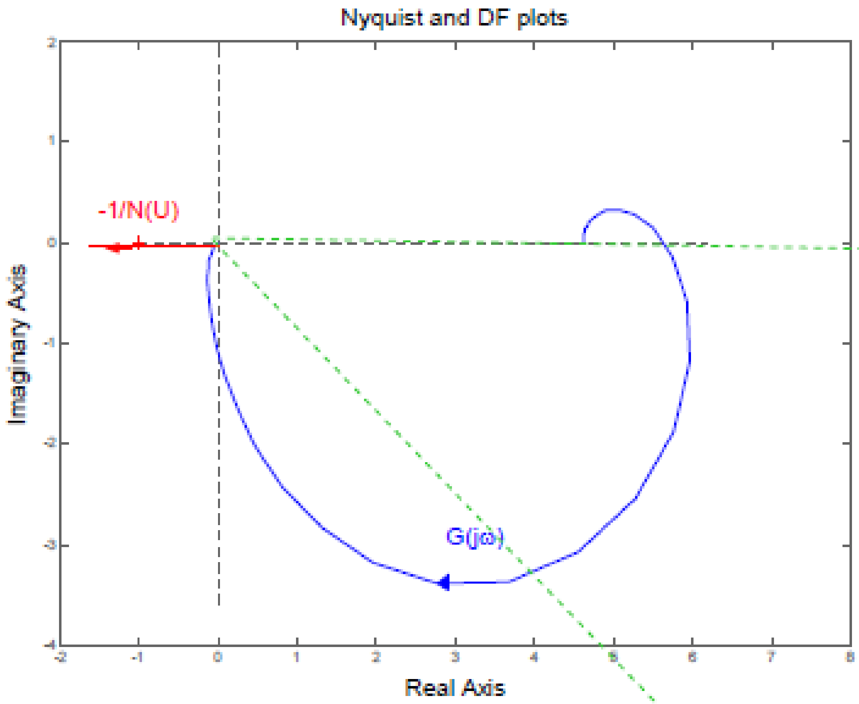
It is very close to a limit cycle

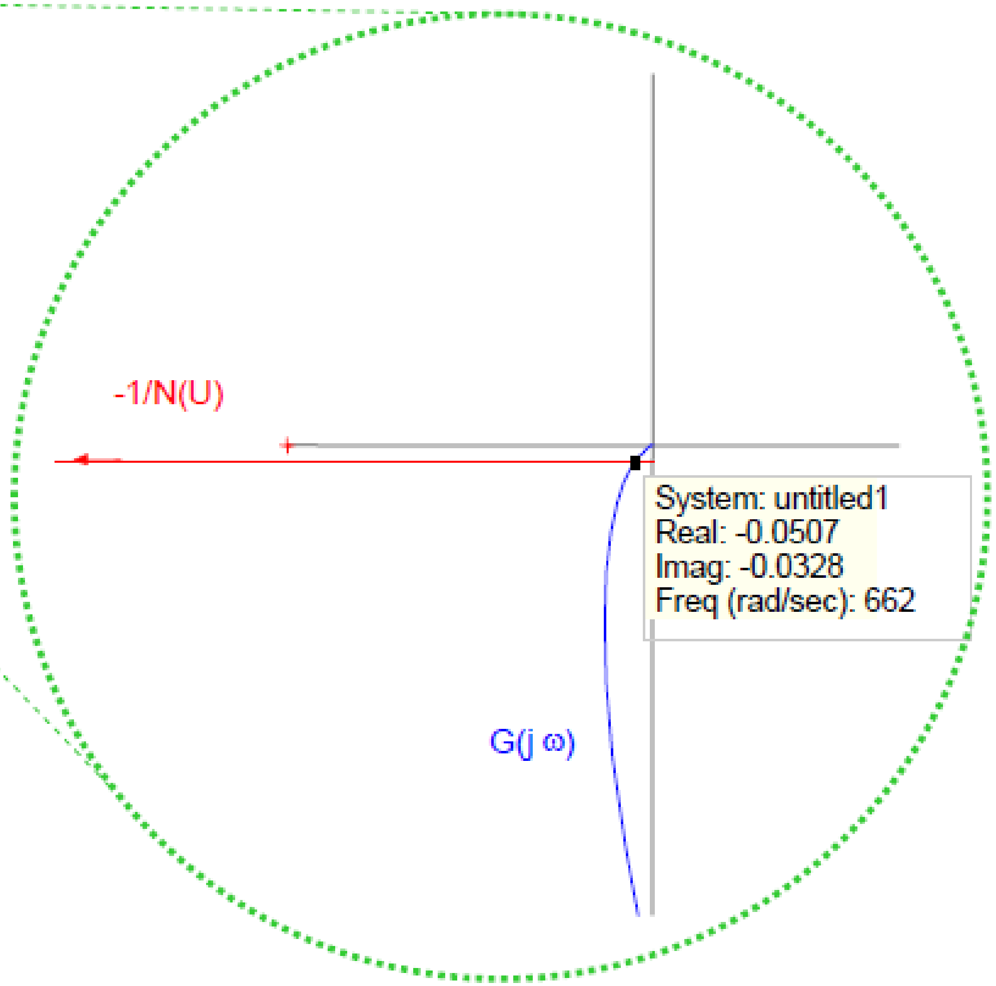
L3 – Effect of the parasitic dynamics

Consider the motor drive as a hysteretic switching device plus a time constant τ_a=0.1 s



L3 – Effect of the parasitic dynamics





Nyquist plot parameters

ω_{1c}=662 rad/s

U_{1m}=1.1833 rad/s^2

U_{1s} represents the magnitude of the oscillation of the sliding variable

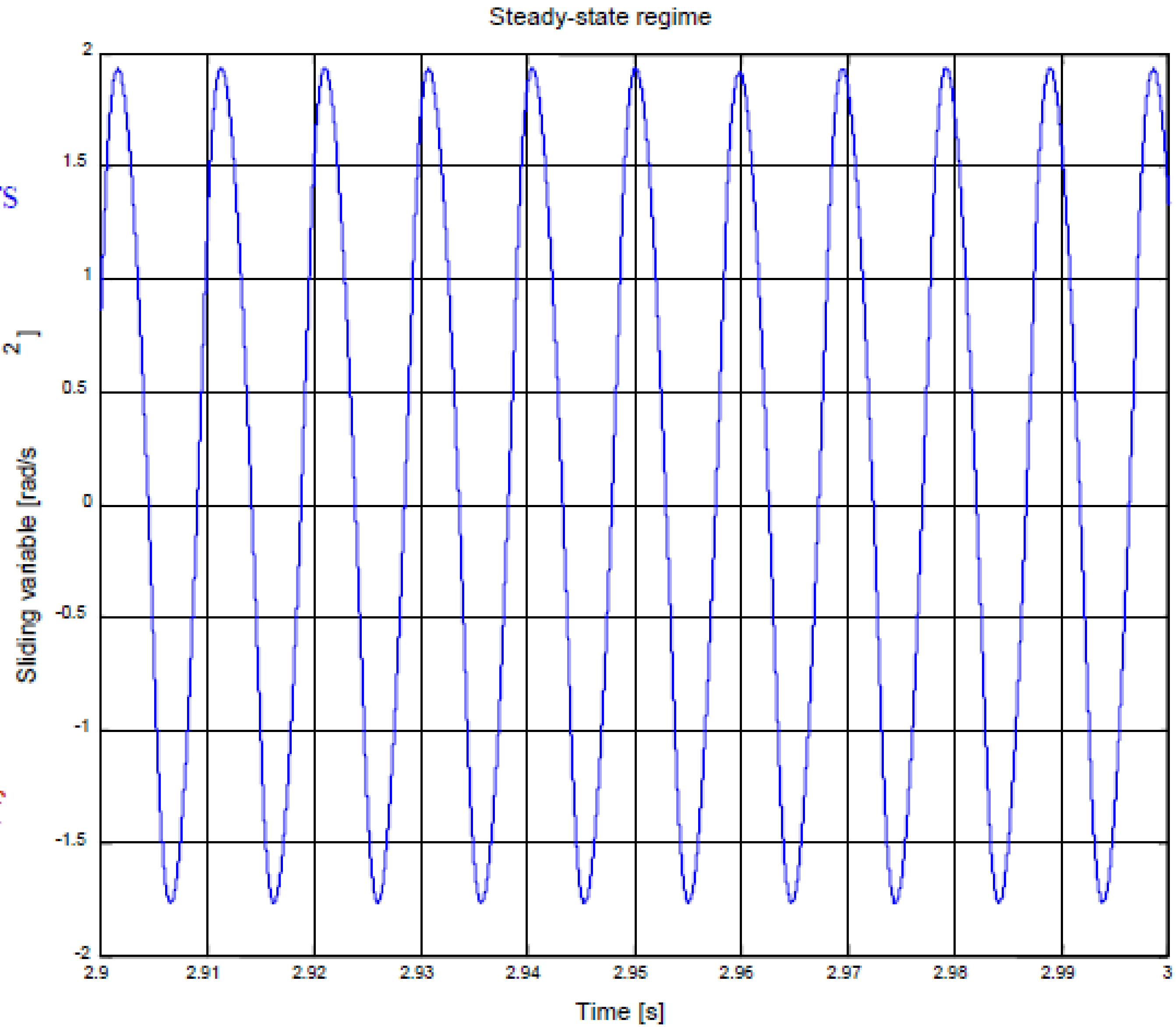
L3 – Effect of the parasitic dynamics

Nyquist plot parameters
ω_{1c}=662 rad/s
U_{1m}=1.1833 rad/s^2

Simulation results
T_{1s}=9.7 ms
ω_{1c}=646 rad/s
U_{1m}=1.8544 rad/s^2

Differences are due to:

- Numerical solution of the simulation
- Approximation of the DF approach



L3 – Effect of the parasitic dynamics

$$\begin{bmatrix} \dot{\omega} \\ \dot{i}_r \end{bmatrix} = \begin{bmatrix} \dfrac{B}{J} & \dfrac{k_t}{J} \\ -\dfrac{k_e}{L_r} & -\dfrac{R_r}{L_r} \end{bmatrix} \begin{bmatrix} \omega \\ i_r \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{L_r} \end{bmatrix} v_r$$

Full system dynamics

$$\sigma = \begin{bmatrix} 2 + \dfrac{B}{J} & \dfrac{k_t}{J} \end{bmatrix} \begin{bmatrix} \omega \\ i_r \end{bmatrix}$$

$$\frac{k_t}{J} i_r = \sigma - \left(2 + \frac{B}{J} \right) \omega$$

Internal reduced-order dynamics $\dot{\omega} = -2\omega + s$

Input-output dynamics

$$\dot{\sigma} = \left(2 + \frac{B}{J} - \frac{R_r}{L_r} \right) \sigma + \left(\left(2 + \frac{B}{J} \right) \left(\frac{R_r}{L_r} - 2 \right) - \frac{k_e}{L_r} \frac{k_t}{J} \right) \omega + \frac{k_t}{JL_r} v_r$$

L3 – Effect of the parasitic dynamics

Internal reduced-order dynamics $\dot{\omega} = -2\omega + \sigma$ $\Rightarrow$ $\Omega(s) = \dfrac{1}{s+2}\Sigma(s)$

At steady-state

$$\sigma(t) = \sigma_0 + U\sin(\omega_{lc} t) \Rightarrow \omega(t) = \omega_0 + U\left|\frac{1}{j\omega_{lc}+2}\right|\sin\left(\omega_{lc} t + \arg\left(\frac{1}{j\omega_{lc}+2}\right)\right)$$

$$\left|\frac{1}{j\omega_{ls}+2}\right| \cong 0.0015$$

Nyquist plot parameters

ω_{lc}=662 rad/s

U_{ls}=1.1833 rad/s^2

$\Delta\omega_{ls}$=0.0018 rad/s

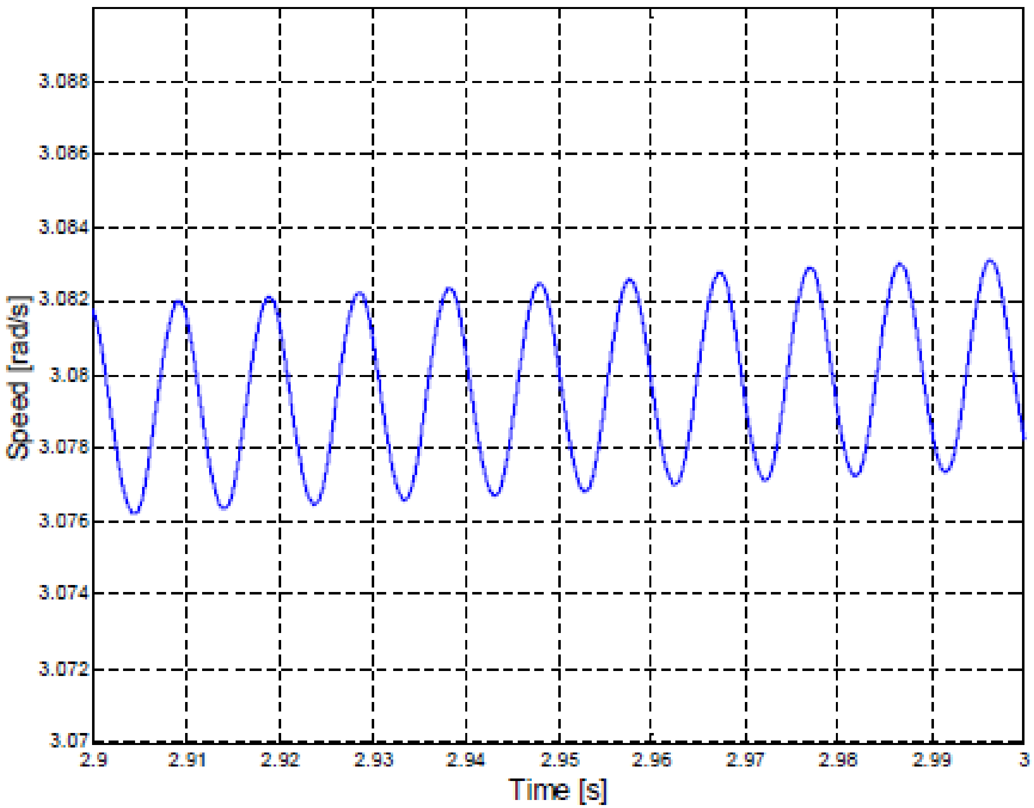
Simulation results

T_{ls}=9.7 ms

ω_{lc}=646 rad/s

U_{ls}=1.8544 rad/s^2

$\Delta\omega_{ls}$=0.0029 rad/s



L3 – Effect of the Measurement noise

A measurement noise super-imposed on the ideal sliding variable

$$\hat{\sigma}(t) = \sigma(t) + n(t), \quad |n(t)| \le \delta$$

⇓

1-SMC $\quad |\sigma(t)| = O(\delta)$

2-SMC $\quad |\sigma(t)| = O(\delta)$, $\quad |\dot{\sigma}(t)| = O\left(\sqrt{\delta}\right)$ $\quad$ Possibly not convergent

r-SMC $\quad \left|\sigma^{(i)}\right| = O\left(\delta^{r-i/r}\right), \quad i = 0,1,2,\ldots$ $\quad$ Possibly not convergent

L3 – Effect of the Measurement noise

Robust sliding mode differentiators can make the HOSM convergent even in the presence of noise

For the generalized sub-optimal a peculiar adaptation of the anticipation parameter β allows for implementing a noise robust 2-SMC

$$u = -\alpha(t) U \operatorname{sgn}(\hat{\sigma} - \beta \hat{s}_{ex}), \qquad \alpha(t) = \begin{cases} 1 & if \;\; \hat{\sigma}_{ex}(\hat{s} - \beta \hat{\sigma}_{ex}) \geq 0 \\ \alpha^* > 1 & if \;\; \hat{\sigma}_{ex}(\hat{\sigma} - \beta \hat{\sigma}_{ex}) < 0 \end{cases}$$

$$\beta \in [0;1)$$

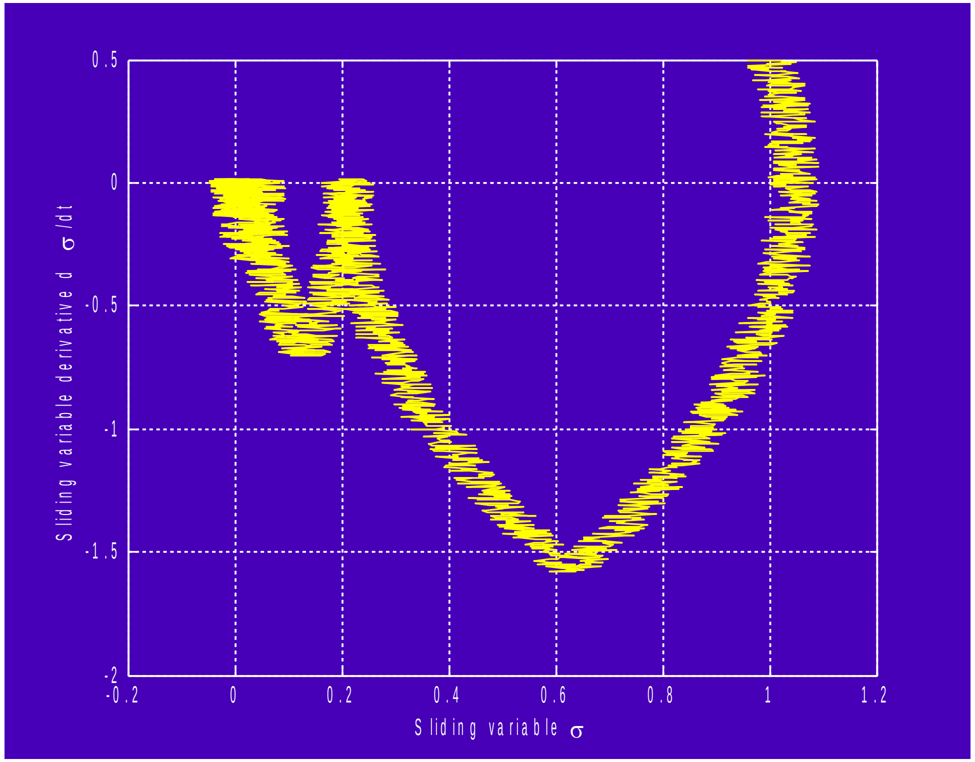
$$\hat{\sigma}_{ex} \text{ is the last } \hat{\sigma}(t_{ex}) \ni \dot{\hat{\sigma}}(t_{ex}) = 0$$

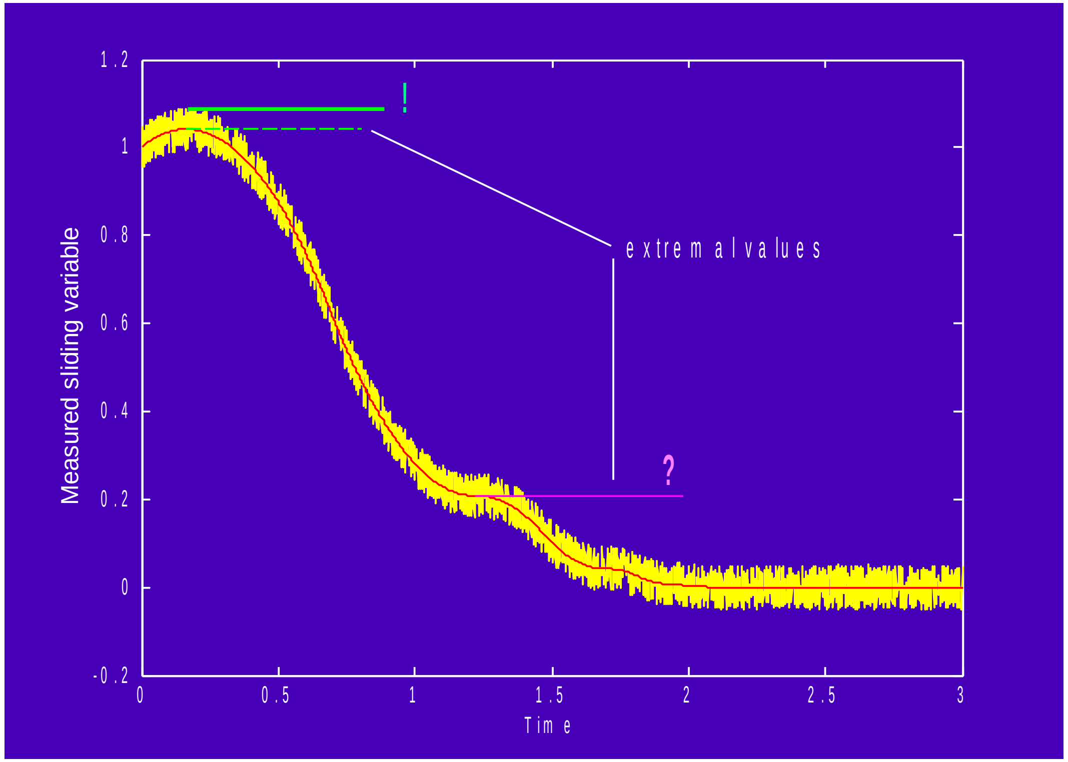
$$U > \frac{F}{G_m}$$

$$\beta \in \left[\frac{2F + (G_M - \alpha^* G_m) U}{(G_M + G_m) U}, 1 \right]$$

L3 – Effect of the Measurement noise

Is it possible to estimate the sequence of the extremal values
by inspection of the measured values of σ in a proper time window if the
measurements are noisy?



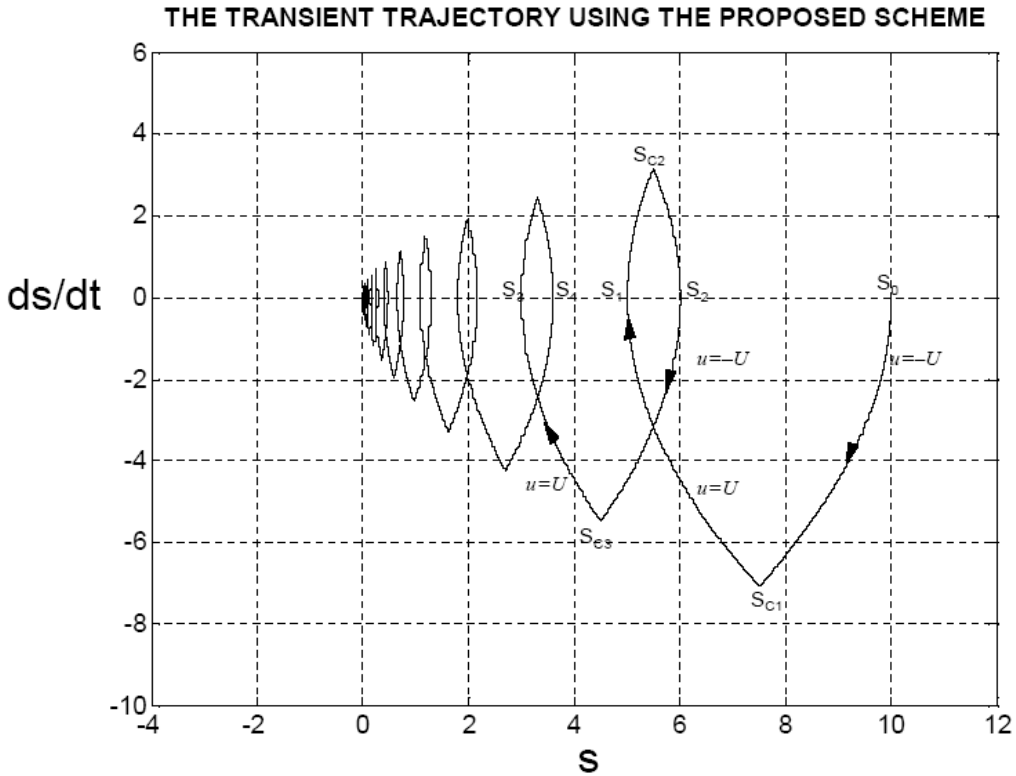


L3 – Effect of the Measurement noise

Since the main problem is the detection of the flex points of the sliding variable $\sigma(t)$, the control switchings needed as local minima are reached could be postponed by a fixed ratio of the distance between two subsequent extremal values, i.e., a local maximum and a local minimum

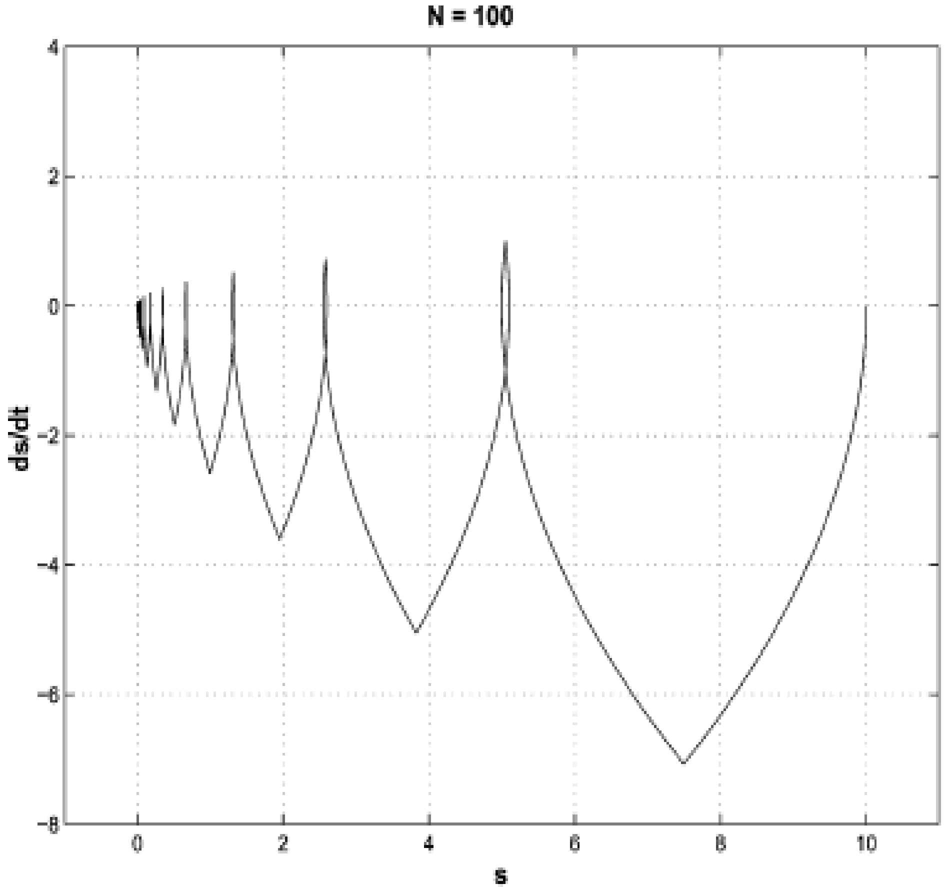
$$\hat{\sigma} - \hat{\sigma}_m \leq \frac{\hat{\sigma}_M - \hat{\sigma}_m}{N}, \quad N > 1$$

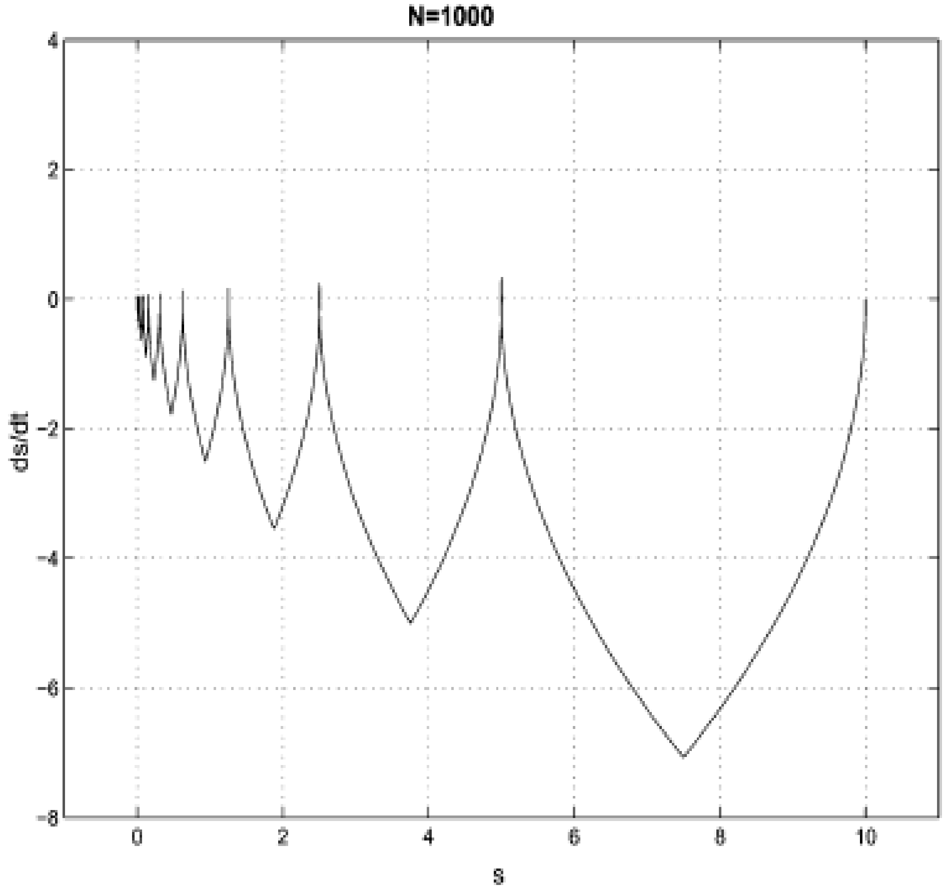
This choice guarantees the reduction of the estimation error of the flex points up to δ as the approximate sliding mode is reached



L3 – Effect of the Measurement noise

N affects the magnitude of the loop, and therefore the ultimate accuracy

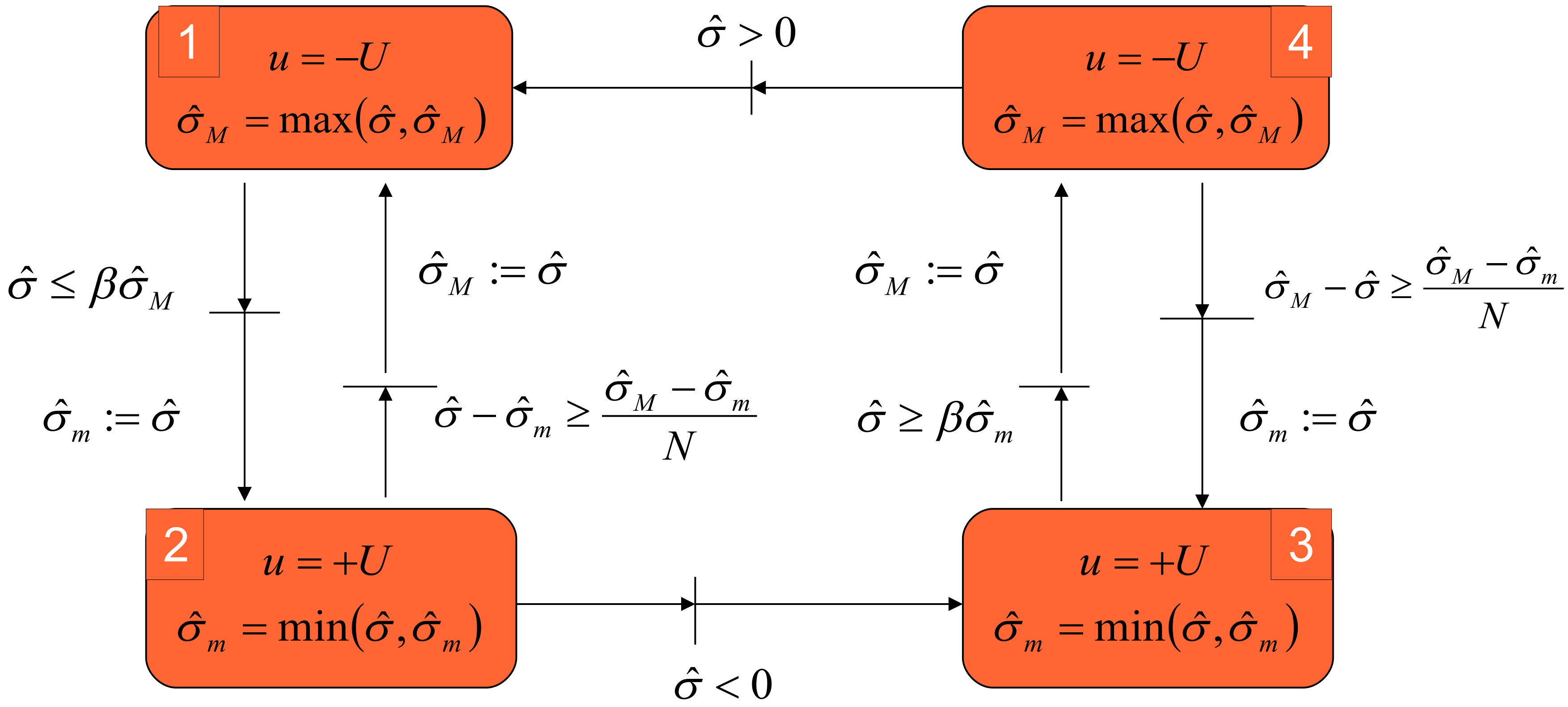




L3 – Effect of the Measurement noise

Implementation of the switching logic require an automaton and modified stability conditions

$$\beta_n = \frac{1+\beta_{so}}{2}, \quad \frac{1}{N} = 1-\beta_n$$



L3 – Effect of the Measurement noise

With exact measurements and ideal switching the 2-Sliding set is reached in a finite time T_∞

$$\sigma \underset{t\to T_\infty}{\to} 0, \qquad \dot{\sigma} \underset{t\to T_\infty}{\to} 0$$

With noisy measurements and switching delays only a boundary layer of the 2-Sliding set can be reached in a finite time T_∞

$$\sigma \underset{t\to T_\infty}{\to} k_0'\left(\tfrac{U_M}{F}, F\right)\delta + k_1'\left(\tfrac{U_M}{F}, F\right)\tau^2$$

$$\dot{\sigma} \underset{t\to T_\infty}{\to} k_2'\sqrt{k_0'\left(\tfrac{U_M}{F}, F\right)\delta + k_1'\left(\tfrac{U_M}{F}, F\right)\tau^2}$$

In general it can be assumed that in the presence of noise and "equivalent" delays the boudary layer of a r[th] Order Sliding Mode will be such that

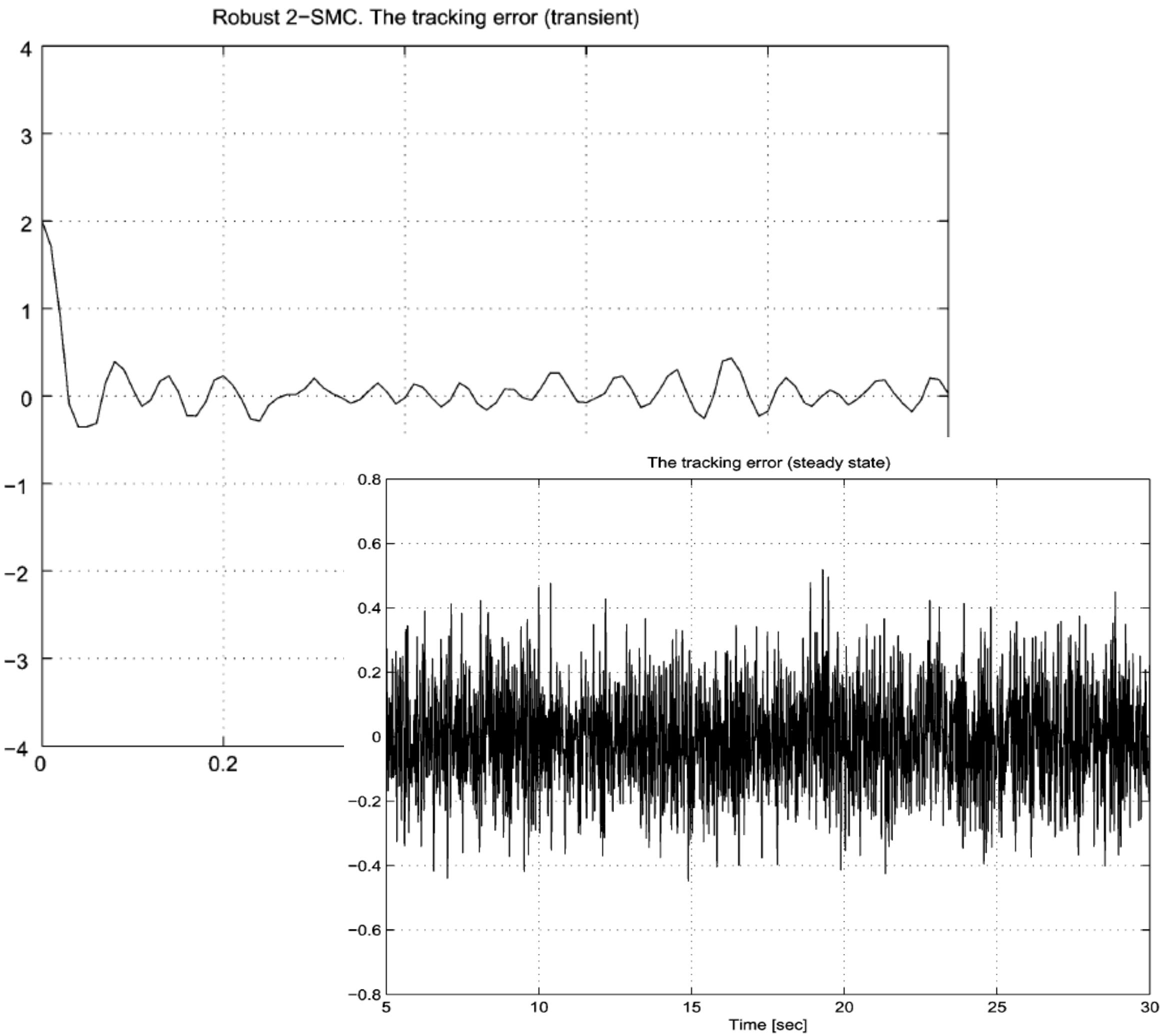
$$|\sigma^{(k)}| \le max\left\{k_1\tau^{(r-k)}, k_2\delta^{\frac{r-k}{r}}\right\}, \quad k=0,1,\cdots,r-1$$

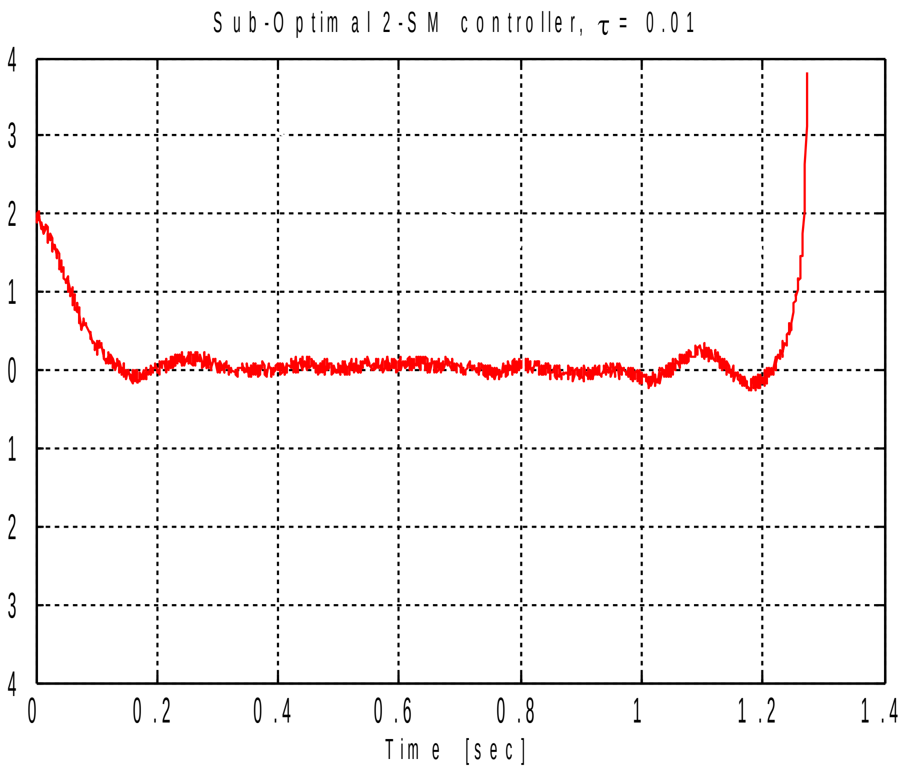
L3 – Effect of the Measurement noise

Example

$$\ddot{y} = y^2 + \dot{y}^2 + \sin(3t) + u, \quad \hat{y} = y + n, \quad |n| \le 0.2$$

$$y_d = 2\sin(5t), \quad \sigma = y - y_d$$





L3 – Chattering Attenuation

What is Chattering ?

Many definitions can be found:

- Discontinuous control
- Not precise attainment of the sliding
- Oscillation of the system state due to unmodelled dynamics

Continuous control can produce large and unpredictable state oscillations

Not precise attainment of the sliding can be due to design errors of the sliding surface

Unmodelled dynamics is always present

⇓

Chattering cannot be eliminated but only attenuated!

L3 – Chattering Attenuation

Chattering depends on the ultimate accuracy of the sliding motion

$$\left|\sigma^{(k)}\right| \leq \upsilon_k\left(\Sigma_{k+1}\right)T^{r-k} \qquad k = 0,1,\ldots,r-1 \qquad \upsilon_k \in \mathsf{K}_\infty$$

v_k is an increasing function of Σ_{k+1}, and in particular, by a chain rule, of Σ_r, that depends on the discontinuous control magnitude

The shape of v_k depends on the overall closed loop dynamics

T can be either:

- the switching delay of a relay device
- the sampling time in a discrete time implementation
- the equivalent time constant of a dynamic actuator

L3 – Chattering Attenuation

Chattering can be attenuated by means of:

Smooth approximations of the discontinuous function

It is effective only if the matching uncertainty vanishes on the sliding surface

Implementation of HOSM

It will require the knowledge of a number of time derivatives of the sliding variable, apart from the Super-Twisting and Sub-Optimal algorithms that are effective for Single-Input systems and only some class of Multi-Input systems

Using much higher sampling frequency

It can be sensible to high frequency noise (*Aliasing phenomenon*)

L3 – Chattering Attenuation

Chattering can be attenuated by means of:

Adaptation of the switching control magnitude

It will require more complicated control schemes

Tuning the parameter of a 2-Order Sliding Mode Controller

It is quite easy but some drawback on the reaching phase will follow, and it is quite simple for Single-Input linear systems only

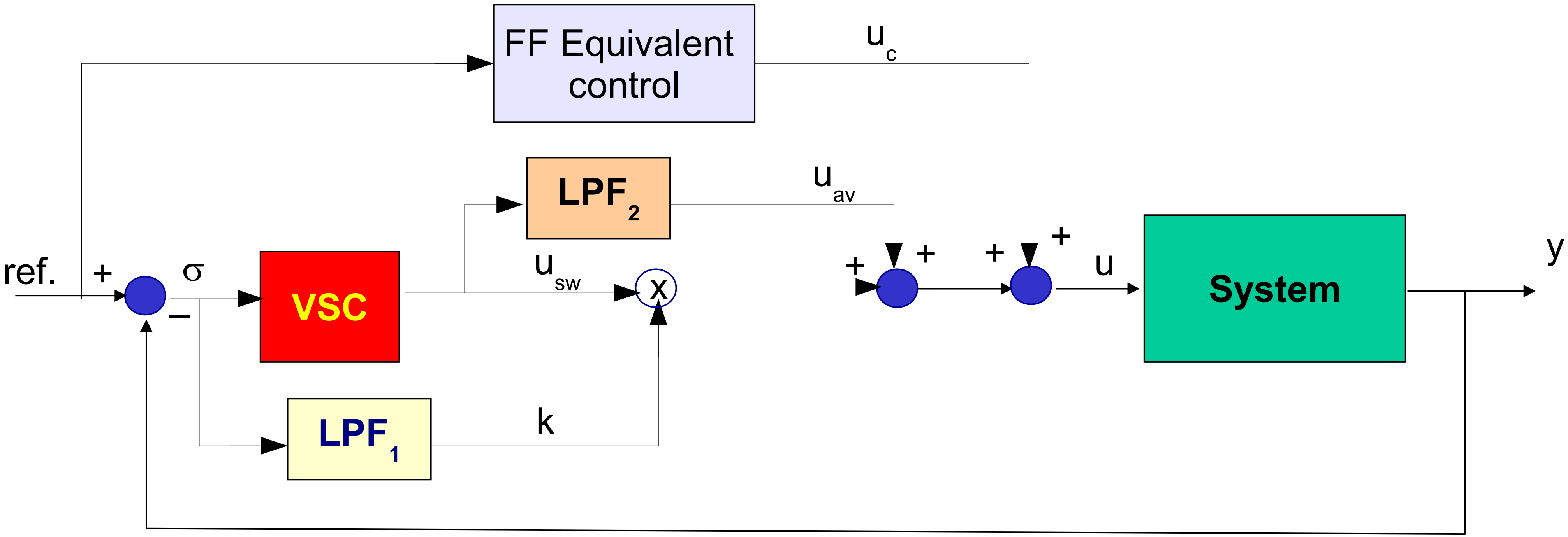
Shaping the dynamics of the system

It requires an additional filter it is quite simple for Single-Input linear systems only

L3 – Control Magnitude Adaptation

Adaptation of the switching control magnitude

Most of the adaptation algorithms resort to the estimation of the equivalent control, and define the system input as a proper combination of the averaged control and a switching control



L3 – Control Magnitude Adaptation

Adaptation of the switching control magnitude

$$
\begin{aligned}
&u(t) = u_c(t) + u_{av}(t) + k(\sigma, t) u_{sw}(t) \\
&u_c(t) = -\frac{\bar{\varphi}(\mathbf{x}(t), t)}{\bar{\gamma}(\mathbf{x}(t))} \\
&u_{sw}(t) = -U \operatorname{sgn}(\sigma) \\
&\tau_{av} \dot{u}_{av}(t) + u_{av}(t) = u_{sw}(t) \\
&\tau_{ad} \dot{k}(t) + k(t) = \kappa |\sigma|^{\alpha}
\end{aligned}
$$

If the parameter are properly chosen the system input asymptotically approaches the ideal equivalent control and the accuracy is improved

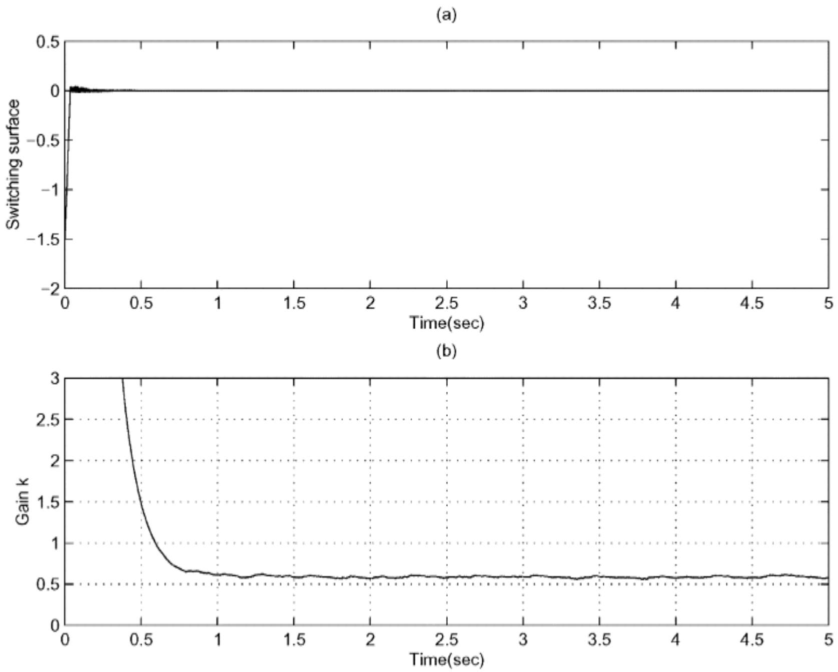
L3 – Control Magnitude Adaptation

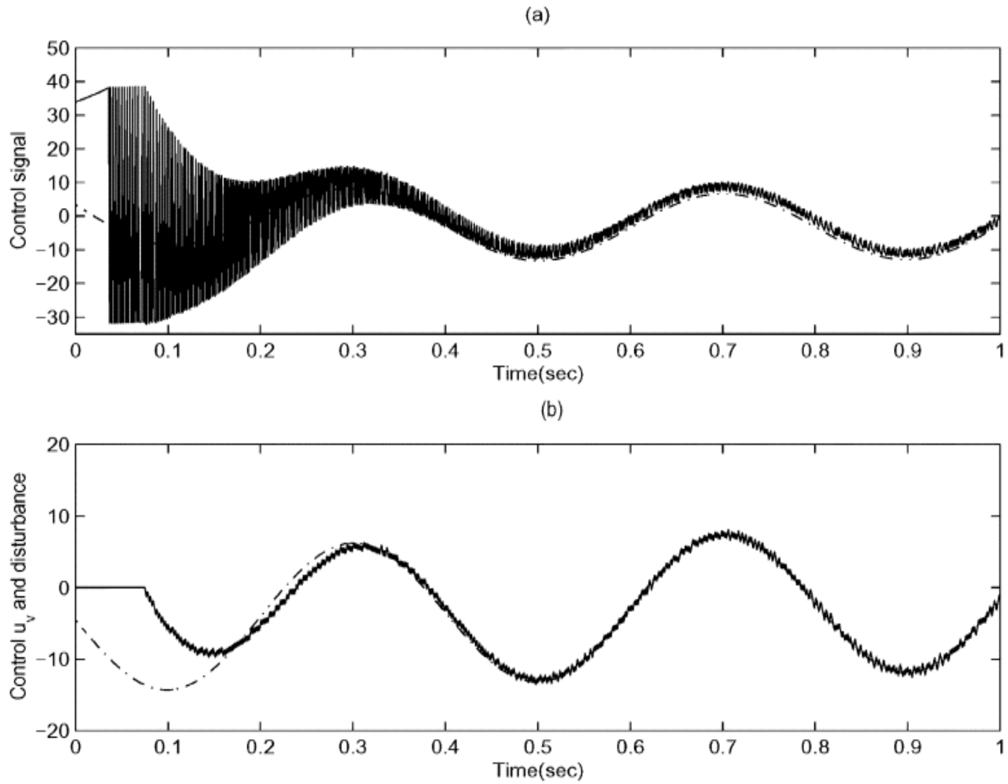
Adaptation of the switching control magnitude

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -2x_1 + \mu\left(1 - x_1^2\right)x_2 + D_1 \sin(\omega t) + D_2 \sin\left(a\sqrt{t+1}\right) + u \end{cases}$$

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = -2z_1 + \mu\left(1 - z_1^2\right)z_2 \end{cases}$$

$$\sigma = (z_2 - x_2) + c(z_1 - x_1)$$





L3 – Control Magnitude Adaptation

Another approach for adapting the switching control magnitude is not based on the on the estimation of the continuous equivalent control but on avoiding too large control authority

A real r-order sliding mode *(r-sliding)* in which the r-th derivative of the sliding variable σ is always separated from zero is characterised by

$$|\sigma| \leq k_0 \tau^r, \quad |\dot{\sigma}| \leq k_1 \tau^{r-1}, \quad \dots \quad \left|\sigma^{(r-1)}\right| \leq k_{r-1} \tau,$$
$$0 < R_1 \leq \left|\sigma^{(r)}\right| \leq R_2$$

τ is the switching delay

In real-sliding mode the derivatives of the sliding variable must be zero in media and the discontinuous control switches at very high, but finite, frequency

$$f_{sw} \in \left[\frac{R_1}{2k_{r-1}\tau}, \frac{1}{\tau} \right]$$

L3 – Control Magnitude Adaptation

Adaptation of the switching control magnitude

Lemma: Let σ be the sliding variable of a variable structure control system, and let the discontinuous control be always separated from zero. Assume that $\sigma^{(i)}$ $(i=0,1,\dots,r\text{-}1)$ are continuous functions. Consider a time interval T^* of length $N\tau$, $N\in R^+$, and let N_{sw} be the number of zero crossings of σ during the time T^*. If $N_{sw} \geq r$ then, over the time interval T^*, the system trajectories are confined within the domain

$$\mathrm{D} = \left\{ \mathbf{x} \in \mathbf{R}^{\mathbf{n}} : \left| \sigma^{(r-i)}(\mathbf{x}) \right| \leq \frac{R_2}{i\,!} N^i \tau^{\mathrm{i}} \;\;,\; i = 0, 1, \dots, r-1 \right\}$$

The time interval between two subsequent zero-crossing of the sliding variable in the steady state varies over $[T_m\ ,\ T_M]$ that depend on the specific VSC algorithm, its parameters and τ

A proper choice for N in order to detect the real sliding could be related to the maximum "cycle-time" T_M

$$N\tau = r\,T_M$$

L3 – Control Magnitude Adaptation

Adaptation of the switching control magnitude

First order sliding modes

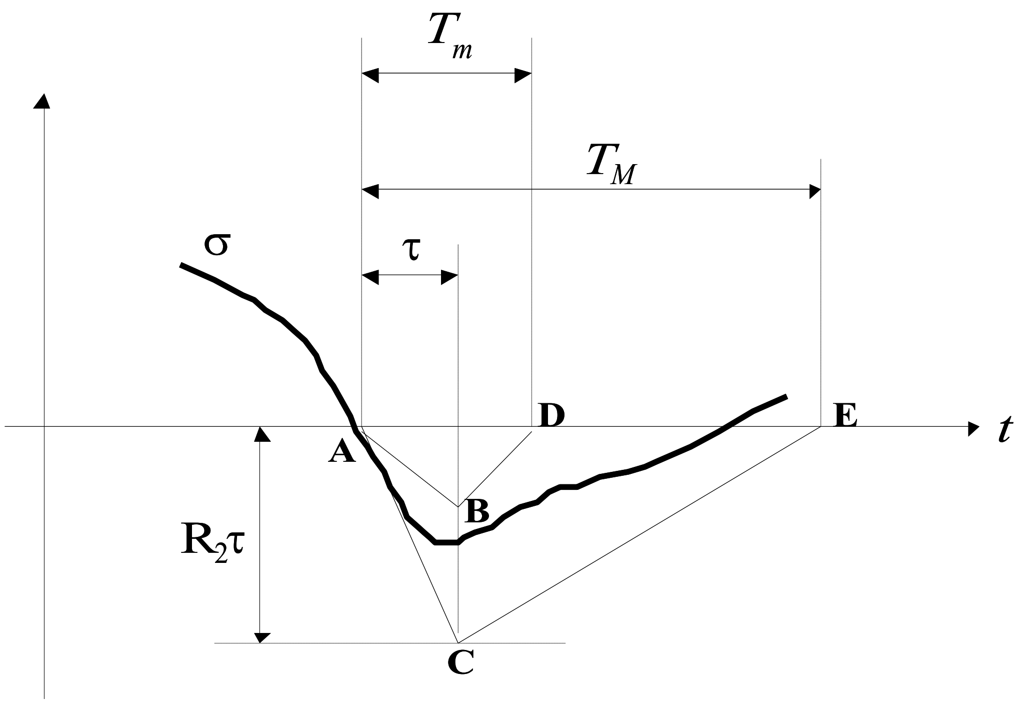
$$\dot{\sigma} = \varphi_1(\mathbf{x},t) + \gamma_1(\mathbf{x},t)u$$

$$|\varphi_1(\mathbf{x},t)| \le \Phi$$

$$0 < \Gamma_1 \le \gamma_1(\mathbf{x},t) \le \Gamma_2$$

$u(t)=-U_M \operatorname{sign}(\sigma(t_i))$

$t_i \le t < t_{i+1} \quad (t_{i+1} - t_i = \tau) \quad i=0,1,2, \ldots$



$$T_m = \left(1 + \frac{R_1}{R_2}\right)\tau$$

$$T_M = \left(1 + \frac{R_2}{R_1}\right)\tau$$

$$\frac{R_1}{R_2} = \frac{\eta\Gamma_2 + 1}{\eta\Gamma_1 - 1}$$

$$\eta = \frac{U_M}{\Phi}$$

L3 – Control Magnitude Adaptation

Adaptation of the switching control magnitude

η represents the "*dominance factor*".

The larger the η the larger the boundary layer size and the highest the switching frequency

As far as the actual value of η guarantees the stability of the sliding mode, the switching frequency is higher than a certain minimum value, as well as the number of changes of sign of σ with a certain time interval

U_M can be adapted on-line to maintain a prescribed, desired, "*dominance factor*" η^*

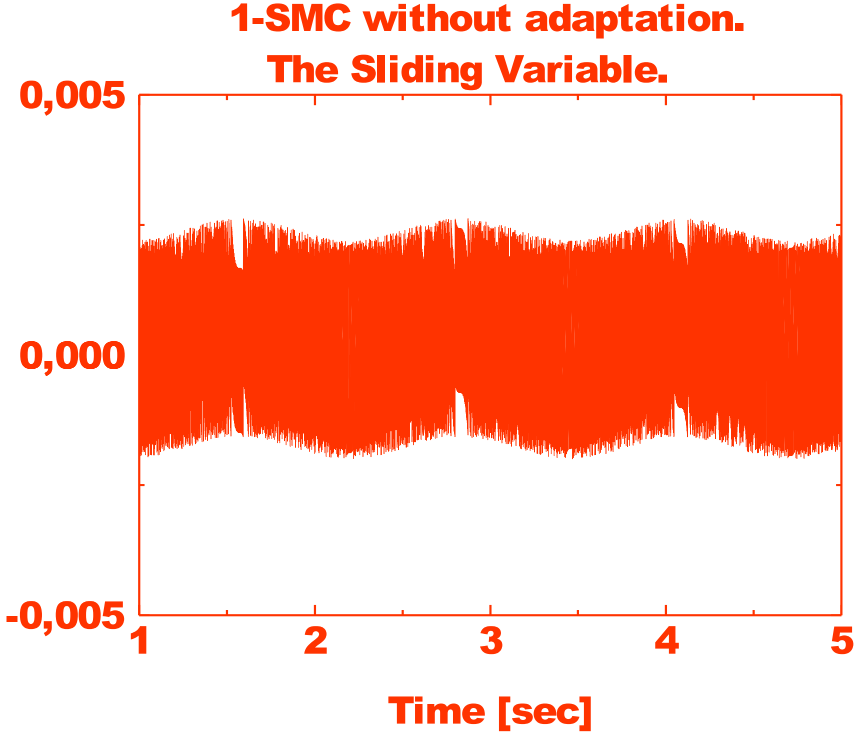
$$U_{Mi+1} = \begin{cases} \max(U_{Mi} - \Lambda N\tau, 0), & N_{\mathrm{sw}} \geq r+1, \\ U_{Mi} + \Lambda N\tau, & N_{\mathrm{sw}} < r+1 \end{cases}$$

L3 – Control Magnitude Adaptation

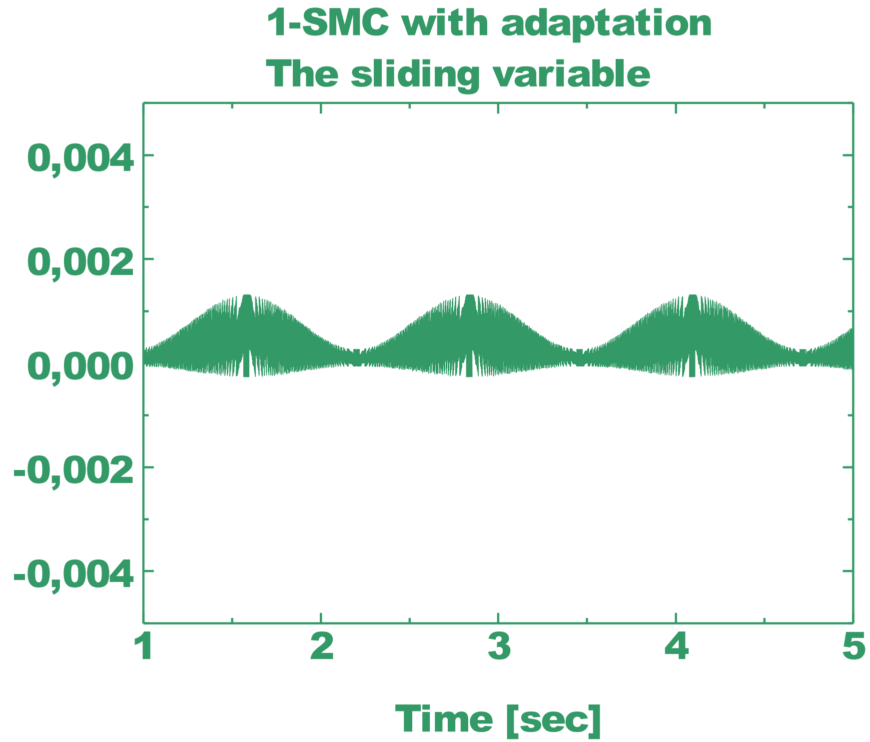
Adaptation of the switching control magnitude

$$\dot{\sigma} = 3 + 2\sin(5t) + u$$

τ =10^{-4}



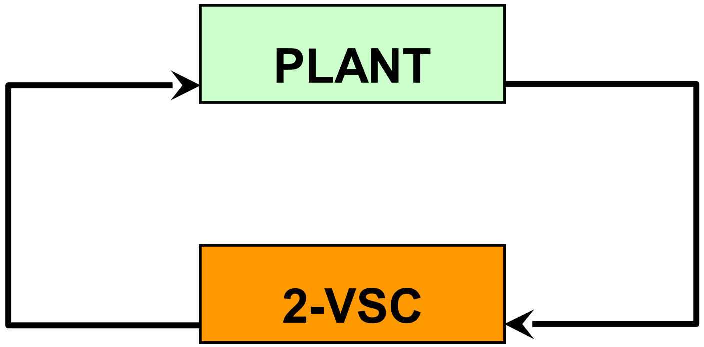
U_M=20



N=10 $\Lambda = 20$

L3 – Parameters Tuning in 2-SMC

A feedback Second-Order Sliding Mode System Scheme is simply represented



σ - Sliding variable

If the plant is stable linear system with low-pass properties

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \\ \sigma(t) = \mathbf{C}\mathbf{x}(t) \end{cases} \qquad \begin{array}{l} \mathbf{x} \in \mathrm{R}^n \\ u \in \mathrm{R} \\ \sigma \in \mathrm{R} \end{array}$$

The controller is the Generalized Sub-Optimal

$$u = -\alpha(t) U \operatorname{sgn}(\hat{\sigma} - \beta\hat{\sigma}_{ex}),$$

$$\alpha(t) = \begin{cases} 1 & if \ \ \hat{\sigma}_{ex}(\hat{\sigma} - \beta\hat{\sigma}_{ex}) \geq 0 \\ \alpha^* > 1 & if \ \ \hat{\sigma}_{ex}(\hat{\sigma} - \beta\hat{\sigma}_{ex}) < 0 \end{cases}$$

$$\beta \in [0;1)$$

L3 – Parameters Tuning in 2-SMC

The steady state analysis of the system can be approximately carried out by means of the Describing Function approach

The Describing Function of the Generalized Sub-Optimal controller

$$N\left(y_M^p\right) = \frac{2U_M}{\pi y_M^p}\left\{\left(\alpha^* + 1\right)\sqrt{1-\beta^2} + j\left[\left(\alpha^* - 1\right) + \beta\left(\alpha^* + 1\right)\right]\right\}$$

The harmonic response of the system

$$G(j\omega) = \mathbf{C}\left(j\omega\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B}$$

The Harmonic Balance equation

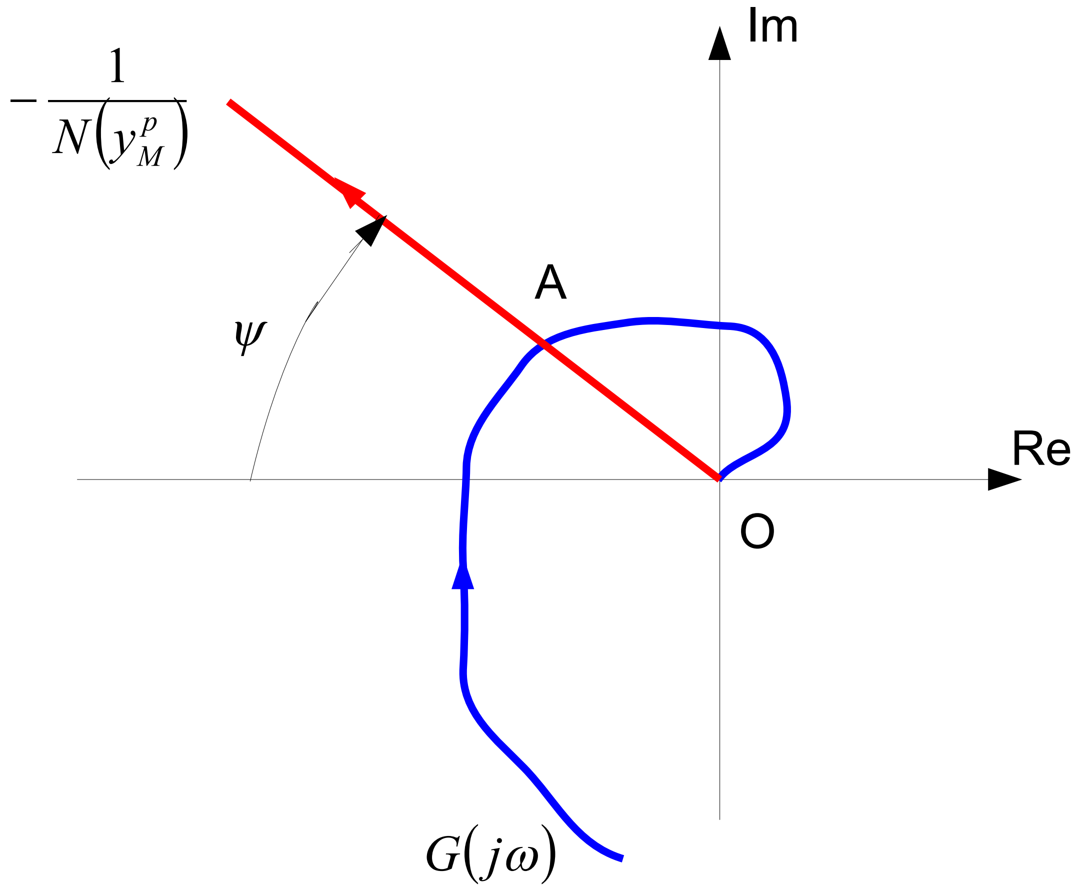
$$G(j\omega) = \frac{\pi y_M^p}{4U_M}\,\frac{-\left(\alpha^* + 1\right)\sqrt{1-\beta^2} + j\left[\left(\alpha^* - 1\right) + \beta\left(\alpha^* + 1\right)\right]}{\alpha^{*2}\left(1+\beta\right) + 1 - \beta}$$

L3 – Parameters Tuning in 2-SMC

The steady state analysis of the system can be approximately carried out by means of the Describing Function approach

$$\psi = \arctan \frac{(\alpha^* - 1) + \beta(\alpha^* + 1)}{(\alpha^* + 1)\sqrt{1 - \beta^2}}$$

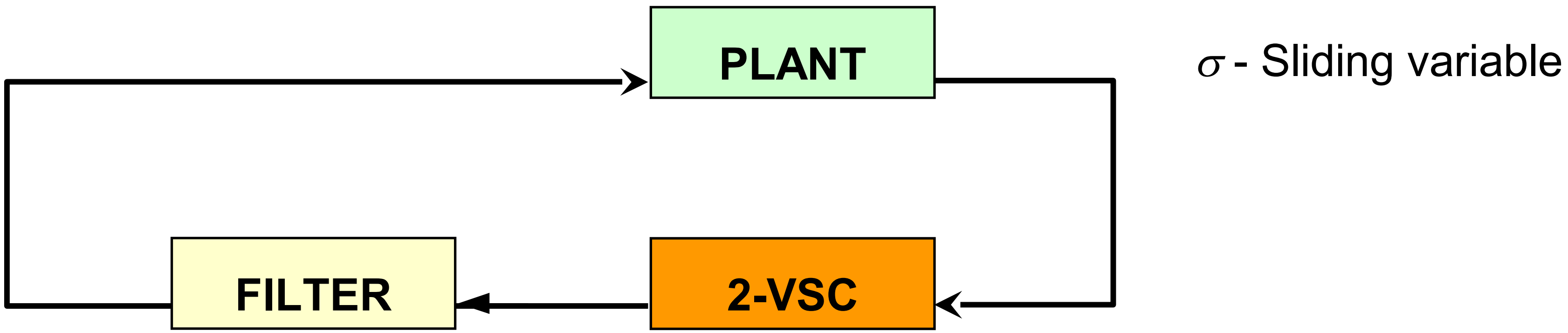
$$\frac{|\mathrm{AO}| U_M}{y_M^p} = \frac{\pi}{2\sqrt{2}\sqrt{\alpha^{*2}(1+\beta) + 1 - \beta}}$$



Parameters α^* and β can be tuned in order to minimize y^{p}_{M}

L3 – System Dynamics Shaping

The Describing Function approach can be used to properly shaping the Nyquist plot of the linear system by introducing a proper linear filter



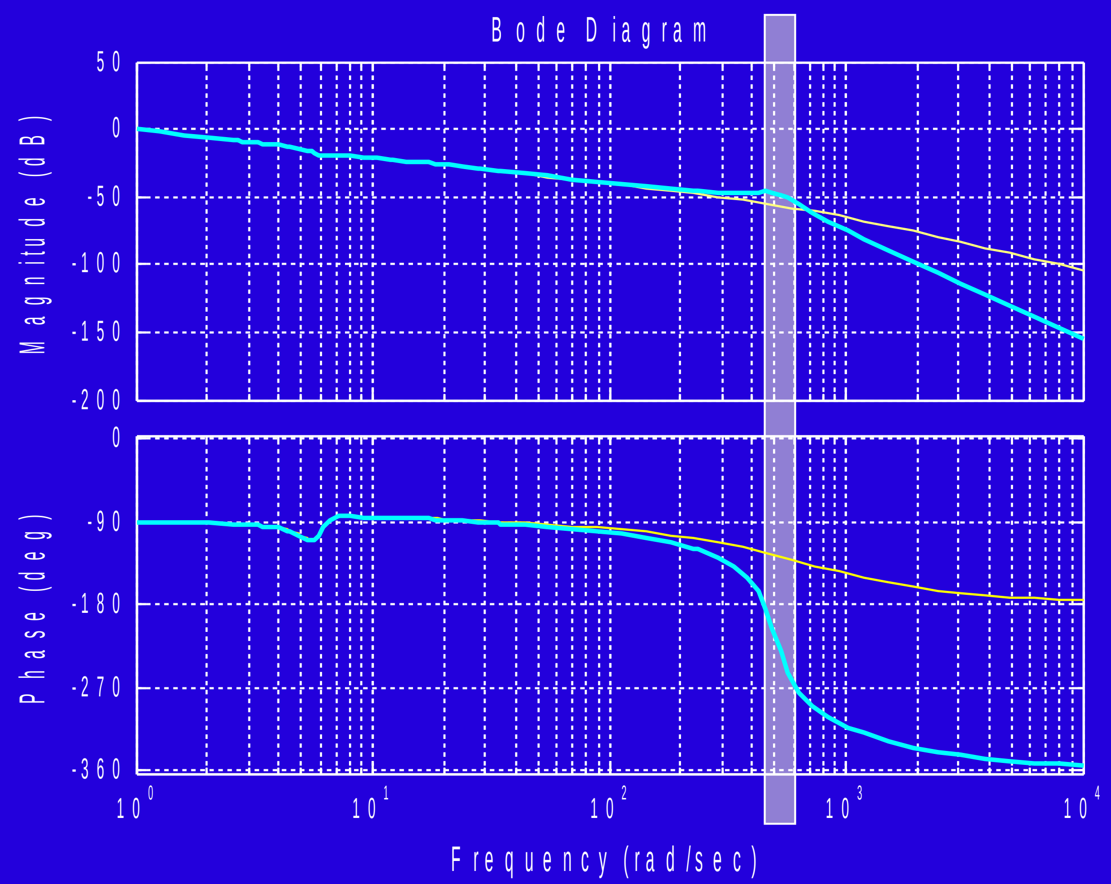
With reference to the Generalized Sub-Optimal controller only parameter β can be considered since optimization is not much sensitive with respect to α^*

L3 – System Dynamics Shaping

The introduction of a low-pass filter “compensates for” the actuator dynamics in the range of frequency of the steady state oscillations

Nominal plant

Nominal plant plus parasitic dynamics



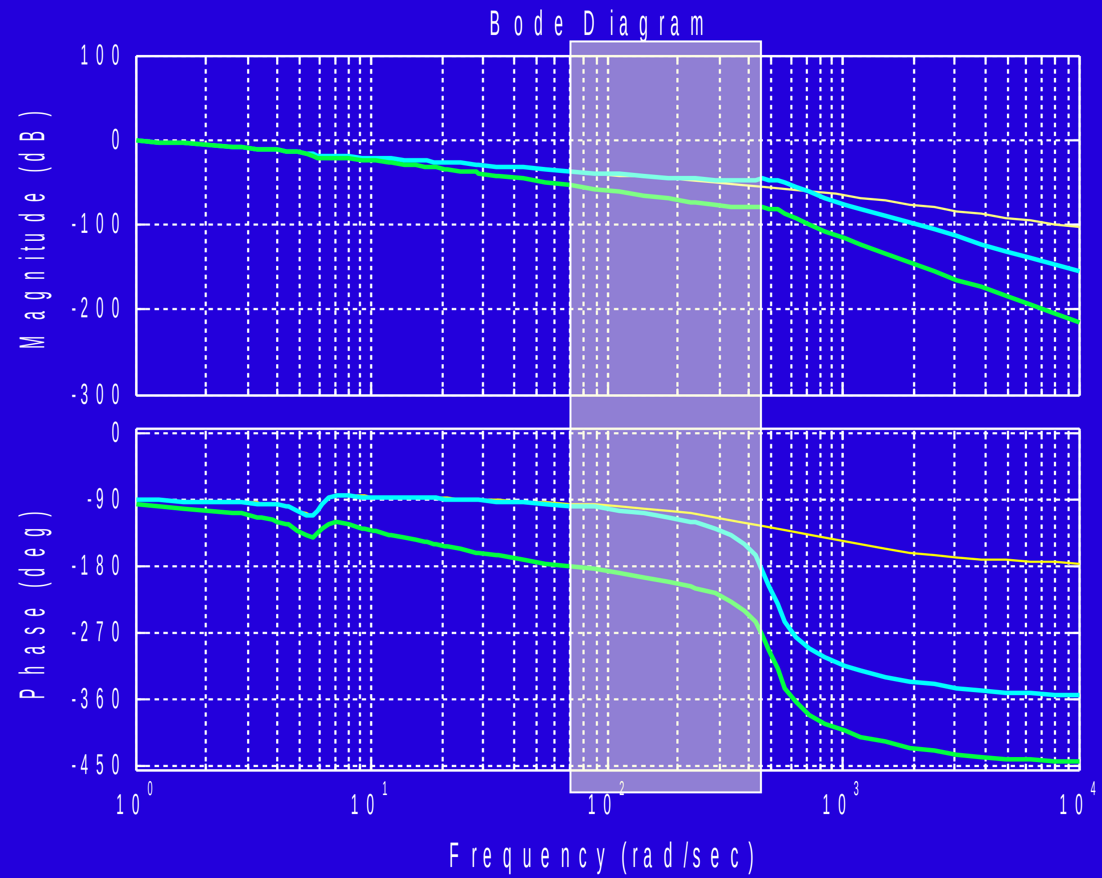
L3 – System Dynamics Shaping

The introduction of a low-pass filter "compensates for" the actuator dynamics in the range of frequency of the steady state oscillations

Nominal plant

Nominal plant plus parasitic dynamics

Overall Harmonic Response with the *shaping* low-pass filter τ=0.1s

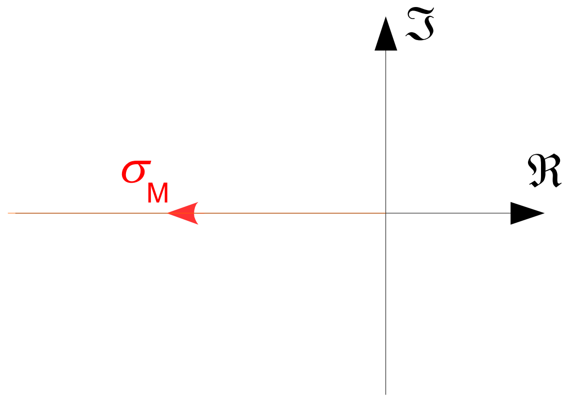


L3 – System Dynamics Shaping

The compensating filter design has to take into account that each sliding mode controller has its own DF:

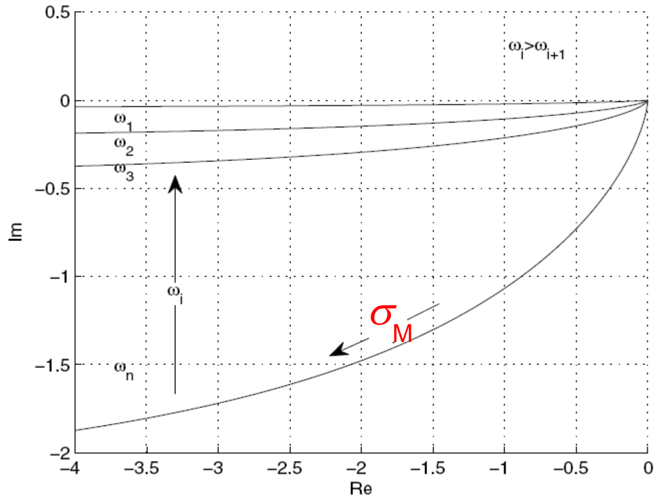
Relay control - $u = -U\, sign(\sigma)$

$$N(\omega, U) = \frac{4U}{\pi \sigma_M}$$



Super-Twisting - $\begin{aligned} u &= -\lambda |\sigma|^{0.5} sign(\sigma) + v \\ \dot{v} &= -\gamma\, sign(\sigma) \end{aligned}$

$$N(\omega, U) = \frac{2\gamma |\sigma_M|^{0.5} \Gamma(1.25)}{\sigma_M \sqrt{\pi}\, \Gamma(1.75)} - j \frac{4\gamma}{\omega \pi \sigma_M}$$



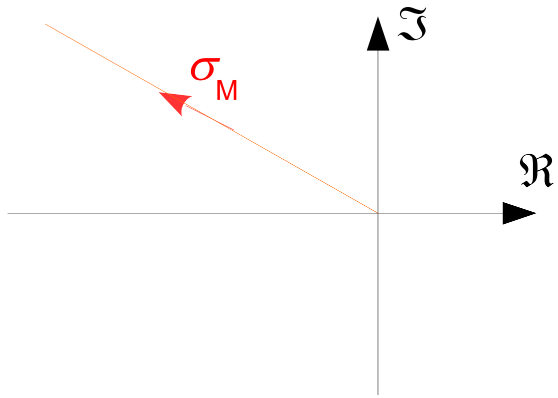
L3 – System Dynamics Shaping

The compensating filter design has to take into account that each sliding mode controller has its own DF:

Twisting - $$u = -c_1 \, sign(\sigma) - c_2 \, sign(\dot{\sigma})$$

$$c_1 > c_2$$

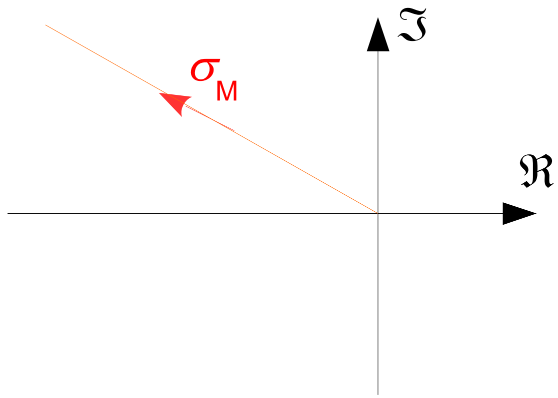
$$N(\omega, U) = -\frac{4}{\pi \sigma_M}(c_1 + j c_2)$$



Generalised Sub-Optimal - $$u = -U \, sign(\sigma - \beta \sigma_{ex})$$

$$0 \leq \beta < 1$$

$$N(\omega, U) = -\frac{4U}{\pi \sigma_M}\left(\sqrt{1-\beta^2} + j \beta\right)$$



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