Institut für Regelungs- und Automatisierungstechnik
Graz University of Technology

Sliding Mode Control: Basic Theory, Advances and Applications

Elio USAI

eusai@diee.unica.it

Dept of Electrical and Electronic Eng.
University of Cagliari

Lecture 2

Higher Order Sliding Mode Control
in
Variable Structure Systems

- Higher Order Sliding Modes in Variable Structure Systems
- Control problem statement
- Dynamic Higher Order Sliding Mode Control
- Terminal Higher Order Sliding Mode Control
- Single-Input “direct” 2nd Order Sliding Mode Control
- Multi-Input “direct” 2nd Order Sliding Mode Control
- Single-Input 3rd Order Sliding Mode Control
- Arbitrary Order Sliding Mode Control

L2 – HOSM in VSS

Classical sliding mode are characterized by constraining the state evolution onto a surface of the state space by means of a switching control

$$\dot{\mathbf{x}} = f\left(\mathbf{x},\mathbf{u},t\right) \qquad \mathbf{x} \in \mathcal{G} \subset \mathrm{R}^n \quad \mathbf{u} \in R^q$$

$$\mathcal{G} = \left\{\mathbf{x} \in \mathrm{R}^n : \boldsymbol{\sigma}\left(\mathbf{x},t\right) = 0\right\} \qquad \boldsymbol{\sigma} :\in \mathrm{R}^n \times \mathrm{R}^+ \to R^q$$

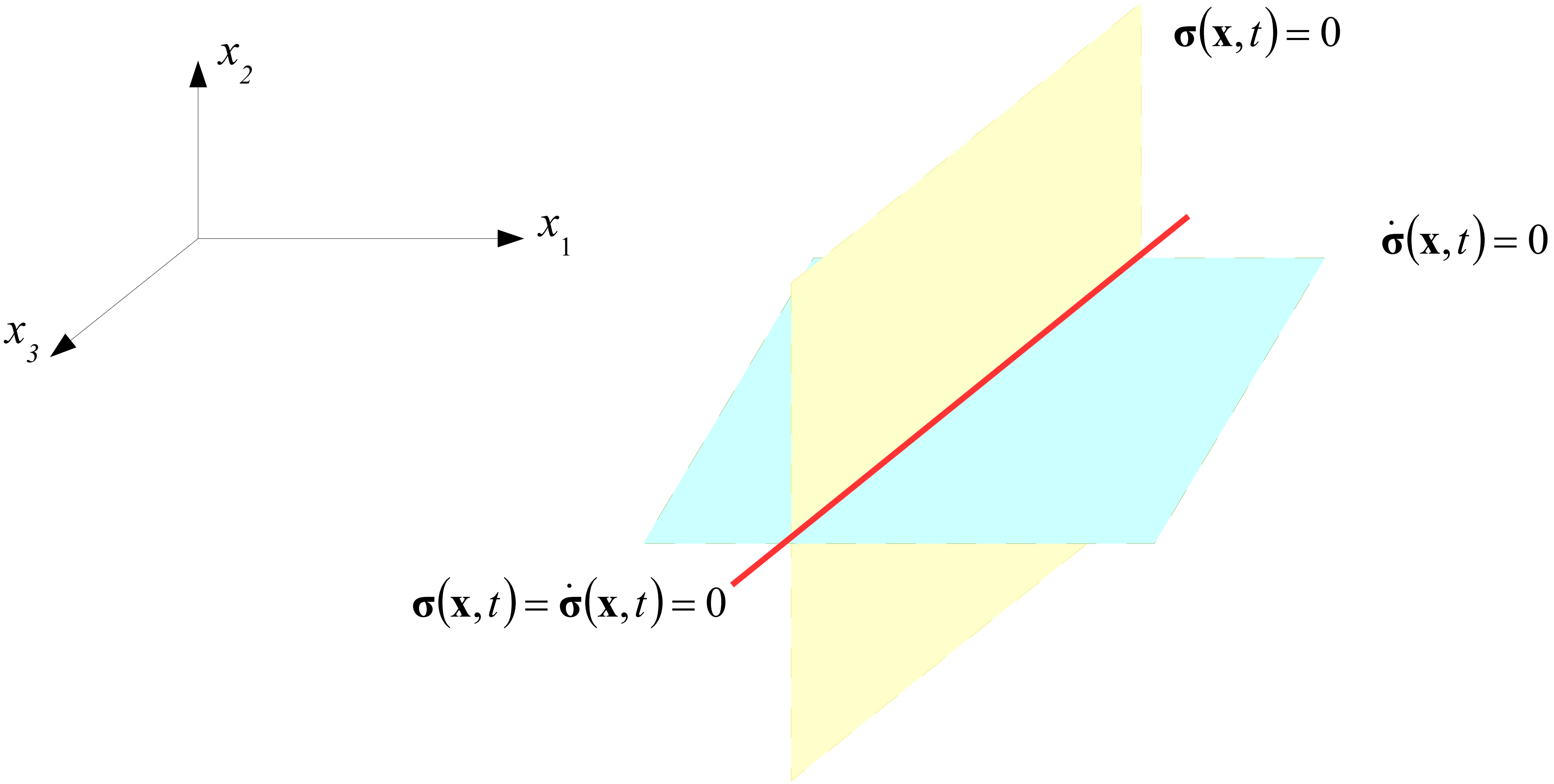
When the differential inclusion defining the closed loop dynamics belongs to the tangential space of the surface $\mathcal{G}$, a 2nd Order Sliding Mode (2-Sliding Mode) appears

$$\dot{\mathbf{x}} = f\left(\mathbf{x},\mathbf{u},t\right) \qquad \mathbf{x} \in \mathcal{G}_2 \subset \mathrm{R}^n \quad \mathbf{u} \in R^q$$

$$\mathcal{G}_2 = \left\{\mathbf{x} \in \mathrm{R}^n : \boldsymbol{\sigma}\left(\mathbf{x},t\right) = \dot{\boldsymbol{\sigma}}\left(\mathbf{x},t\right) = 0\right\} \qquad \boldsymbol{\sigma} :\in \mathrm{R}^n \times \mathrm{R}^+ \to R^q$$

The 2nd Order Sliding Mode is characterized by a n-2q reduced order dynamics

L2 – HOSM in VSS

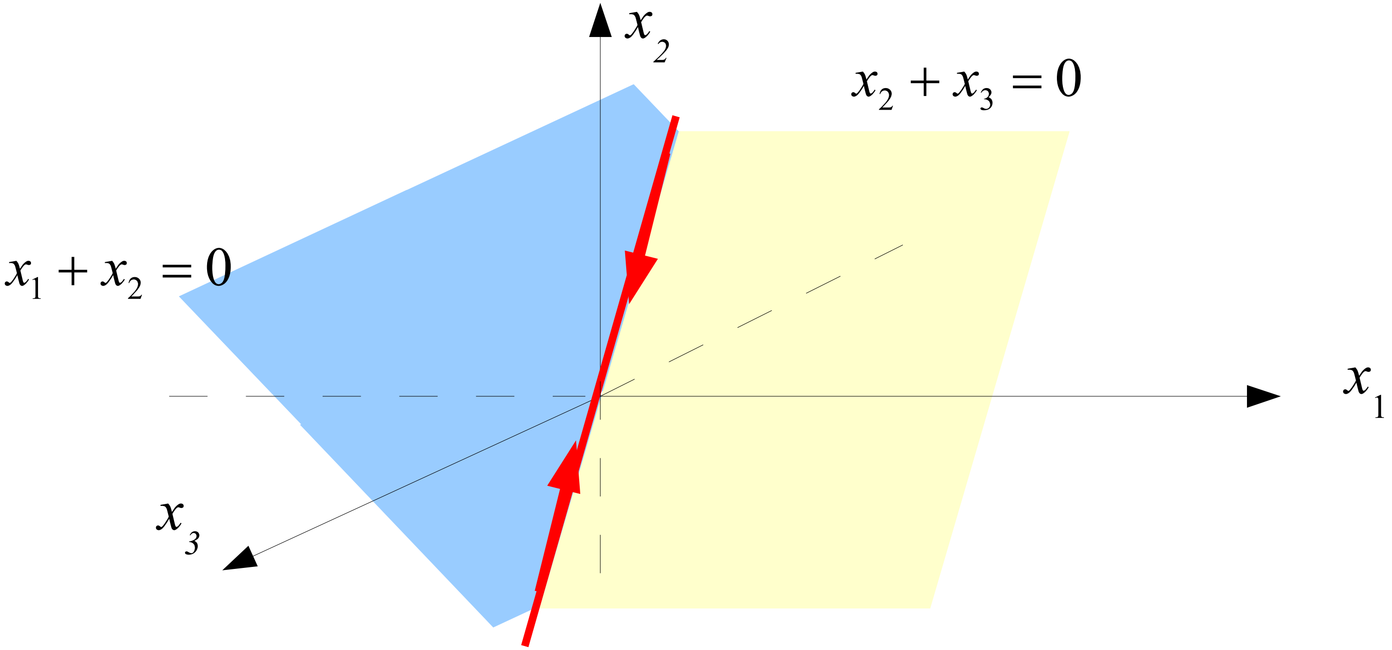


The constraint $\dot{\boldsymbol{\sigma}}(\mathbf{x},t)=0$ depends on the system dynamics as $\dot{\boldsymbol{\sigma}}(\mathbf{x},t)=\frac{\partial\boldsymbol{\sigma}}{\partial\mathbf{x}}\cdot f(\mathbf{x},\mathbf{u},t)+\frac{\partial\boldsymbol{\sigma}}{\partial t}$

L2 – HOSM in VSS

Example

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_3\\ \dot{x}_3 &= f(\mathbf{x},u,t)\\ y &= x_1 + x_2\end{aligned} \;\Rightarrow\; \begin{bmatrix} y \\ \dot{y} \\ w \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \;\Rightarrow\; \begin{cases} \ddot{y} = x_3 + f(y,\dot{y},w,u,t) \\ \dot{w} = -w + y \end{cases}$$



L2 – HOSM in VSS

The definition of Higher Order Sliding Mode can be extended to a r-order sliding surface

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, t) \qquad \mathbf{x} \in \mathcal{G}_r \subset \mathrm{R}^n \quad \mathbf{u} \in R^q$$

$$\mathcal{G}_r = \left\{ \mathbf{x} \in \mathrm{R}^n : \frac{d^k \boldsymbol{\sigma}(\mathbf{x}, t)}{dt^k} = 0, k = 0,1,\ldots,r-1 \right\} \qquad \boldsymbol{\sigma} :\in \mathrm{R}^n \times \mathrm{R}^+ \rightarrow R^q$$

σ is a sufficiently smooth vector function: $\boldsymbol{\sigma} \in \mathcal{C}^{r-1}$

Discontinuity appears in the r$^{\text{th}}$ order time derivative of **σ**

The overall number of constraints must be less than the system order

n>rq

L2 – HOSM in VSS

Definition

Let the r-sliding set $\mathcal{G}_r$ be non-empty and assume that it is locally an integral set in Filippov's sense (i.e. it consists of Filippov's trajectories of the discontinuous dynamic system). Then the corresponding motion of the system state belonging to the set $\mathcal{G}_r$ is called r-Sliding Mode (r-SM) with respect to the constraint function σ.

L2 – HOSMC: Problem statement

A Higher–Order Sliding Mode Control (HOSMC) system is implemented when the control $\mathbf{u}$ is able to constrain the system state onto the set $\mathcal{G}_r$ starting from any point in a ε-vicinity of the set $\mathcal{G}_r$.

Finding a control law such that it is able to enforce a r-SM on a surface in the state space is a difficult task for generic nonlinear systems

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, t)$$

A simpler system dynamics is usually considered

L2 – HOSMC: Problem statement

$$\dot{\bar{\mathbf{x}}} = \bar{\mathbf{f}}\left(\bar{\mathbf{x}},\bar{\mathbf{u}},t\right)$$

$$\Downarrow$$

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}} \\ \dot{\hat{x}}_0 \\ \dot{\bar{\mathbf{u}}} \\ \dot{\hat{\mathbf{x}}} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{x}} \\ 1 \\ \mathbf{0} \\ \dfrac{\partial \bar{f}\left(\bar{\mathbf{x}},\bar{\mathbf{u}},\hat{x}_0\right)}{\partial \bar{\mathbf{x}}} \cdot \hat{\mathbf{x}} + \dfrac{\partial \bar{f}\left(\bar{\mathbf{x}},\bar{\mathbf{u}},\hat{x}_0\right)}{\partial \hat{x}_0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{I}_q \\ \dfrac{\partial \bar{f}\left(\bar{\mathbf{x}},\bar{\mathbf{u}},\hat{x}_0\right)}{\partial \bar{\mathbf{u}}} \end{bmatrix} \mathbf{u}$$

$$\Downarrow$$

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}\right) + \mathbf{g}\left(\mathbf{x}\right)\mathbf{u}$$

$\mathbf{f}(\mathbf{x})$: drift vector function

$\mathbf{g}(\mathbf{x})$: gain matrix function

$\mathbf{x}$: state variables vector

$\mathbf{u}$: control variables vector

L2 – HOSMC: Problem statement

Notation

$$h(\mathbf{x}): \mathrm{R}^n \to \mathrm{R} \quad \Rightarrow \quad \nabla h(\mathbf{x}) = \frac{\partial h}{\partial \mathbf{x}}$$

Gradient vector

$$\mathbf{h}(\mathbf{x}): \mathrm{R}^n \times \mathrm{R}^q \to \mathrm{R}^{\mathrm{p}} \quad \Rightarrow \quad \nabla \mathbf{h}(\mathbf{x}) = \begin{bmatrix} \nabla h_1(\mathbf{x}) \\ \vdots \\ \nabla h_1(\mathbf{x}) \end{bmatrix} = \mathbf{J}^{\mathbf{h}}$$

Jacobian matrix

$$\begin{array}{l} h(\mathbf{x}): \mathrm{R}^n \to \mathrm{R} \\ \mathbf{g}(\mathbf{x}): \mathrm{R}^n \to \mathrm{R}^n \end{array} \quad \Rightarrow \quad \begin{array}{c} L_{\mathbf{g}} h = \nabla h(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x}) \\ L_{\mathbf{g}}^k h = \nabla\left(L_{\mathbf{g}}^{k-1} h\right) \cdot \mathbf{g}(\mathbf{x}) \quad i = 1,2,\ldots \quad L_{\mathbf{g}}^0 h = h(\mathbf{x}) \end{array}$$

Lie derivatives

$$\begin{array}{l} \mathbf{h}(\mathbf{x}): \mathrm{R}^n \to \mathrm{R}^{\mathrm{p}} \\ \mathbf{g}(\mathbf{x}): \mathrm{R}^n \to \mathrm{R}^n \end{array} \quad \Rightarrow \quad \begin{array}{c} L_{\mathbf{g}} \mathbf{h} = \nabla \mathbf{h}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x}) = \mathbf{J}^{\mathbf{h}} \cdot \mathbf{g}(\mathbf{x}) \\ L_{\mathbf{g}}^k \mathbf{h} = \nabla\left(L_{\mathbf{g}}^{k-1} \mathbf{h}\right) \cdot \mathbf{g}(\mathbf{x}) \quad i = 1,2,\ldots \quad L_{\mathbf{g}}^0 \mathbf{h} = \mathbf{h}(\mathbf{x}) \end{array}$$

L2 – HOSMC: Problem statement

The Higher Order Sliding Mode Control problem is to define a suitable output function $\boldsymbol{\sigma}(t)$ and a proper control law $\mathbf{u}(t)$ such that the state trajectory of the dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}$$

$$\mathbf{x} \in \mathrm{R}^n \quad \mathbf{u} \in R^q$$

$$\mathbf{f} : \mathrm{R}^n \to R^n \quad \mathbf{g} : \mathrm{R}^n \to \mathrm{R}^n \times R^q$$

is constrained onto the r-order sliding surface

$$G_r = \left\{ \mathbf{x} \in \mathrm{R}^n : \frac{d^k \boldsymbol{\sigma}(\mathbf{x})}{dt^k} = 0, k = 0,1,\ldots,r-1 \right\} \quad \boldsymbol{\sigma} :\in \mathrm{R}^n \times \mathrm{R}^+ \to R^q \quad \mathbf{u} \in R^q$$

from a time instant t_∞ on, possibly in finite time ($t_\infty \leq T < \infty$).

L2 – HOSMC: Problem statement

Given the control affine system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}$$

$$\mathbf{x} \in \mathrm{R}^n \quad \mathbf{u} \in R^q$$

$$\mathbf{f} : \mathrm{R}^n \to R^n \quad \mathbf{g} : \mathrm{R}^n \to \mathrm{R}^n \times R^q$$

A r-order Sliding Mode Control on the surface $\boldsymbol{\sigma}(\mathbf{x})=\mathbf{0}$ can be designed if the following conditions are fulfilled

$$L_{\mathbf{g}} L_{\mathbf{f}}^{k} \boldsymbol{\sigma} = 0 \quad i = 0,2,\ldots,r-2$$

The control does not appear in the first r-1 derivatives of the system output $\boldsymbol{\sigma}(\mathbf{x})$

$L_{\mathbf{g}} L_{\mathbf{f}}^{r-1} \boldsymbol{\sigma}$ has full rank

The output variable $\boldsymbol{\sigma}(\mathbf{x})$ is completely controllable by the $\mathbf{u}(\mathbf{x})$ control

L2 – HOSMC: Problem statement

Several HOSMC algorithms were presented in the literature, here are some

relative degree	sliding order	feedback signals	VSC	main “*inventors*”
1	1	$\boldsymbol{\sigma}$	classic 1-SMC	*Utkin et al.*
	2	$\boldsymbol{\sigma}$	super-twisting	*Emilyanov, Levant*
2	2	$\boldsymbol{\sigma}\,;\,\dot{\boldsymbol{\sigma}}$	terminal SM	*Zhihong, Yu*
		$sign(\boldsymbol{\sigma});\, sign(\dot{\boldsymbol{\sigma}})$	twisting	*Emilyanov, Levant*
		$\boldsymbol{\sigma}\,;\,\boldsymbol{\sigma}(t-\tau)$	sub-optimal	*Bartolini et al.*
3	3	$sign(\sigma);\, sign(\dot{\sigma});\, sign(\ddot{\sigma})$	hybrid-VSC	*Bartolini et al.*
		σ	3-SMO	*Moreno & Dochain*
r	r	$sign(\sigma);\cdots;\, sign\left(\sigma^{(r-1)}\right)$	hybrid-VSC	*Polyakov*
		$\boldsymbol{\sigma}\,;\cdots;\,\boldsymbol{\sigma}^{(r-1)}$	dynamic SM	*Sira-Ramirez*
		$\sigma\,;\cdots;\,\sigma^{(r-2)};\, sign\left(\sigma^{(r-1)}\right)$	universal HOSM	*Levant*

L2 – Dynamic HOSMC

Dynamic Higher Order sliding Mode Control is based on the definition of an augmented output variable

$$\mathbf{s}(\mathbf{x}) = \boldsymbol{\sigma}^{(r-1)} + \sum_{i=0}^{r-2} c_i \boldsymbol{\sigma}^{(i)}(\mathbf{x})$$

Coefficients c_i are chosen so that the polynomial $P(p)$ has all roots with negative real part

$$P(p) = p^{r-1} + \sum_{i=0}^{r-2} c_i p^i$$

The control $\mathbf{u}(t)$ affects the time derivative of $\mathbf{s}(\mathbf{x})$, and if it is designed so that $\mathbf{s}(\mathbf{x})$ is stabilized to zero, each output variable σ_i, and its *r-1* time derivatives, are stabilized to zero asymptotically

L2 – Dynamic HOSMC

Theorem.

Consider the system dynamics $$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}$$

Chose the sliding variable set $\boldsymbol{\sigma}(\mathbf{x})$ such that the internal dynamics corresponding to the output variables $\boldsymbol{\sigma}, \dot{\boldsymbol{\sigma}}, \ddot{\boldsymbol{\sigma}}, \ldots, \boldsymbol{\sigma}^{(r-1)}$ is IIS.

Assume that the uncertainties are bounded by a known function, and that the gain matrix of the input-output dynamics is definite positive

$$\begin{aligned} &\left\| L_{\mathbf{f}}^{k}\boldsymbol{\sigma} \right\| \le \Lambda_k(\mathbf{x}) && k = 1,2,\ldots,r \\ &L_{\mathbf{g}}L_{\mathbf{f}}^{k}\boldsymbol{\sigma} = \mathbf{0} && k = 0,1,\ldots,r-2 \\ &0 < \Lambda_m \le \min\left\{ eig\left[L_{\mathbf{g}}L_{\mathbf{f}}^{r-1}\boldsymbol{\sigma} \right] \right\} && \forall \mathbf{x} \in \mathsf{V}_{\varepsilon}, \forall t \end{aligned}$$

Define an auxiliary output as a *stable* linear combination of the sliding variable and its r-1 time derivatives

$$\mathbf{s}(\mathbf{x}) = \boldsymbol{\sigma}^{(r-1)} + \sum_{i=0}^{r-2} c_i \boldsymbol{\sigma}^{(i)}(\mathbf{x})$$

The state feedback control law assures the asymptotic stability of the r-SM on the sliding surface $\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{0}$

$$\mathbf{u} = -\frac{\Lambda_r(\mathbf{x}) + \sum_{i=0}^{r-2} c_i \Lambda_{i+1}(\mathbf{x}) + \eta}{\Lambda_m} \frac{\mathbf{s}}{\left\| \mathbf{s} \right\|_2} \qquad \eta > 0$$

L2 – Dynamic HOSMC

Proof.

Auxiliary sliding variable dynamics
$$\dot{\mathbf{s}}(t) = L_{\mathbf{f}}^{r}\boldsymbol{\sigma} + \sum_{i=0}^{r-2} c_i L_{\mathbf{f}}^{i+1}\boldsymbol{\sigma} + L_{\mathbf{g}}L_{\mathbf{f}}^{r-1}\boldsymbol{\sigma}\cdot\mathbf{u}(t)$$

Lyapunov function
$$V(\mathbf{s}) = \frac{1}{2}\mathbf{s}^{T}\cdot\mathbf{s}$$

$$\begin{aligned}\dot{V} &= \boldsymbol{s}^{\boldsymbol{T}}\ \boldsymbol{s} \\ &= \mathbf{s}^{T}\cdot\left[L_{\mathbf{f}}^{r}\boldsymbol{\sigma} + \sum_{i=0}^{r-2} c_i L_{\mathbf{f}}^{i+1}\boldsymbol{\sigma} - \frac{\Lambda_r(\mathbf{x}) + \sum_{i=0}^{r-2} c_i\Lambda_{i+1}(\mathbf{x}) + \eta}{\Lambda_m} L_{\mathbf{g}}L_{\mathbf{f}}^{r-1}\boldsymbol{\sigma}\cdot\frac{\mathbf{s}}{\|\mathbf{s}\|_2}\right] \\ &\le -\frac{\eta}{\Lambda_m\|\mathbf{s}\|_2}\mathbf{s}^{T}\cdot L_{\mathbf{g}}L_{\mathbf{f}}^{r-1}\boldsymbol{\sigma}\cdot\mathbf{s} \le -\eta\|\mathbf{s}\|_2 < -\eta\sqrt{V} < 0\end{aligned}$$

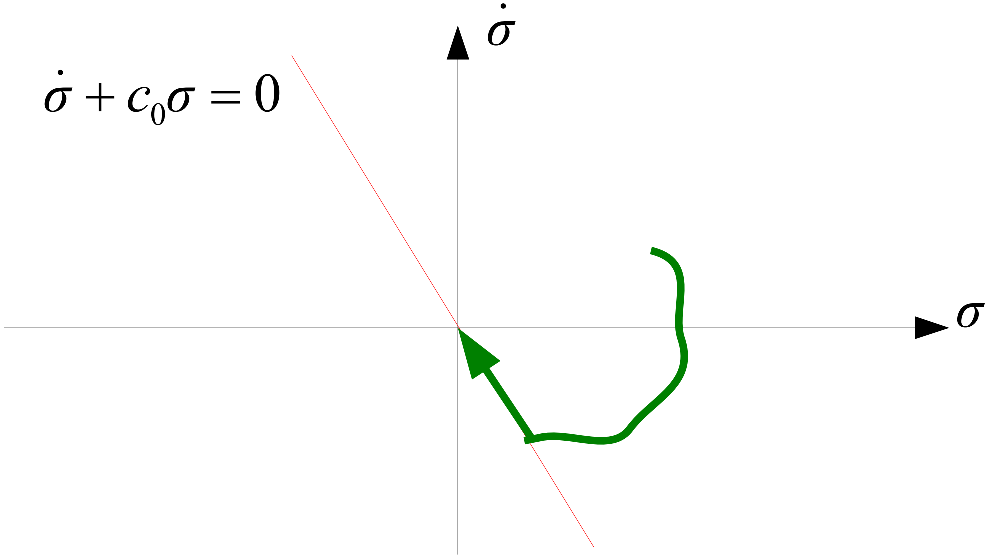
Once the condition $\mathbf{s}=\mathbf{0}$ is fulfilled, each sliding variable motion is characterized by a **asymptotically** stable linear dynamics.

L2 – Dynamic HOSMC

Dynamic Higher Order Sliding Modes are an extension of the classic 1-SM

The Higher Order Sliding Mode behavior is reached asymptotically

Perfect knowledge of the time derivatives of the sliding variable up to the r-1 order are needed



L2 – Terminal 2-SMC

Terminal 2nd-Order sliding Mode Control is based on the properties of finite time control by means of the definition of an augmented nonlinear output variable

$$\mathbf{s}(\mathbf{x}) = \boldsymbol{\sigma}(\mathbf{x}) + \frac{1}{\beta}\dot{\boldsymbol{\sigma}}^{p/m}$$

Coefficients p and m ($p>m$) are odd numbers so that the motion on the hyper-surface $\mathbf{s} = \mathbf{0}$ in the phase space is characterized by the finite time reaching of the origin

The control $\mathbf{u}(t)$ affects the time derivative of $\mathbf{s}(\mathbf{x})$, and if it is designed so that $\mathbf{s}(\mathbf{x})$ is stabilized to zero, each output variable σ_i, and its time derivative, is stabilized to zero in a finite time

L2 – Terminal 2-SMC

Theorem.

Consider the system dynamics

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}$$

Chose the sliding variable set $\boldsymbol{\sigma}(\mathbf{x})$ such that the internal dynamics corresponding to the output variables $\boldsymbol{\sigma}, \dot{\boldsymbol{\sigma}}$ is BIBS stable.

Assume that the uncertainties are bounded by a known function and the gain matrix of the input-output dynamics is definite positive

$$\begin{cases} \left\| L_{\mathbf{f}}^{2}\boldsymbol{\sigma} \right\| \le \Lambda_2(\mathbf{x}) \\ L_{\mathbf{g}}\boldsymbol{\sigma} = \mathbf{0} \\ \exists \ \left[L_{\mathbf{g}} L_{\mathbf{f}}^{r-1}\boldsymbol{\sigma} \right]^{-1} \end{cases} \qquad \forall \mathbf{x} \in V_{\varepsilon}, \forall t$$

Define an auxiliary output as a *stable* nonlinear combination of the sliding variable and its time derivatives

$$\mathbf{s}(\mathbf{x}) = \boldsymbol{\sigma}(\mathbf{x}) + \frac{1}{\beta}\dot{\boldsymbol{\sigma}}^{p/m}$$

The state feedback control law assures the finite time stability of the 2-SM on the sliding surface $\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{0}$

$$\mathbf{u} = -\left[L_{\mathbf{g}} L_{\mathbf{f}}^{r-1}\boldsymbol{\sigma} \right]^{-1} \left[\beta \frac{m}{p} \dot{\boldsymbol{\sigma}}^{2-p/m} + (\Lambda_2 + \eta)\operatorname{sgn}(\mathbf{s}) \right] \qquad \eta > 0$$

L2 – Terminal 2-SMC

Proof.

Auxiliary sliding variable dynamics
$$\dot{\mathbf{s}}(t) = \dot{\boldsymbol{\sigma}} + \frac{1}{\beta}\frac{p}{m} diag\left\{\dot{\sigma}_i^{2-p/m}\right\}\left(L_{\mathbf{f}}^2\boldsymbol{\sigma} + L_{\mathbf{g}}L_{\mathbf{f}}\boldsymbol{\sigma}\cdot\mathbf{u}(t)\right)$$

Lyapunov function
$$V(\mathbf{s}) = \frac{1}{2}\mathbf{s}^T\cdot\mathbf{s}$$

$$\begin{aligned}
\dot{V} &= \frac{1}{\beta}\frac{p}{m}\mathbf{s}^T\cdot diag\left\{\dot{\sigma}_i^{2-p/m}\right\}\left(L_{\mathbf{f}}^2\boldsymbol{\sigma} + L_{\mathbf{g}}L_{\mathbf{f}}\boldsymbol{\sigma}\cdot\mathbf{u}(t)\right)\\
&= \frac{1}{\beta}\frac{p}{m}\mathbf{s}^T\cdot diag\left\{\dot{\sigma}_i^{2-p/m}\right\}\left[L_{\mathbf{f}}^2\boldsymbol{\sigma} - \left(\beta\frac{m}{p}\dot{\boldsymbol{\sigma}}^{2-p/m} + (\Lambda_2+\eta)\operatorname{sgn}(\mathbf{s})\right)\right]\\
&\le -\frac{\eta}{\beta}\frac{p}{m}\left|\mathbf{s}^T\right|\cdot\dot{\boldsymbol{\sigma}}^{p/m-1} < 0 \quad \forall\dot{\boldsymbol{\sigma}}\neq\mathbf{0}\\
\ddot{\boldsymbol{\sigma}} &\le -\eta\operatorname{sgn}(\boldsymbol{\sigma}) \quad \forall\dot{\boldsymbol{\sigma}}=\mathbf{0}
\end{aligned}$$

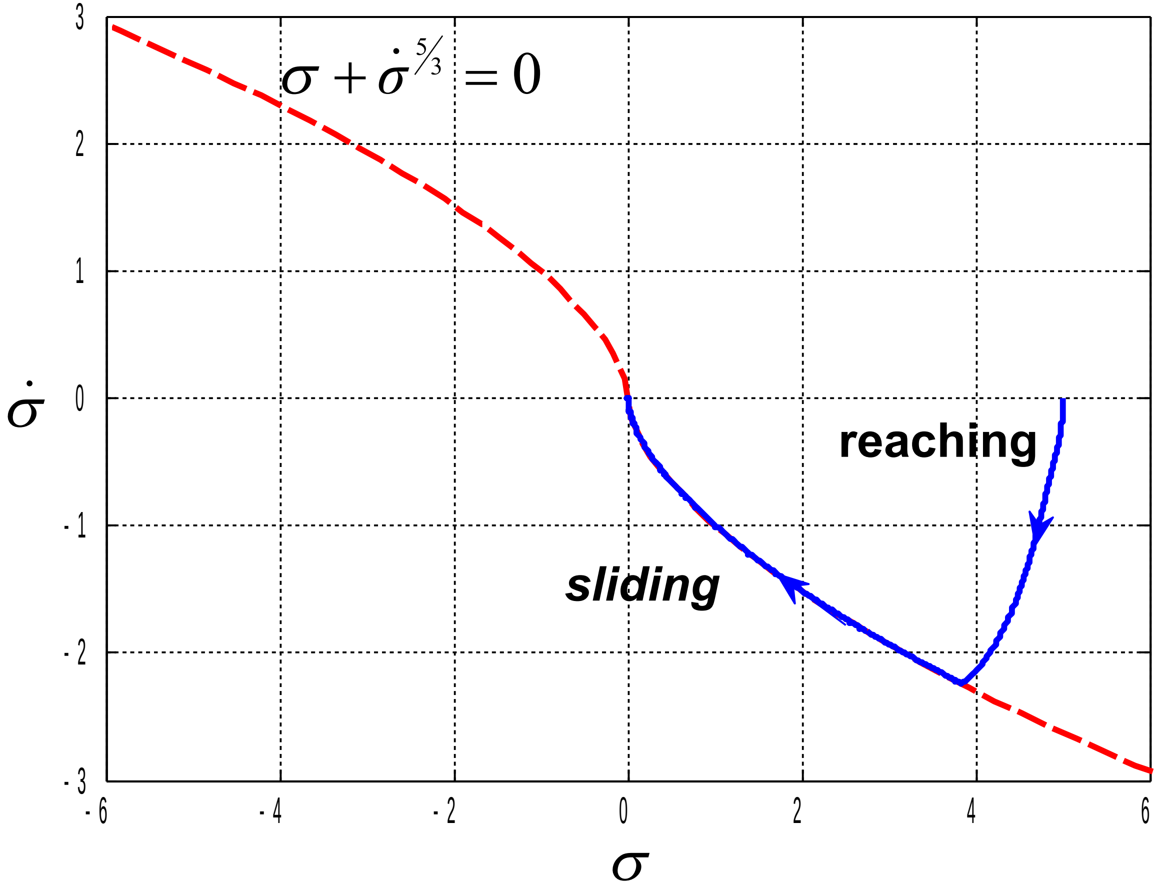
Once the condition $\mathbf{s}=\mathbf{0}$ is fulfilled, each sliding variable motion is characterized by a **finite time** stable nonlinear dynamics.

L2 – Terminal 2-SMC

Terminal 2nd Order Sliding Modes are the multi-variable non-singular implementation of the "*prescribed law of variation*" 2-SMC

The 2nd Order Sliding Mode behavior is reached in finite time

Perfect knowledge of the first time derivatives of the sliding variable order is needed



L2 – Single Input 2-SMC

Both dynamic HOSM and terminal 2-SM are implementations of the classical sliding mode control approach but with a peculiar sliding manifold

Several algorithms that implement a 2nd Order Sliding Mode directly have been presented in the literature

They are based on peculiar trajectories on the $(\sigma\,;\dot{\sigma})$ plane

Because of the difficulties in defining Lyapunov functions for proving their finite time stability, a usual first step presenteation refers to Single Input systems

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u \qquad \mathbf{x} \in \mathcal{G} \subset \mathbb{R}^n \quad u \in \mathbb{R}$$
$$\mathcal{G} = \{\mathbf{x} \in \mathbb{R}^n : \sigma(\mathbf{x},t) = 0\} \qquad \sigma :\in \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$$

L2 – Single Input 2-SMC

There are two main families of "*direct*" 2nd order sliding mode control algorithms:

SuperTwisting: it is designed for relative degree 1 sliding surface (the control u appears in the first time derivative of the sliding variable)

The parameter tuning is based on the knowledge of an upper bound for the time derivative of the drift term

Only the sliding variable has to be available

Sub-Optimal: it is designed for relative degree 2 sliding surface (the control u appears in the second time derivative of the sliding variable)

The parameter tuning is based on the knowledge of an upper bound for the time derivative of the drift term and it can be dynamically set to have global convergence properties

The sliding variable has to be available as well as its past values to evaluate local extremal values

L2 – Single Input 2-SMC – Supertwisting

$$\dot{\sigma} = \varphi(\bullet) + \gamma(\bullet)u$$

$$u = -\lambda\sqrt{|\sigma|}\,\mathrm{sgn}(\sigma) - \alpha\int \mathrm{sgn}(\sigma)\,dt$$

$$\alpha > \frac{\Phi}{\Gamma_m}$$

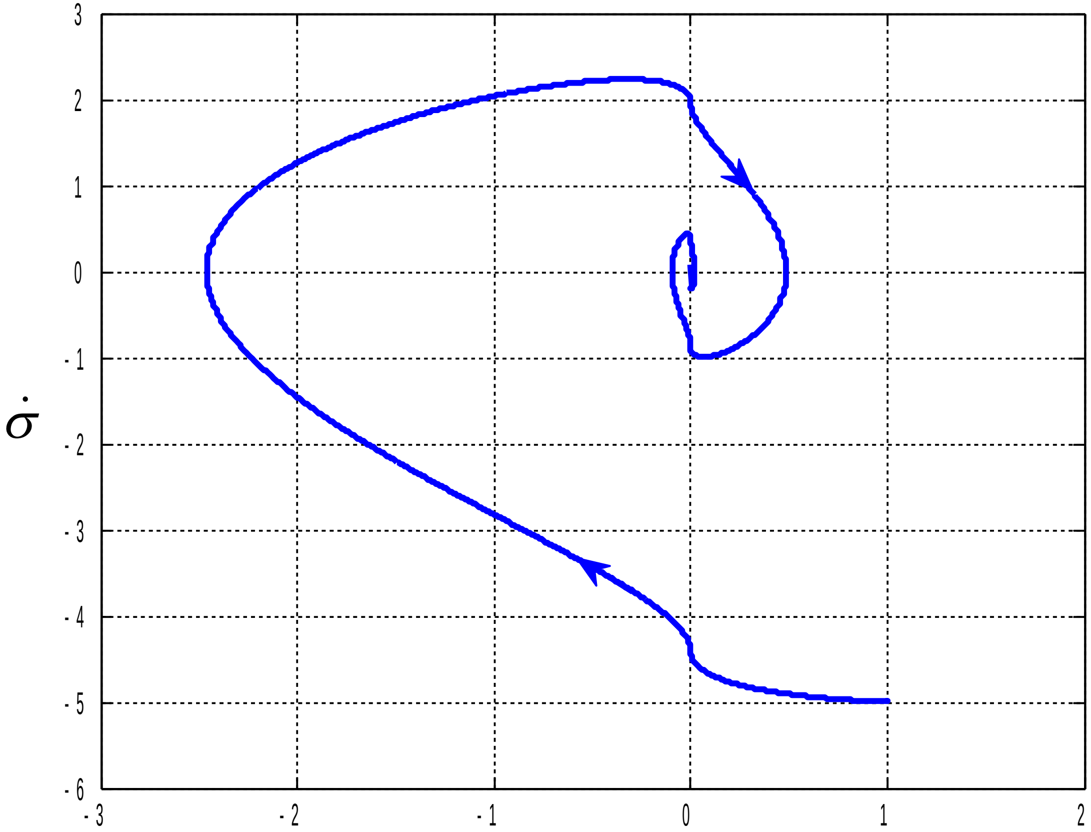
$$\lambda^2 > 2\frac{\Phi + \alpha\Gamma_M}{\Gamma_m}$$

Φ: upper bound of the “drift” term

Γ_m, Γ_M: positive lower and upper bounds of the gain term

The control is continuous

The control is based on the measure of the sliding variable only



L2 – Single Input 2-SMC – Supertwisting

Theorem.

Consider the system dynamics

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u$$

Chose the sliding variable set $\sigma(\mathbf{x})$ such that the internal dynamics corresponding to the output variables $\sigma, \dot{\sigma}$ is BIBS stable.

Assume that that the sliding dynamics has relative degree 1 and constant bounds for the uncertain drift and gain vector are known

$$\begin{cases} \left| L_{\mathbf{f}}^2\sigma + \left(L_{\mathbf{f}}L_{\mathbf{g}}\sigma + L_{\mathbf{g}}L_{\mathbf{f}}\sigma\right)u + u^2 L_{\mathbf{g}}^2\sigma \right| \le \Phi \\ 0 < \Gamma_m \le L_{\mathbf{g}}\sigma \le \Gamma_M \end{cases}$$

$$\forall \mathbf{x} \in \mathsf{V}_\varepsilon, \forall t$$

Set the controller parameters such that the convergence conditions are fulfilled

$$\alpha > \frac{\Phi}{\Gamma_m}, \quad \lambda^2 > 2\frac{\Phi + \alpha\Gamma_M}{\Gamma_m}$$

The state feedback control law assures the asymptotic stability of the 2-SM on the sliding surface $\sigma(\mathbf{x}) = 0$

$$u = -\lambda\sqrt{|\sigma|}\,\mathrm{sgn}(\sigma) - \alpha\int \mathrm{sgn}(\sigma)\,dt$$

L2 – Single Input 2-SMC – Supertwisting

Proof.

Differential inclusion

$$\ddot{\sigma} \in \left[\alpha\Gamma_m - \Phi, \alpha\Gamma_M + \Phi\right]\operatorname{sgn}(\sigma) - \frac{\lambda}{2}\left[\Gamma_m, \Gamma_M\right]\frac{\dot{\sigma}}{\sqrt{|\sigma|}}$$

$$U > \frac{\Phi}{\Gamma_m}$$

Dominance condition

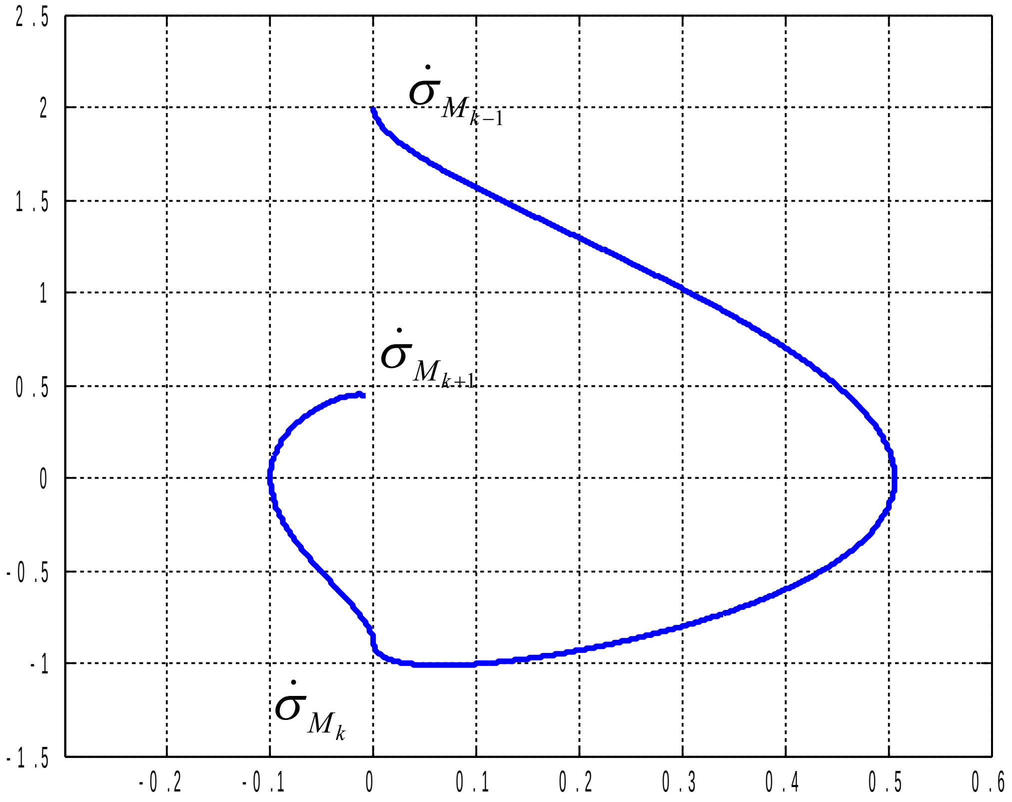
Convergence condition

$$\begin{cases} \left| L_{\mathbf{f}}^2\sigma + \left(L_{\mathbf{f}}L_{\mathbf{g}}\sigma + L_{\mathbf{g}}L_{\mathbf{f}}\sigma\right)u + u^2 L_{\mathbf{g}}^2\sigma \right| \le \Phi \\ 0 < \Gamma_m \le L_{\mathbf{g}}\sigma \le \Gamma_M \end{cases}$$

$$\alpha > \frac{\Phi}{\Gamma_m}, \quad \lambda^2 > 2\frac{\Phi + \alpha\Gamma_M}{\Gamma_m}$$

$$\Downarrow$$

$$\left|\dot{\sigma}_{M_{k+1}}\right| \le \rho\left|\dot{\sigma}_{M_k}\right|, \quad \rho \in (0,1)$$



A geometric convergent series of extremal points is generated and the 2-SM is achieved in a finite time

L2 – Single Input 2-SMC – Supertwisting

The Supertwisting 2nd Order Sliding Mode Control algorithm is based on mixing a nonlinear finite time first order feedback and an asymptotic second order switching logic

The reaching phase are characterized by a smooth twisting around the origin of the $\sigma,\dot{\sigma}$ plane

The Supertwisting algorithm is effective for SISO systems having relative degree 1

The 2_SM feature allows for easy implementation of an exact finite time convergent first order differentiator

L2 – Single Input 2-SMC – Supertwisting

For the Super-Twisting applied to a class of uncertain systems it is available a convergence proof based on a strict Lyapunov function.

Consider the system $\dot{\boldsymbol{x}}(t)=\boldsymbol{f}\left(\boldsymbol{x}(t)\right)+\boldsymbol{g}\left(\boldsymbol{x}(t)\right)u(t)$

Define the sliding variable $\sigma(t)=\sigma\left(\boldsymbol{x}(t)\right)$

such that $0<\Gamma_m\leq L_g\sigma\left(\boldsymbol{x}(t)\right)\leq\Gamma_M$

Then the sliding dynamics is $\dot{\sigma}(t)=L_{\boldsymbol{f}}\sigma\left(\boldsymbol{x}(t)\right)+L_{\boldsymbol{g}}\sigma\left(\boldsymbol{x}(t)\right)u(t)$

Defining the control as the STA the system dynamics results

$$\begin{aligned}\dot{\sigma}(t)&=L_{\boldsymbol{f}}\sigma-\lambda|\sigma|^{\frac{1}{2}}sign(\sigma)L_{\boldsymbol{g}}\sigma+L_{\boldsymbol{g}}\sigma*v(t)\\ \dot{v}(t)&=-k_2\,sign(\sigma)\end{aligned}$$

L2 – Single Input 2-SMC – Supertwisting

Define the auxiliary variables

$$z_1(t)=|\sigma|^{\frac{1}{2}}\,sign(\sigma(t))$$
$$z_2(t)=L_f\sigma(\boldsymbol{x}(t))+L_g\sigma(\boldsymbol{x}(t))*v(t)$$

The sliding dynamics is represented by

$$\dot{\boldsymbol{z}}(t)=\boldsymbol{A}(t)\boldsymbol{z}(t)+\boldsymbol{G}(t)$$

where

$$\boldsymbol{z}(t)=\begin{bmatrix} z_1(t) & z_2(t)\end{bmatrix}^T$$
$$\boldsymbol{A}(t)=\frac{1}{\sqrt{|\sigma|}}\begin{bmatrix} \frac{-\lambda\gamma(t)}{2} & \frac{1}{2} \\ -\alpha\gamma(t) & 0 \end{bmatrix}$$
$$\boldsymbol{G}(t)=\begin{bmatrix} 0 \\ \varphi(t)\end{bmatrix}$$
$$\varphi(t)=L_f^2\sigma(\boldsymbol{x}(t))+\left(L_f L_g\sigma(\boldsymbol{x}(t))+L_g L_f\sigma(\boldsymbol{x}(t))\right)u(t)+u^2(t)\,L_g^2\sigma(\boldsymbol{x}(t))$$
$$\gamma(t)=L_g\sigma(\boldsymbol{x}(t))$$

L2 – Single Input 2-SMC – Supertwisting

Define the scalar function $V(t)=\boldsymbol{z}^{\boldsymbol{T}}(t)\,\boldsymbol{P}\,\boldsymbol{z}(t);\quad \boldsymbol{P}=\boldsymbol{P}^{\boldsymbol{T}}>0$

Its time derivative results to be

$$\dot{V}(t)=\boldsymbol{z}^{\boldsymbol{T}}\left[\boldsymbol{A}^{\boldsymbol{T}}(t)\,\boldsymbol{P}+\boldsymbol{P}\,\boldsymbol{A}(t)\right]\boldsymbol{z}(t)+2\,\boldsymbol{G}(t)\,\boldsymbol{P}\,\boldsymbol{z}(t)$$

It will be a Lyapunov function provided that the following conditions are fulfilled

$$\boldsymbol{z}^{\boldsymbol{T}}\boldsymbol{Q}(t)\,\boldsymbol{z}(t)-\boldsymbol{G}(t)\,\boldsymbol{P}\,\boldsymbol{z}(t)>0\,;\quad \forall\,\boldsymbol{z}(t)$$

$$\boldsymbol{A}^{\boldsymbol{T}}(t)\,\boldsymbol{P}+\boldsymbol{P}\,\boldsymbol{A}(t)=-2\,\boldsymbol{Q}(t);\quad \boldsymbol{Q}(t)=\boldsymbol{Q}^{\boldsymbol{T}}(t)>0$$

Such a “robust” Lyapunov equation has been solved for the case of constant gain, which is useful for observers design, i.e.,

$$\gamma=L_g\,\sigma(\boldsymbol{x})=\Gamma_m=const.$$

L2 – Single Input 2-SMC – Sub Optimal

$$\ddot{\sigma} = \varphi(\bullet) + \gamma(\bullet)u$$

$$u = -\alpha(t)U\,\mathrm{sgn}(\sigma - \beta\sigma_M)$$

$$\alpha(t) = \begin{cases} 1 & (\sigma - \beta\sigma_M)\sigma_M \geq 0 \\ \bar{\alpha} & (\sigma - \beta\sigma_M)\sigma_M < 0 \end{cases}$$

$$\beta \in (0;1)$$

$$U > \frac{\Phi}{\Gamma_m}$$

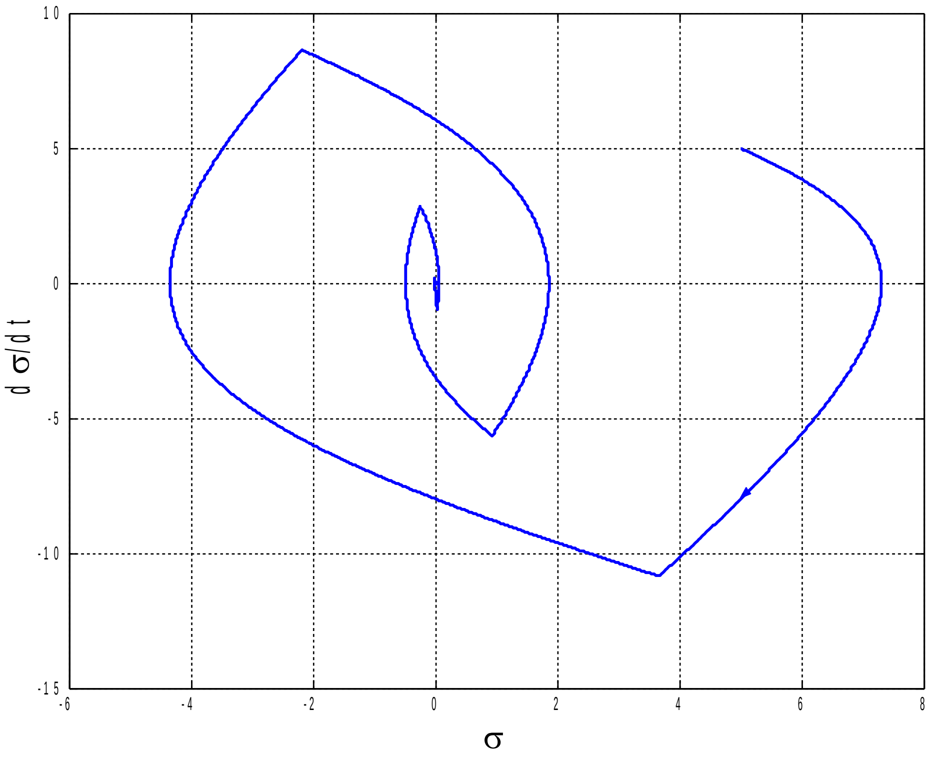
$$\bar{\alpha} \in [1;+\infty) \cap \left(\frac{2\Phi + \Gamma_M U(1-\beta)}{\Gamma_m U(1+\beta)};+\infty \right)$$

Φ: upper bound of the drift term

Γ_m, Γ_M: positive lower and upper bounds of the gain term

The control is discontinuous

The control is based on the estimation of the local maxima, minima and first order flex points



L2 – Single Input 2-SMC – Sub Optimal

$$u = -\alpha(t) U \operatorname{sgn}(\sigma - \beta\sigma_M)$$

$$\alpha(t) = \begin{cases} 1 & (\sigma - \beta\sigma_M)\sigma_M \geq 0 \\ \overline{\alpha} & (\sigma - \beta\sigma_M)\sigma_M < 0 \end{cases}$$

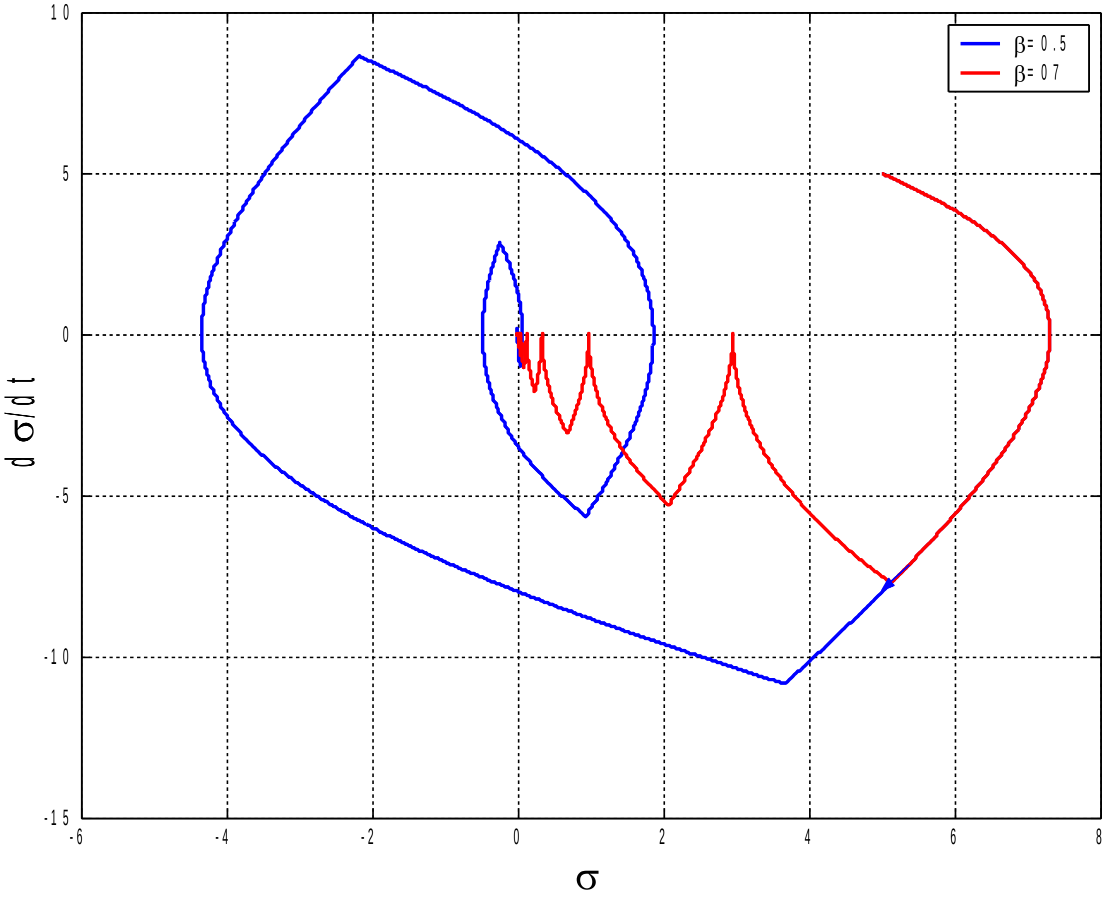
$$\beta \in (0;1)$$

α: modulation parameter

β: anticipation parameter

U: gain parameter

The proper tuning of the anticipation parameter allows for counteracting the peaking phenomenon



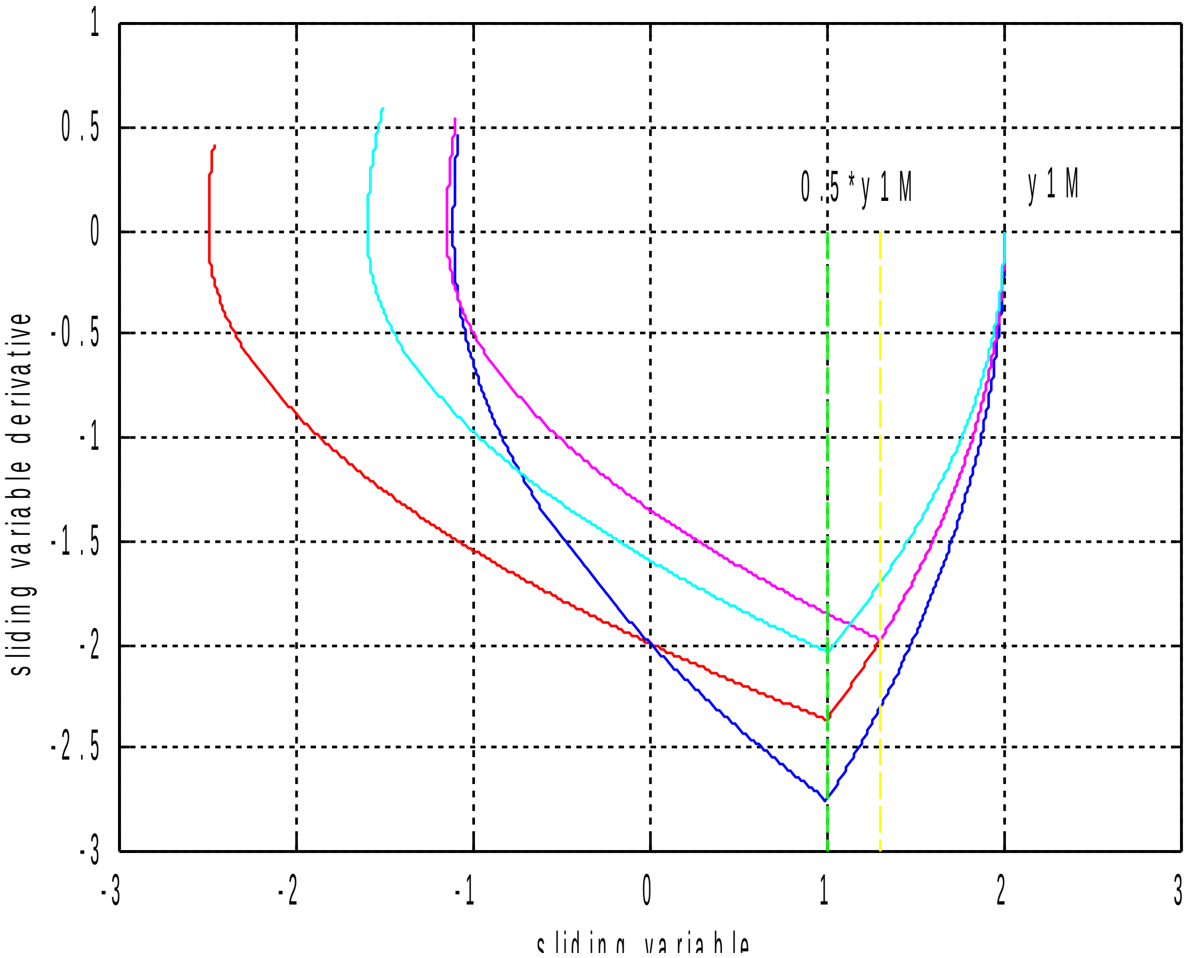
L2 – Single Input 2-SMC – Sub Optimal

$$u = -\alpha(t) U \operatorname{sgn}(\sigma - \beta\sigma_M)$$

$$\alpha(t) = \begin{cases} 1 & (\sigma - \beta\sigma_M)\sigma_M \geq 0 \\ \overline{\alpha} & (\sigma - \beta\sigma_M)\sigma_M < 0 \end{cases}$$

$$\beta \in (0;1)$$

- — increasing V_M
- — anticipation β
- — modulation $1/\alpha$
- — reference α



L2 – Single Input 2-SMC – Sub Optimal

Theorem.

Consider the system dynamics $$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u$$

Chose the sliding variable set $\sigma(\mathbf{x})$ such that the internal dynamics corresponding to the output variables $\sigma, \dot{\sigma}$ is ISS.

Assume that that the sliding dynamics has relative degree 2 and constant bounds for the uncertain drift and gain vector are known

$$\begin{cases} \left\| L_{\mathbf{f}}^2 \sigma \right\| \le \Phi \\ L_{\mathbf{g}} \sigma = 0 \\ 0 < \Gamma_m \le L_{\mathbf{g}} L_{\mathbf{f}} \sigma \le \Gamma_M \end{cases} \qquad \forall \mathbf{x} \in \mathsf{V}_{\varepsilon}, \forall t$$

Set the controller parameters such that the convergence conditions are fulfilled

$$U > \frac{\Phi}{\Gamma_m}, \quad \bar{\alpha} \in [1;+\infty) \cap \left(\frac{2\Phi + \Gamma_M U (1-\beta)}{\Gamma_m U (1+\beta)} ;+\infty \right)$$

The state feedback control law assures the asymptotic stability of the 2-SM on the sliding surface $\sigma(\mathbf{x}) = 0$

$$u = -\alpha(t) U \operatorname{sgn}(\sigma - \beta \sigma_M)$$

$$\beta \in (0;1), \quad \alpha(t) = \begin{cases} 1 & (\sigma - \beta\sigma_M)\sigma_M \ge 0 \\ \bar{\alpha} & (\sigma - \beta\sigma_M)\sigma_M < 0 \end{cases}$$

L2 – Single Input 2-SMC – Sub Optimal

Proof.

$$U > \frac{\Phi}{\Gamma_m}$$ Dominance condition

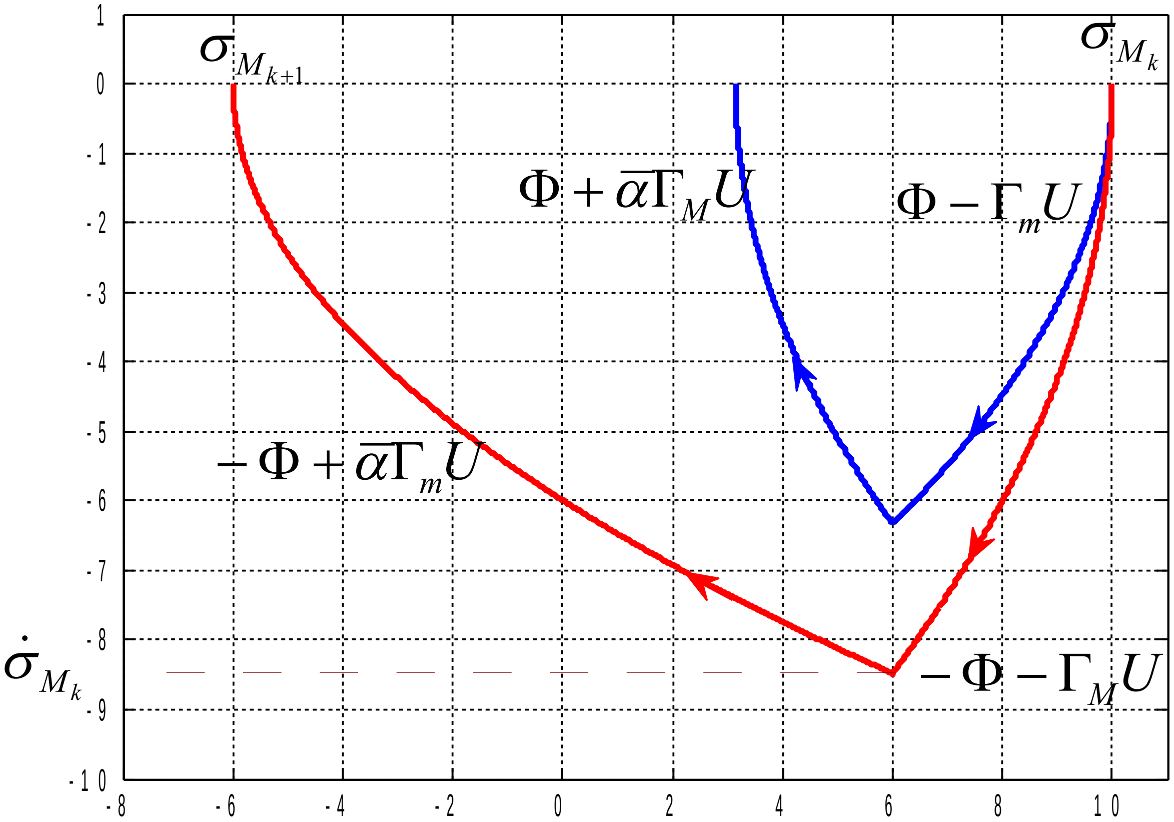
Convergence condition

$$\overline{\alpha} \in [1;+\infty) \cap \left(\frac{2\Phi + \Gamma_M U (1-\beta)}{\Gamma_m U (1+\beta)} ;+\infty \right)$$

⬇

$$\left|\sigma_{M_{k+1}}\right| \le \rho \left|\sigma_{M_k}\right|, \quad \rho \in (0,1)$$

$$\left|\dot{\sigma}_{M_k}\right| \le \sqrt{2(1-\beta)(\Gamma_M U + \Phi)\left|\sigma_{M_k}\right|},$$



A geometric convergent series of extremal points is generated and the 2-SM is achieved in a finite time

L2 – Single Input 2-SMC – Sub Optimal

The Generalized Sub-Optimal 2nd Order Sliding Mode Control algorithm is based on a switching logic based on the memory of past values of the sliding variable

The values $\sigma_{M,k}$ can be ideally evaluated by looking at the past, or by checking the time instant at which the time derivative of the sliding variable vanishes

The reaching phase can be characterized by twisting around the origin of the $\sigma,\dot{\sigma}$ plane, or by monotonic convergence of the sliding variable

The Generalized Sub-Optimal algorithm can implement the *Sub-Optimal* (β =½) and the *Twisting* algorithms by proper tuning of the parameters

Condition for monotonic convergence

$$\bar{\alpha} \in [1;+\infty) \cap \left(\frac{\Phi + (1-\beta)\Gamma_M U}{\beta \Gamma_m U};+\infty \right)$$

Condition for *Twisting* algorithm

$$\beta = 0$$

$$\bar{\alpha} > \frac{2\Phi + \Gamma_M U}{\Gamma_m U}$$

L2 – Single Input 2-SMC – Sub Optimal

The parameters of the Generalized Sub-Optimal 2nd Order Sliding Mode Control algorithm can be adjusted to guarantee the global convergence property

Global convergence can be lost because of the peaking phenomenon that is usually related to large values of the time derivative of the controlled output

Anticipating the switching will cause a reduction of the values

$$\left|\dot{\sigma}_{M_k}\right| \le \sqrt{2(1-\beta)(\Gamma_M U + \Phi)\left|\sigma_{M_k}\right|}$$

Very small values of β will cause very slow convergence to the 2-SM

$$\tau_\infty \le \tau_0 + U\,\frac{\overline{\alpha}\Gamma_m + \Gamma_M}{\overline{\alpha}\Gamma_m U - \Phi}\sqrt{\frac{2(1-\beta)\left|\sigma_{M,1}\right|}{\Phi + \Gamma_M U}}\;\frac{1}{1-\sqrt{\dfrac{\left|\Phi + \left[(1-\beta)\Gamma_M - \overline{\alpha}\beta\Gamma_m\right]U\right|}{\overline{\alpha}\Gamma_M U - \Phi}}}$$

L2 – Single Input 2-SMC – Sub Optimal

The gain U and the anticipation β of the Generalized Sub-Optimal 2nd Order Sliding Mode Control algorithm are adjusted at any local extremal value of σ

$$\left|L_{\mathbf{f}}^{2}\sigma\right| \le \phi(\sigma,\dot{\sigma})$$
$$0 < \Gamma_m \le L_{\mathbf{g}}L_{\mathbf{f}}\sigma \le \Gamma_M(\sigma,\dot{\sigma})$$

$$u = -U_{M_k}\,\mathrm{sign}\left(\sigma - \beta_k\sigma_{M_k}\right)$$

$$\beta_k = \max\left(\frac{1}{2};1-\frac{\eta^2}{2\left(\bar{\phi}_k + \bar{\Gamma}_{M_k}U_{M_k}\right)}\right)$$

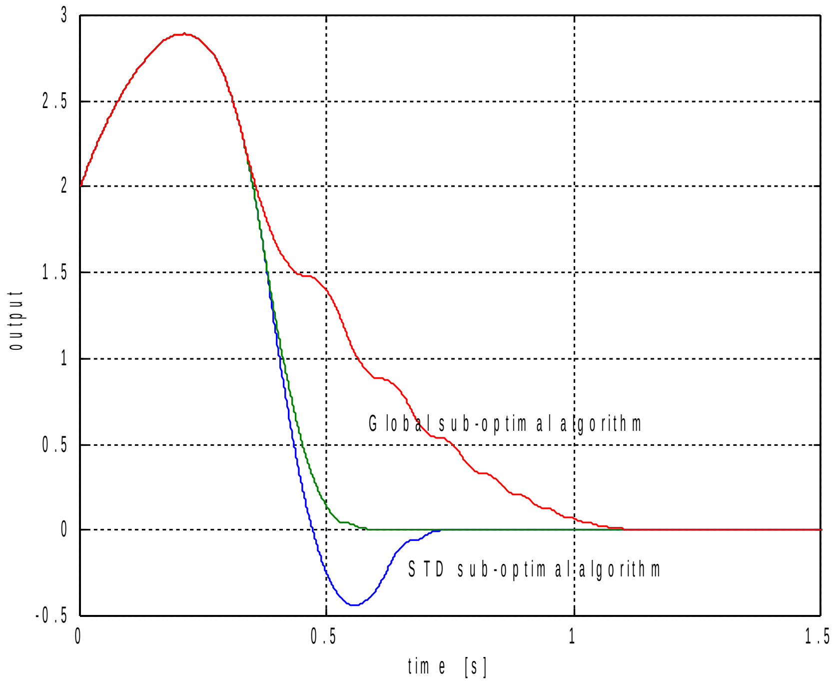
$$U_{M_k} > \frac{1}{\bar{\Gamma}_{M_k}}\left(\bar{\phi}_k + \tfrac{1}{3}\eta^2\right)$$

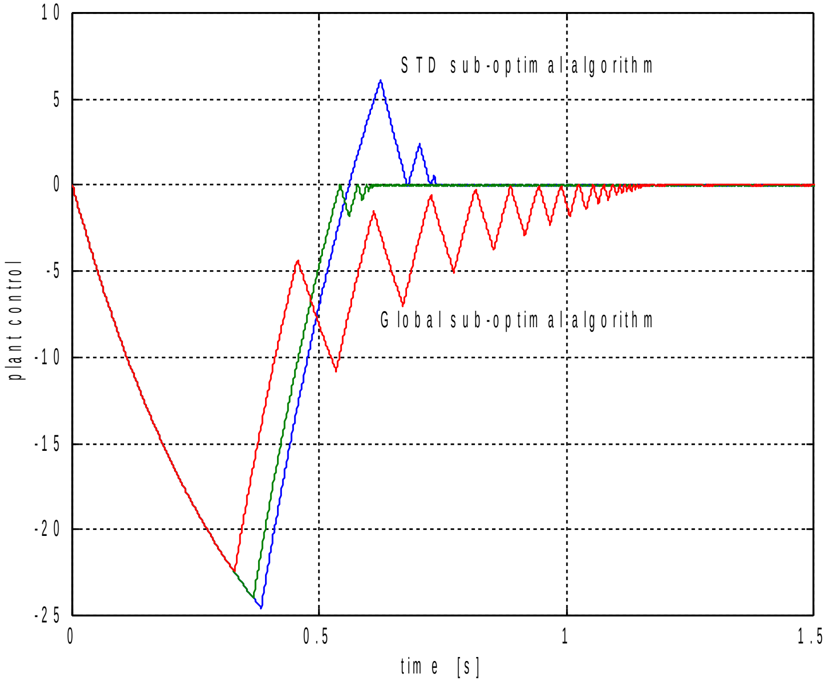
$$\bar{\phi}_k = \phi\left(\sigma_{M_k},\eta\sqrt{\sigma_{M_k}}\right)$$
$$\bar{\Gamma}_{M_k} = \Gamma_M\left(\sigma_{M_k},\eta\sqrt{\sigma_{M_k}}\right)$$

A prediction of the steepest transient is performed to evaluate the right control gain and anticipation in order to limit the magnitude of the time derivative of σ

L2 – Single Input 2-SMC – Sub Optimal

An illustrative example





The red line shows the case of a more restrictive condition on the time derivative of the sliding variable

L2 – MIMO 2-SMC – Supertwisting

The case of MIMO systems with a Super-Twisting-like algorithm was solved for the case of known perturbed systems, such that the sliding dynamics can be reduced to

$$\begin{aligned}\dot{\boldsymbol{\sigma}}(t) &= -k_1 \frac{\boldsymbol{\sigma}(t)}{\|\boldsymbol{\sigma}(t)\|^{0.5}} + \boldsymbol{z}(t) - k_2 \boldsymbol{\sigma}(t) + \boldsymbol{\gamma}(t, \boldsymbol{\sigma}) \\ \dot{\boldsymbol{z}}(t) &= -k_3 \frac{\boldsymbol{\sigma}(t)}{\|\boldsymbol{\sigma}(t)\|} - k_4 \boldsymbol{\sigma}(t) + \phi(t)\end{aligned}$$

$$\begin{gathered}\boldsymbol{\sigma}\,; \boldsymbol{z} \in \mathbb{R}^q \\ \|\boldsymbol{\gamma}\| \le \delta_1 \|\boldsymbol{\sigma}\| \\ \|\boldsymbol{\phi}\| \le \delta_2\end{gathered}$$

$$V(\boldsymbol{\sigma}, \boldsymbol{z}) = 2k_3 \|\boldsymbol{\sigma}(t)\| + k_4 \boldsymbol{\sigma}^T \boldsymbol{\sigma} + \frac{1}{2} \boldsymbol{z}^T \boldsymbol{z} + \left[k_1 \frac{\boldsymbol{\sigma}(t)}{\|\boldsymbol{\sigma}(t)\|^{0.5}} - \boldsymbol{z}(t) + k_2 \boldsymbol{\sigma}\right]^T \left[k_1 \frac{\boldsymbol{\sigma}(t)}{\|\boldsymbol{\sigma}(t)\|^{0.5}} - \boldsymbol{z}(t) + k_2 \boldsymbol{\sigma}\right]$$

is a Lyapunov function for the sliding dynamics provided that

$$k_1 > \sqrt{2\delta_2}; \quad k_2 > 2\delta_2; \quad k_3 > max\left\{k_3^\Omega, k_3^\Psi\right\} \quad k_4 > max\left\{k_4^\Omega, k_4^\Psi\right\}$$

$$k_3^\Omega = 3\delta_2 + \frac{2\delta_2^2}{k_1^2}; \quad k_3^\Psi = \frac{\frac{9}{16} k_1^2 \delta_1^2}{k_2(k_2 - 2\delta_1)} + \frac{\frac{1}{2} k_1^2 \delta_1 - 2k_1^2 k_2 + k_2 \delta_2}{(k_2 - 2\delta_1)};$$

$$k_4^\Omega = \frac{\left(1.5k_1^2 + 3\delta_2 k_2\right)^2}{k_3 k_1^2 - 2\delta_2^2 - 3\delta_2 k_1^2} + 2\mathrm{k}_2^2 + \frac{3}{2} k_2 \delta_1; \quad k_4^\Psi = \frac{4.5 k_1^2 \delta_2^2 (2k_2 + \delta_1)^2}{k_2(k_2 - 2\delta_1)\left[16 k_2^2 (k_3 + 2\mathrm{k}_1^2 - \delta_2) - k_2(32k_3 + 8k_1^2)\delta_1 - 9k_1^2 \delta_1^2\right]} + \frac{2k_2^2 \delta_1 + 0.25 k_2 \delta_1^2}{k_2 - 2\delta_1}$$

L2 – MIMO 2-SMC – Sub Optimal

The case of MIMO systems with the Sub-Optimal algorithm was solved for a class of unknown systems, such that the sliding dynamics can be reduced to

$$\ddot{\boldsymbol{\sigma}}(t)=\boldsymbol{F}(\boldsymbol{x})+\boldsymbol{G}(\boldsymbol{x})\boldsymbol{u}(t) \qquad \boldsymbol{\sigma},\boldsymbol{u}\in\mathbb{R}^q; \quad \begin{matrix}|f_i|<\bar{F}_i \\ 0<\underline{g_{ij}}\le g_{ij}\le\overline{g_{ij}}\end{matrix}, \quad i,j=1,\cdots,q$$

Choosing $\boldsymbol{u}(t)=V_M\, diag\left\{\alpha_i(t)\, sign\left(\sigma_i-\frac{1}{2}\sigma_{M_i}\right)\right\}$; $\alpha_i(t)=\begin{cases}\underline{\alpha}_i & if\left[\sigma_i-0.5\sigma_{M_i}\right]\left[\sigma_{M_i}-\sigma_i\right]>0 \\ 1 & if\left[\sigma_i-0.5\sigma_{M_i}\right]\left[\sigma_{M_i}-\sigma_i\right]\le 0\end{cases}$

the sliding dynamics is stabilised if the gain matrix is “sufficiently” dominant diagonal”, i.e.,

$$|g_{ij}|>\sum_{j=1,\, j\ne i}^{q}|g_{ij}|;\quad \overline{g_{ij}}<\frac{3\underline{g_{ii}}}{\overline{g_{ii}}+4\underline{g_{ii}}};\quad \forall i,j=1,2,\cdots,q$$

and

$$V_M\ge\max_{i=1,\cdots,q}\left\{\frac{\bar{F}_i}{\underline{\alpha}\,\underline{g_{ii}}-\sum_{j=1,\, j\ne i}^{q}\overline{g_{ij}}};\frac{4\bar{F}_i}{3\underline{g_{ii}}-\left(\underline{\alpha}\,\overline{g_{ii}}+4\sum_{j=1,\, j\ne i}^{q}\overline{g_{ij}}\right)}\right\};\qquad \underline{\alpha}_i\in[\varepsilon;1]\cap\left(\frac{\sum_{j=1,j\ne i}^{q}\overline{g_{ij}}}{\underline{g_{ii}}};\frac{3\underline{g_{ii}}-4\sum_{j=1,j\ne i}^{q}\overline{g_{ij}}}{\overline{g_{ii}}}\right),\ \varepsilon>0$$

L2 – MIMO 2-SMC – Twisting

The case of MIMO systems with the Twisting algorithm was solved for a class of unknown systems, such that the system dynamics can be reduced to

$$\dot{\boldsymbol{w}}(t)=\boldsymbol{h}\left(\boldsymbol{w}(t),\boldsymbol{\sigma}(t),\dot{\boldsymbol{\sigma}}(t)\right)$$
$$\ddot{\sigma}(t)=\boldsymbol{B}^{-1}\left(\boldsymbol{w}(t),\boldsymbol{\sigma}(t)\right)\left[\boldsymbol{\Psi}\left(\boldsymbol{w}(t),\boldsymbol{\sigma}(t),\dot{\boldsymbol{\sigma}}(t)\right)+\boldsymbol{u}(t)\right]$$

$$\boldsymbol{w}\in\mathbb{R}^{m},\ \boldsymbol{\sigma}\in\mathbb{R}^{q},\ n=m+q$$

satifying the following assumptions

$$\|\boldsymbol{\sigma}(0)\|_1\leqslant\sigma_0,\qquad \|\dot{\boldsymbol{\sigma}}(0)\|_1\leqslant\dot{\sigma}_0,\qquad \|\mathbf{w}(0)\|_1\leqslant w_0$$

Initial conditions are within a known bounded domain

$$|b_{i,j}(\boldsymbol{\sigma},\mathbf{w})|\leqslant\rho_{\mathbf{B}}\left(\|\boldsymbol{\sigma}\|_1+\|\mathbf{w}\|_1\right)+\rho_{\mathbf{B}_0}$$
$$0<\gamma_0\leqslant\lambda^{i}_{\mathbf{B}^s}(\boldsymbol{\sigma},\mathbf{w})\leqslant m_{\mathbf{B}^s}\left(\|\boldsymbol{\sigma}\|_1+\|\mathbf{w}\|_1\right)+\gamma_1$$

The elements of B and the eigenvalues of its simmetric part are bounded by know class-*K* functions

$$\|\boldsymbol{\Psi}(\boldsymbol{\sigma},\dot{\boldsymbol{\sigma}},\mathbf{w})\|_1\ \leqslant\ \kappa_0+\kappa_1\left(\|\boldsymbol{\sigma}\|_1+\|\dot{\boldsymbol{\sigma}}\|_1+\|\mathbf{w}\|_1\right)$$
$$\|\dot{\mathbf{B}}(\boldsymbol{\sigma},\dot{\boldsymbol{\sigma}},\mathbf{w})\|_1\ \leqslant\ \kappa_2+\kappa_3\left(\|\boldsymbol{\sigma}\|_1+\|\dot{\boldsymbol{\sigma}}\|_1+\|\mathbf{w}\|_1\right)$$

The drift term and the time variation of B are bounded by know class-*K* functions

$$\|\mathbf{w}(t)\|_1\leqslant\kappa_4\left(\|\mathbf{w}(0)\|_1\right)+\kappa_5\left(\sup_{0\leqslant\tau\leqslant t}\|\boldsymbol{\sigma}(\tau)\|_1+\sup_{0\leqslant\tau\leqslant t}\|\dot{\boldsymbol{\sigma}}(\tau)\|_1\right)$$

The internal dynamics are ISS

L2 – MIMO 2-SMC – Twisting

The case of MIMO systems with the Twisting algorithm was solved for a class of unknown systems, such that the system dynamics can be reduced to

$$\begin{aligned}\dot{\boldsymbol{w}}(t)&=\boldsymbol{h}\big(\boldsymbol{w}(t),\boldsymbol{\sigma}(t),\dot{\boldsymbol{\sigma}}(t)\big)\\ \ddot{\sigma}(t)&=\boldsymbol{B}^{-1}\big(\boldsymbol{w}(t),\boldsymbol{\sigma}(t)\big)\big[\boldsymbol{\Psi}\big(\boldsymbol{w}(t),\boldsymbol{\sigma}(t),\dot{\boldsymbol{\sigma}}(t)\big)+\boldsymbol{u}(t)\big]\end{aligned} \qquad \boldsymbol{w}\in\mathbb{R}^m,\boldsymbol{\sigma}\in\mathbb{R}^q,n=m+q$$

Under the previous conditions the system is stabilizable by means of

$$\mathbf{u}(t) = -a\,\mathbf{sign}(\boldsymbol{\sigma}) - b\,\mathbf{sign}(\dot{\boldsymbol{\sigma}})$$

$$b > \Gamma$$

$$a > b + \Gamma$$

$$\begin{aligned}\Gamma &= \kappa_0+\kappa_1(\varphi_0)+(\kappa_2+\kappa_3(\varphi_0))\sqrt{\frac{2nR}{\gamma_0}},\\ \varphi_0 &= \frac{R}{a}+\sqrt{\frac{2nR}{\gamma_0}}+\kappa_4(w_0)+\kappa_5\left(\frac{R}{a}+\sqrt{\frac{2nR}{\gamma_0}}\right),\\ R &= \eta R_0=\eta\left[\frac{1}{2}\left[m_{\mathrm{B^s}}(\sigma_0+w_0)+\gamma_1\right]\dot{\sigma}_0^2+a\sigma_0\right], \quad \eta>1\end{aligned}$$

L2 – Single Input 3-SMC

The 3rd Order Sliding Mode Control refers to systems whose sliding variable has relative degree 3 with respect to the (discontinuous) control variable

$$L_{\mathbf{g}} L_{\mathbf{f}}^{k} \sigma = 0 \quad i = 0,1 \qquad\qquad L_{\mathbf{g}} L_{\mathbf{f}}^{2} \sigma \neq 0$$

The main point is to design a control law with the less measurement demand as possible

Find a finite-time stabilizer that depends only on the sign of the phase variables

$$u(t) = u\left[sign(\sigma), sign(\dot{\sigma}), sign(\ddot{\sigma})\right]$$

L2 – Single Input 3-SMC

The considered 3rd order dynamics dynamics has known constant bounds

$$\left|L_{\mathbf{f}}^{3}\sigma\right| \le \Phi \qquad L_{\mathbf{g}}L_{\mathbf{f}}^{k}\sigma = 0 \quad i = 0{,}1 \qquad 0 < \Gamma_m \le L_{\mathbf{g}}L_{\mathbf{f}}^{2}\sigma \le \Gamma_M$$

A “switched” sliding-mode controller commuting between two structures:

Anosov unstable
[AU]

$$u = -U\,sign(\sigma)$$

$$U > \Phi / \Gamma_m$$

?

Generalized Twisting
[G-TW]

$$u = -U_1 sign(\dot{\sigma}) - U_2 sign(\ddot{\sigma})$$

$$\Lambda_m\left[U_1 + U_2\right] - \Phi > \Gamma_M\left[U_1 - U_2\right] + \Phi$$

L2 – Single Input 3-SMC

What is the system behaviour under the separate action of the two considered "structures" ?

Anosov unstable AU

$$u = -U_0 sign(\sigma)$$

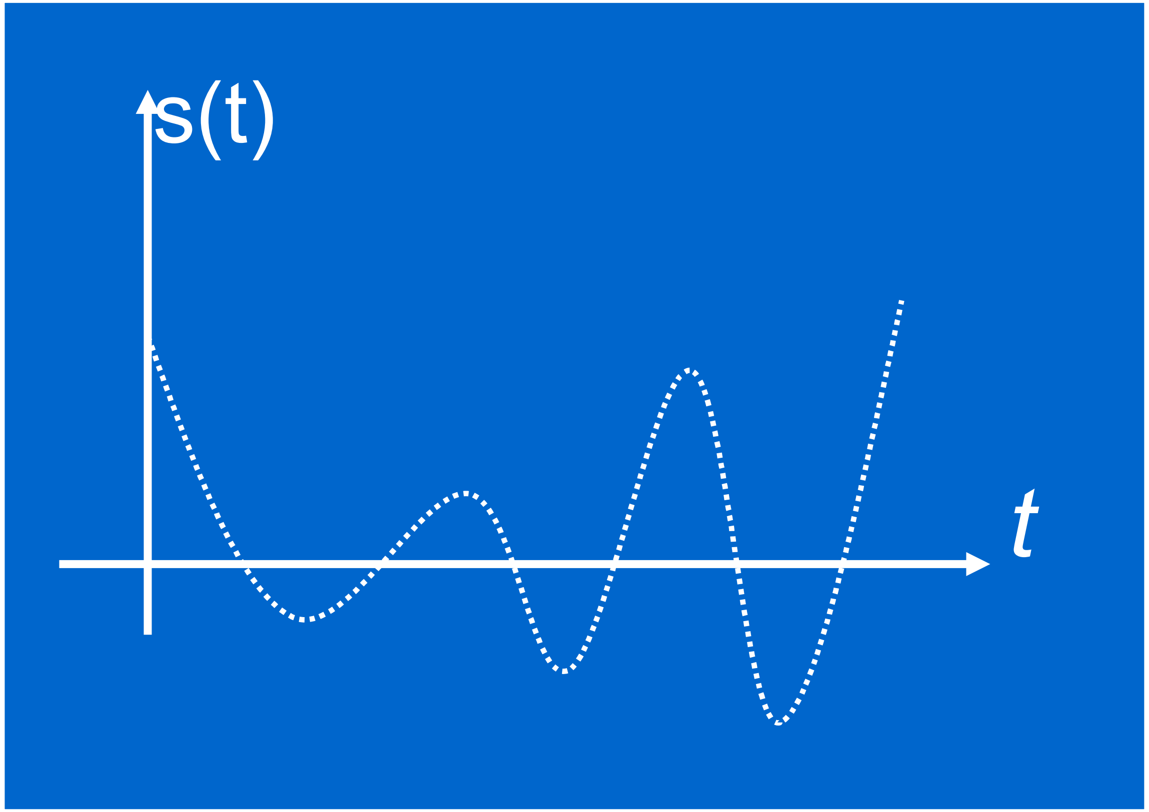
The appearance of a sequence of zero crossings is ensured.

$$\exists t_{z_i} \qquad i = 1,2,\ldots,\infty$$

$$: \qquad \sigma\left(t_{z_i}\right) = 0$$

The sliding variable features unstable oscillations around zero.

$$\dot{\sigma}\left(t_{z_i}\right), \ddot{\sigma}\left(t_{z_i}\right) \rightarrow \infty$$

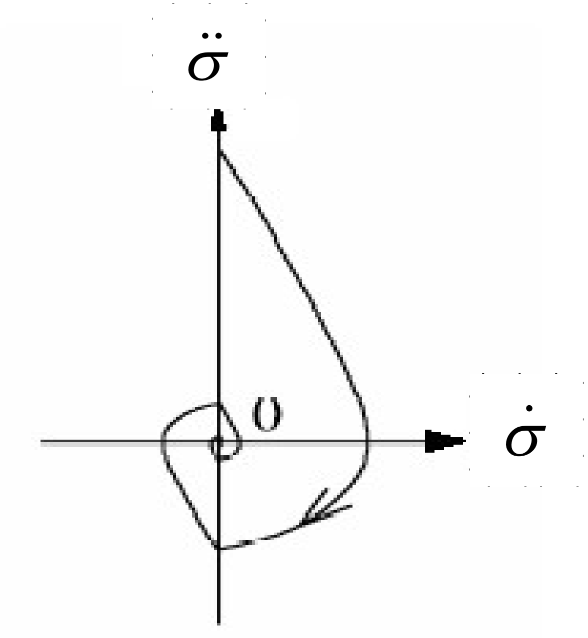


L2 – Single Input 3-SMC

What is the system behaviour under the separate action of the two considered "structures" ?

Generalized Twisting
G-TW

$$u = -\frac{1}{2}\left(U_1 + U_2\right) sign(\dot{s}) - \frac{1}{2}\left(U_1 - U_2\right) sign(\ddot{s})$$



$\dot{\sigma}$ and $\ddot{\sigma}$ both converge to zero in finite time

σ converges to a constant value σ_e (not necessarily zero)

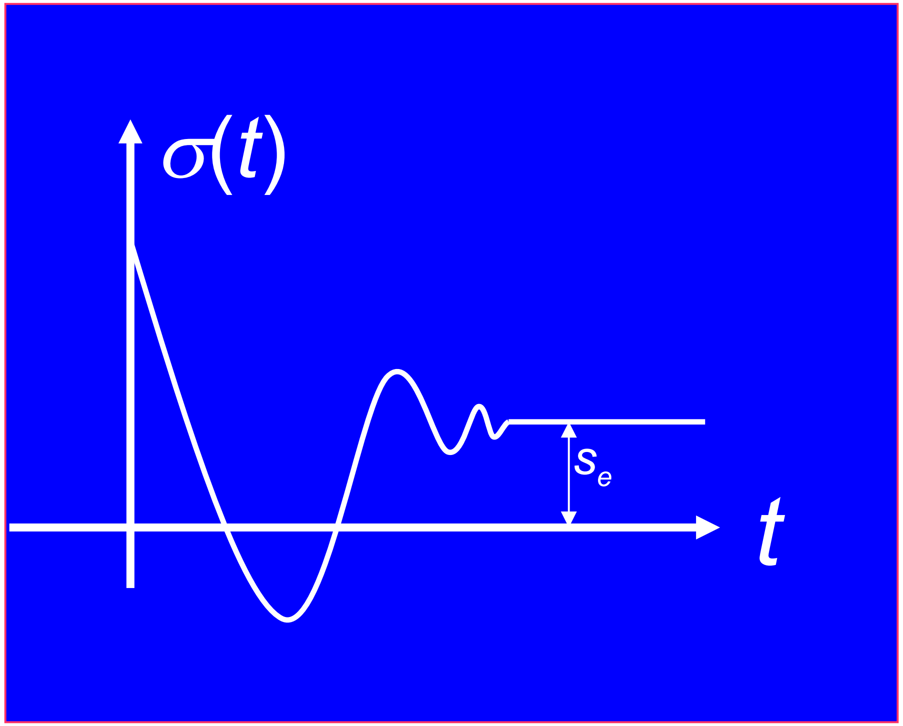
$(\sigma_e,0,0)$ is said to be an "equilibrium point"

L2 – Single Input 3-SMC

What is the system behaviour under the separate action of the two considered "structures" ?

Generalized Twisting
G-TW

$$u = -\frac{1}{2}\left(U_1 + U_2\right) sign(\dot{s}) - \frac{1}{2}\left(U_1 - U_2\right) sign(\ddot{s})$$



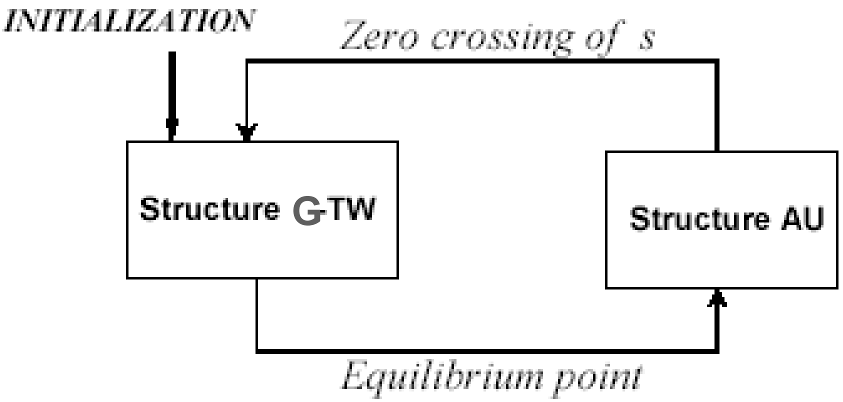
$\dot{\sigma}$ and $\ddot{\sigma}$ both converge to zero in finite time

σ converges to a constant value σ_e (not necessarily zero)

$(\sigma_e,0,0)$ is said to be an "equilibrium point"

L2 – Single Input 3-SMC

The 3rd Order Sliding Mode Controller switches between the two structure according the represented automaton



AU always leads to a zero crossing

G-TW always leads to an eq. point

→

The "event-driven" controller is never "blocked"

L2 – Single Input 3-SMC

Theorem.

Consider the system dynamics $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u$

Chose the sliding variable set $\sigma(\mathbf{x})$ such that the internal dynamics corresponding to the output variables $\sigma, \dot{\sigma}, \ddot{\sigma}$ is BIBS stable.

Assume that that the sliding dynamics has relative degree 3 and constant bounds for the uncertain drift and gain vector are known

$$\begin{cases} \left\| L_{\mathbf{f}}^{3}\sigma \right\| \leq \Phi \\ L_{\mathbf{g}}\sigma = L_{\mathbf{g}}L_{\mathbf{f}}\sigma = 0 \\ 0 < \Gamma_m \leq L_{\mathbf{g}}L_{\mathbf{f}}^{2}\sigma \leq \Gamma_M \end{cases} \qquad \forall \mathbf{x} \in \mathrm{V}_{\varepsilon}, \forall t$$

Define the parameters Δ_i

$$\Delta_1 = \frac{\Gamma_M U_0 + \Phi}{\Gamma_m U_1 - \Phi} \qquad \Delta_2 = \frac{\Gamma_M U_0 + \Phi}{\Gamma_m U_2 - \Phi}$$

$$\Delta_3 = \frac{\Gamma_m U_1 - \Phi}{\Gamma_M U_2 + \Phi} \qquad \Delta_4 = \frac{\Gamma_M U_2 + \Phi}{\Gamma_m U_2 - \Phi}$$

Chose the contraction rate $\varepsilon \in (0,1)$

L2 – Single Input 3-SMC

Theorem, *cont.*

The state feedback control law assures the asymptotic stability of the 3-SM on the sliding surface $\sigma(\mathbf{x})=0$, if the controller parameters are chosen according the following design steps

$$u=-\frac{1}{2}\left(U_1+U_2\right)sign(\dot{s})-\frac{1}{2}\left(U_1-U_2\right)sign(\ddot{s})$$

1) Set

$$U_0>\frac{\Phi}{\Gamma_m}$$

2) Set U_2 sufficiently large so that

$$0<2\sqrt{\Delta_2}<\varepsilon$$

3) Set U_1 sufficiently large so that

$$\Delta_1\left(3+2\Delta_1\right)+2\sqrt{\Delta_2}\left(1+2\Delta_1\right)^{3/2}<\varepsilon$$

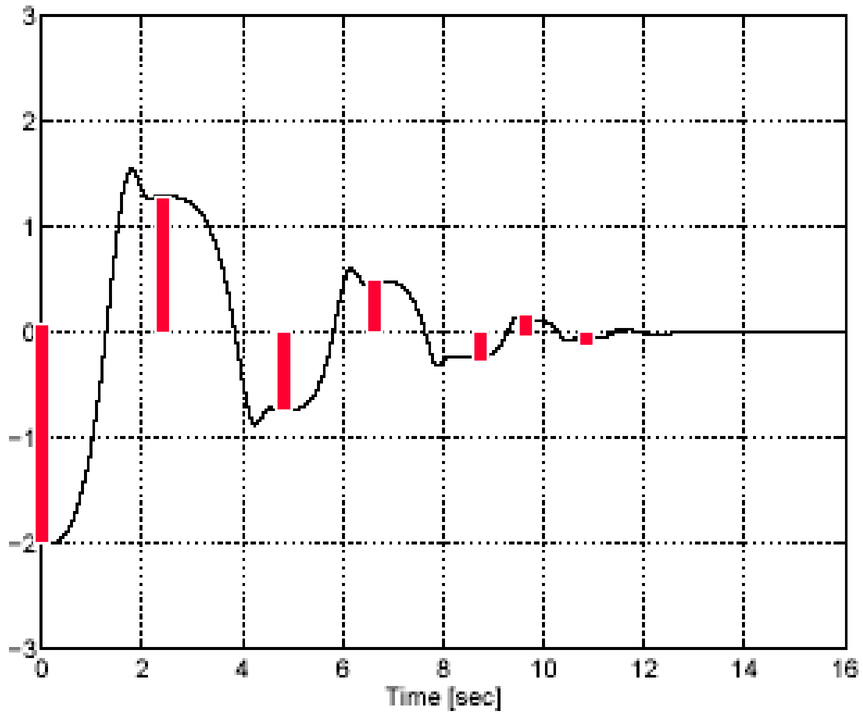
$$\sqrt{\Delta_3}\left(\Delta_3^2-1\right)>\Delta_4^2$$

L2 – Single Input 3-SMC

The proof is base on finding the conditions on the control parameters such that a series of contracting equilibrium points is enforced .

$$\left|\sigma_{e,i+1}\right| \le \varepsilon \left|\sigma_{e,i}\right| \quad i = 0,1,2,\ldots$$

A typical time-evolution of the sliding quantity



L2 – Single Input 3-SMC

The Hybrid 3rd Order Sliding Mode Control algorithm is based on switching between an unstable controller and a stable controller that in general force the closed loop system onto an equilibrium point with $\sigma \neq 0$

A reduced amount of information is required

A specific care must be devoted to identify the reaching of an equilibrium point

Tuning is quite easy if the step of the procedure are carefully performed

The approach has been extended to arbitrary order relative degree systems by a recursive implementation of the simple 3rd order approach

L2 – Arbitrary Order SMC

The Arbitrary Order Sliding Mode Control laws are based on a recursive algorithm deriving form the “*prescribed law of variation*” 2-SMC

A r-SMC algorithm needs the knowledge of the sliding variable, of its r-2 time derivatives and the sign of its r-1 time derivative

It is almost always associated to an arbitrary order differentiator

As for the Dynamical Sliding Mode Control it is assumed that the time derivatives of the sliding variable are available

Several implementation were presented trying to get a smoother reaching phase

L2 – Arbitrary Order SMC

Consider the nonlinear system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u$$

Define a suitable output $\sigma(\mathbf{x})$

$$L_{\mathbf{g}} L_{\mathbf{f}}^{k} \sigma = 0 \quad i = 1,2,\ldots,r-2$$

$$L_{\mathbf{g}} L_{\mathbf{f}}^{r-1} \sigma \neq 0$$

Calculate m as the least common multiplier of $1,2,3,\ldots,r$

Define the quantities

$$N_{i,r} = \left(\sum_{k=0}^{i-1} \left| \sigma^{(k)} \right|^{m/r-k} \right)^{r-1/m} \quad i = 1,2,\ldots,r-1$$

Define the hyper-surfaces

$$\Phi_{0,r} = \sigma$$

$$\Phi_{i,r} = \sigma^{(i)} + \beta_i N_{i,r} \operatorname{sgn}\left(\Phi_{i-1,r}\right) \quad i = 1,2,\ldots,r-1$$

L2 – Arbitrary Order SMC

From the definitions it is clear that

$$\Phi_{i,r} = 0 \Rightarrow \Phi_{i-1,r} \xrightarrow[t \to \tau_{r-1-i}]{} 0 \quad i = 1,2,\dots,r-1$$

Example $r=3$

$$m = 2 \cdot 3 = 6$$

$$N_{1,r} = |\sigma|^{2/3}$$
$$N_{2,r} = \left(|\sigma|^{6/3} + |\dot{\sigma}|^{6/2}\right)^{1/6}$$

$$\Phi_{0,r} = \sigma$$
$$\Phi_{1,r} = \dot{\sigma} + \beta_1 |\sigma|^{2/3} \operatorname{sgn}(\sigma)$$
$$\Phi_{2,r} = \ddot{\sigma} + \beta_2 \left(|\sigma|^{6/3} + |\dot{\sigma}|^{6/2}\right)^{1/6} \operatorname{sgn}\left(\dot{\sigma} + \beta_1 |\sigma|^{2/3} \operatorname{sgn}(\sigma)\right)$$

L2 – Arbitrary Order SMC

If the sliding dynamics is bounded
$$\begin{cases} \left\| L_{\mathbf{f}}^{r}\sigma \right\| \leq \Phi \\ 0 < \Gamma_m \leq L_{\mathbf{g}} L_{\mathbf{f}}^{r-1}\sigma \leq \Gamma_M \end{cases}$$

The differential inclusion defining the sliding variable dynamics is

$$\sigma^{(r)} \in \left[-\Phi, +\Phi\right] + \left[\Gamma_m, \Gamma_M\right] u$$

The controller
$$u = -\alpha \,\mathrm{sgn}\left(\Phi_{r-1,r}\left(\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)}\right)\right)$$

with parameters α and β_i sufficiently large will force $\Phi_{r-1,r}$ to zero in finite time.

Then the “cascade” convegence will be established.

L2 – Arbitrary Order SMC

The coefficients appearing on each $\Phi_{i,r}$ are such that the closed loop dynamics has homogeneity properties

A vector–set field $\mathrm{F}(\mathbf{x}) : \mathrm{R}^n \to \mathrm{R}^n$ is called homogeneous of degree $q \in \mathrm{R}$ with the dilation $\mathrm{d}_\kappa : (x_1, x_2, \ldots, x_n) \to (x_1^{m1}, x_2^{m2}, \ldots, x_n^{mn})$, $mi \in \mathrm{R}^+$, if the following identity holds

$$\mathbf{F}(\mathbf{x}) = \kappa^{-q} d_\kappa^{-1} \mathbf{F}(d_\kappa \mathbf{x}) \quad \forall \kappa > 0$$

A differential inclusion $\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x})$ is called homogeneous of degree $q \in \mathrm{R}$ if it is invariant with respect to the combined time–coordinate transformation $\Delta_\kappa : (t, \mathbf{x}) \to (\kappa^{-q}, \mathrm{d}_\kappa \mathbf{x})$

L2 – Arbitrary Order SMC

The homogeneity property implies the finite time stability once the contraction is proved

Coefficients β_i can be defined once and the only tuning parameter is α depending on the bounds of the uncertain dynamics

$$
\begin{aligned}
&r = 1. \quad u = -\alpha\,\mathrm{sign}\,(s) \\
&r = 2. \quad u = -\alpha\,\mathrm{sign}(\dot{s} + |s|^{1/2}\mathrm{sign}\,s) \\
&r = 3. \quad u = -\alpha\,\mathrm{sign}(\ddot{s} + 2(|\dot{s}|^3 + |s|^2)^{1/6}\mathrm{sign}(\dot{s} + |s|^{2/3}\mathrm{sign}\,s) \\
&r = 4. \quad u = -\alpha\,\mathrm{sign}\{s^{(3)} + 3(\ddot{s}^6 + \dot{s}^4 + |s|^3)^{1/12}\mathrm{sign}[\ddot{s} + \\
&\qquad\qquad (\dot{s}^4 + |s|^3)^{1/6}\mathrm{sign}(\dot{s} + 0.5|s|^{3/4}\mathrm{sign}\,s)]\}
\end{aligned}
$$

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