Institut für Regelungs- und Automatisierungstechnik
Graz University of Technology

Sliding Mode Control:
Basic Theory, Advances and Applications

Elio USAI

eusai@diee.unica.it

Dept of Electrical and Electronic Eng.
University of Cagliari

Summary of the Talks

- Introduction to Sliding Modes in Variable Structure Systems
- Sliding Mode Control of uncertain systems: basics
- Higher-Order sliding modes: basics
- Higher-Order sliding mode control design
- Implementation issues of sliding mode controllers with engineering applications
- Simulation/solution of ODE with Algebraic constraints via VSS
- Real-time differentiation via higher-order sliding modes
- State variable estimation and input reconstruction in dynamical systems via VSS
- Some applications

Lecture 1

Basics on Sliding Mode Control
in
Variable Structure Systems

- Sliding Modes in Variable Structure Systems
- Control problem statement
- Internal and input-output dynamics
- Convergence and stability conditions for systems with **known** gain function
- Convergence and stability conditions for systems with **unknown** gain function
- Invariance and reduced order dynamics
- Equivalent control
- Filippov's solution

L1 - SM in VSS

Variable Structure Systems are dynamical systems such that their behavior is characterised by different dynamics in different domains

$$\dot{\mathbf{x}}(t) = f_i(\mathbf{x}(t), t, \mathbf{u}(t)), \quad \mathbf{x} \in X_i \subseteq \mathsf{R}^n, \quad \mathbf{u} \in U_i \subseteq \mathsf{R}^q, \quad t \in \mathsf{R}^+$$

$$i \in Q \subseteq \mathsf{N}$$

f_i *is a smooth vector field* f_i: $\mathbf{R}^n \mathrm{x} \mathbf{R}^q \mathrm{x} \mathbf{R}^+ \rightarrow \mathbf{R}^n$

The state dynamics is invariant until a switch occurs

The system dynamics is represented by a differential equation with discontinuous right-hand side

L1 - SM in VSS

Switching between different dynamics

$$g_i^{sw}\left(\mathbf{x}\left(\tau_k\right),\tau_k\right)=0$$
$$\Downarrow$$
$$\dot{\mathbf{x}}\left(\tau_k^-\right)=f_i\left(\mathbf{x}\left(\tau_k^-\right),\tau_k^-,\mathbf{u}\left(\tau_k^-\right)\right),\quad \dot{\mathbf{x}}\left(\tau_k^+\right)=f_j\left(\mathbf{x}\left(\tau_k^+\right),\tau_k^+,\mathbf{u}\left(\tau_k^+\right)\right)$$

$$\mathbf{x}\in X_i\subseteq \mathrm{R}^n,\quad \mathbf{u}\in U_i\subseteq \mathrm{R}^m,\quad t\in \mathrm{R}^+$$
$$i,j\in Q\subseteq \mathrm{N}$$

The reaching of the guard g_i^{sw} cause the switching from the dynamics f_i to the dynamics f_j, according to proper rules

L1 - SM in VSS

What does it happen on the guard? $g_i^{sw}\left(\mathbf{x}\left(\tau_k\right),\tau_k\right)=0$

$\mathbf{x}\left(\tau_k^-\right)=\mathbf{x}\left(\tau_k^+\right)$ Continuous state variables

$\mathbf{x}\left(\tau_k^-\right)\neq\mathbf{x}\left(\tau_k^+\right)$ Jumps in state variables

$\mathbf{u}\left(\tau_k^-\right)=\mathbf{u}\left(\tau_k^+\right)$ Continuous control variables

$\mathbf{u}\left(\tau_k^-\right)\neq\mathbf{u}\left(\tau_k^+\right)$ Discontinuous control variables

In Variable Structure Systems there is no jumps in the state variables but there could be discontinuity in the control variables.

The most interesting point is what happens on the guard.

L1 - SM in VSS

Variable Structure Systems may behave very differently from each of the constituting ones.

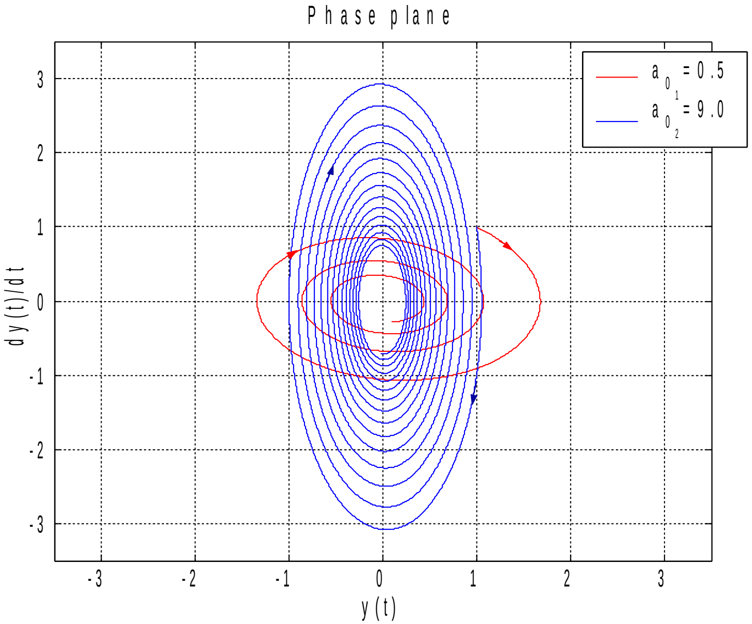
$$\ddot{y}(t)+a_1\dot{y}(t)+a_{01}y(t)=0 \quad system\,1$$

$$\ddot{y}(t)+a_1\dot{y}(t)+a_{02}y(t)=0 \quad system\,2$$

$$0<a_{01}<a_{02}$$

- $a_1>0$ the systems are both asymptotically stable
- $a_1=0$ the systems are both marginally stable
- $a_1<0$ the systems are both unstable

L1 - SM in VSS



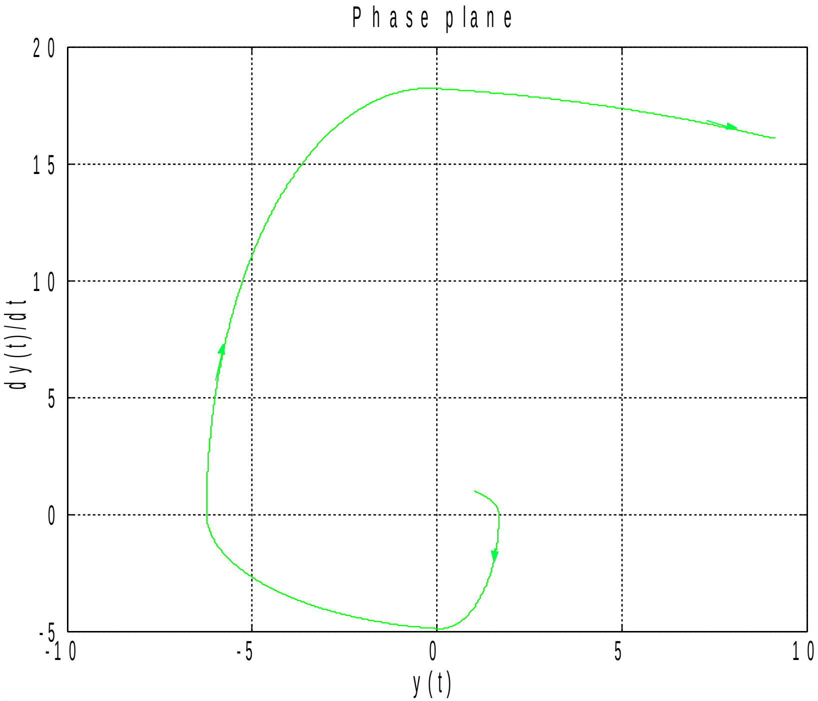
$$a_1 = 0.1$$

Both dynamics are asymptotically stable

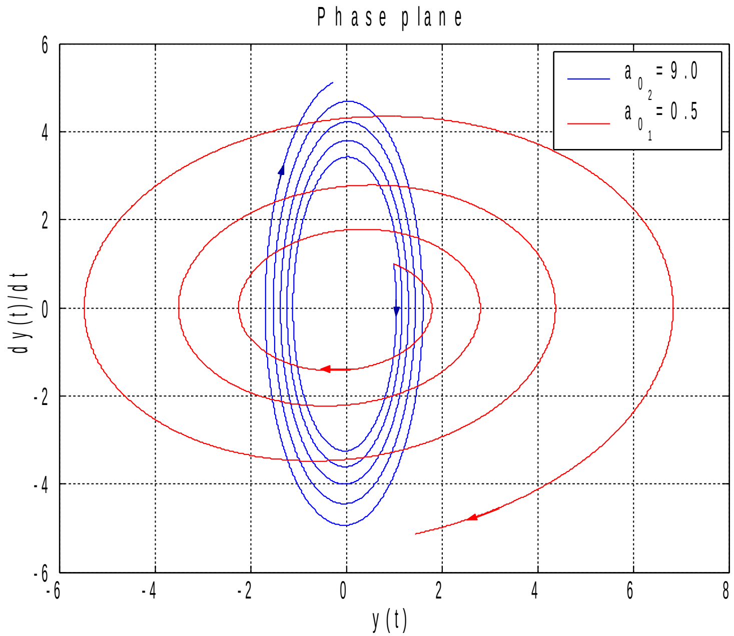
$$\ddot{y} + a_1 \dot{y} + a_0 y - \Delta a_0 |y| \operatorname{sgn}(\dot{y}) = 0$$

$$a_1 = 0.1 \quad a_0 = 0.7 \quad \Delta a_0 = 0.2$$

Switched unstable dynamics

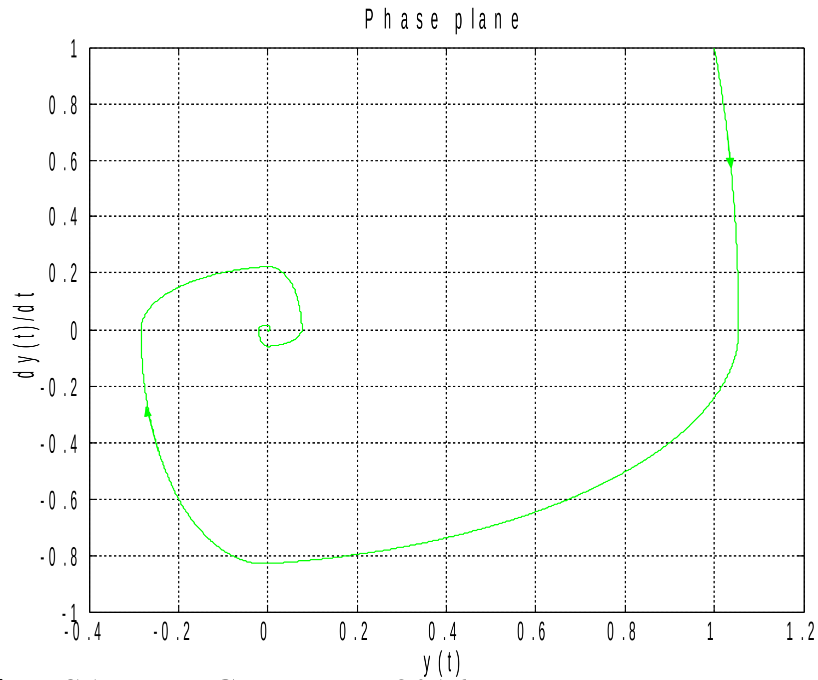


L1 - SM in VSS



$$a_1 = -0.1$$

Both dynamics are unstable



$$\ddot{y} + a_1\dot{y} + a_0 y + \Delta a_0 |y| \operatorname{sgn}(\dot{y}) = 0$$

$$a_1 = -0.1 \quad a_0 = 0.7 \quad \Delta a_0 = 0.2$$

Switched asymptotically stable dynamics

L1 - SM in VSS

A Sliding Mode behavior appears when the switching frequency tends to infinity

$$\left(\tau_{i+1} - \tau_i\right) \underset{i \to \infty}{\longrightarrow} 0$$

If the switching frequency tends to infinity in a finite time, the sliding mode can be considered as a Zeno phenomenon in Hybrid Systems which are characterized by their execution set χ^H

$$\chi^H = \left\{T, In, Ed\right\}$$

$T = \{\tau_i\}_{i \in N}$: set of switching/jump time instants

$In = \{\mathbf{x}_i\}_{i \in N}\ \mathbf{x}_i \subseteq D$: set of initial states sequence

$Ed = \{\eta_i\}_{i \in N}\ \eta_i = (i,j) \subseteq Q \times Q$: set of edge sequence

L1 - SM in VSS

The Zeno phenomenon appears if the execution set is such that

$$\lim_{i\to\infty}\tau_i = \sum_{i=0}^{\infty}\left(\tau_{i+1}-\tau_i\right) = \tau_\infty < \infty$$

$$\Downarrow$$

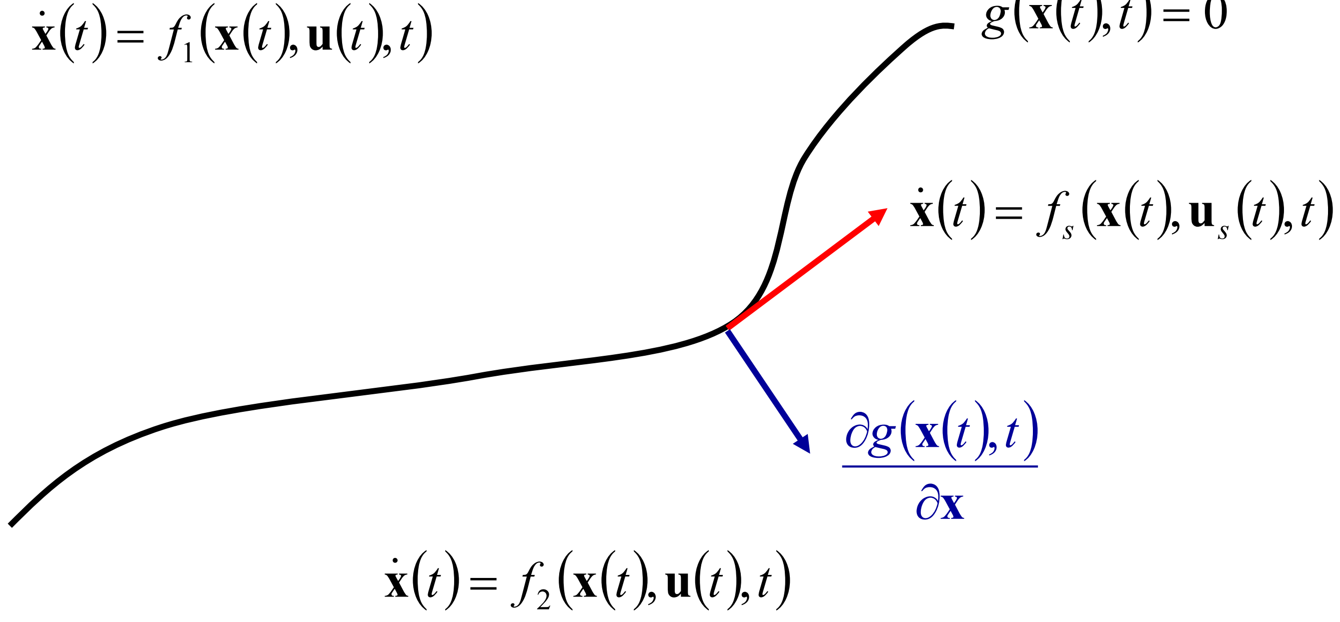
the Sliding Mode behavior is achieved in a finite time

In a Zeno/Sliding Mode condition the system evolves along a guard

$$\frac{\partial g\left(\mathbf{x}(t),t\right)}{\partial \mathbf{x}}\cdot\dot{\mathbf{x}}(t) = 0 \quad \forall t \geq \tau_\infty$$

L1 - SM in VSS

The Zeno phenomenon is mainly related to the switching frequency on a guard, but previous relationship shows the relation between the guard and the system dynamics



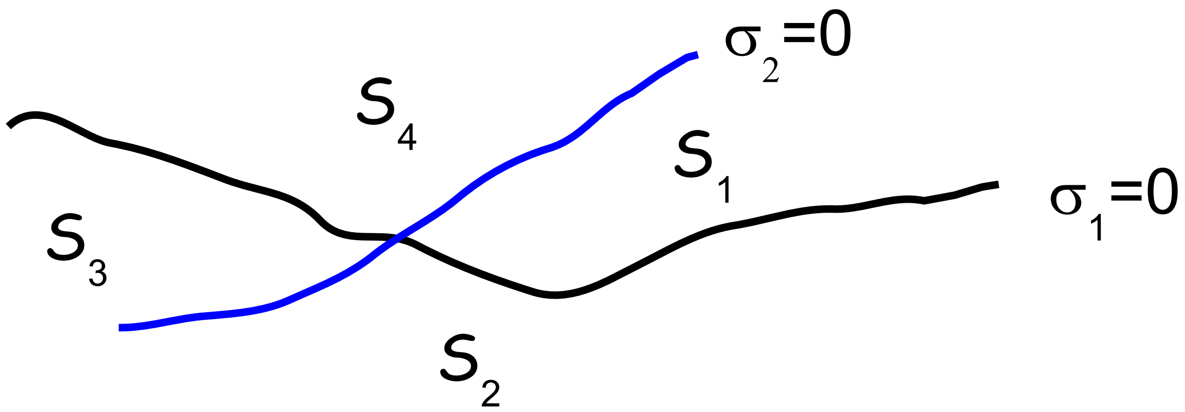
The motion of the system on a discontinuity surface is called Sliding Mode

L1 - SM in VSS

- Sliding Modes are Zeno behaviours in switching systems
- The system is constrained onto a surface in the state space, the *sliding surface*
- When the system is constrained on the sliding surface, the system modes differ from those of the original systems
- The system can be invariant when constrained on the sliding surface
- Systems belonging to a specific class and constrained onto the sliding surface behave the same way

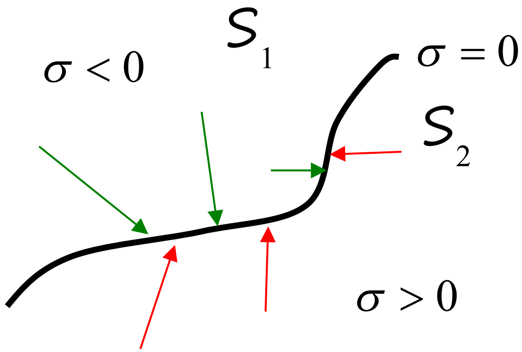
L1 - SM in VSS

$\boldsymbol{\sigma}(\mathbf{x}(t),t)=0$ Represents the boundary between distinct regions S_k of the state space, possibly time varying



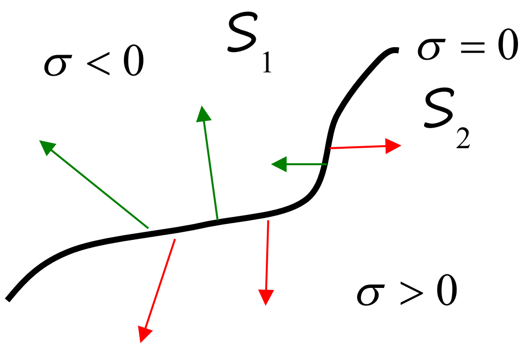
The behavior of the system on/across the guard $\boldsymbol{\sigma}=\mathbf{0}$, defining the regions of the state space, depends on how the dynamics f_k are related to the switching surface

L1 - SM in VSS



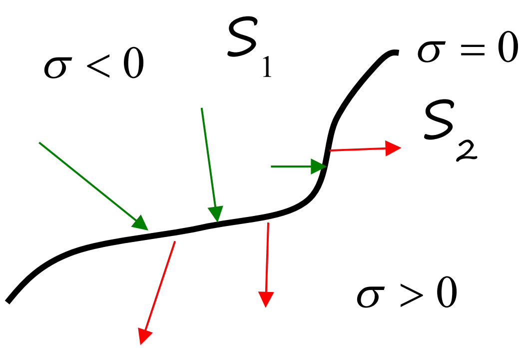
$$\begin{cases} \dfrac{\partial \sigma(\mathbf{x}(t),t)}{\partial \mathbf{x}} \cdot f_2(\mathbf{x}(t),t) < 0 \\ \dfrac{\partial \sigma(\mathbf{x}(t),t)}{\partial \mathbf{x}} \cdot f_1(\mathbf{x}(t),t) > 0 \end{cases}$$

attractive switching surface



$$\begin{cases} \dfrac{\partial \sigma(\mathbf{x}(t),t)}{\partial \mathbf{x}} \cdot f_2(\mathbf{x}(t),t) > 0 \\ \dfrac{\partial \sigma(\mathbf{x}(t),t)}{\partial \mathbf{x}} \cdot f_1(\mathbf{x}(t),t) < 0 \end{cases}$$

repulsive switching surface



$$\begin{cases} \dfrac{\partial \sigma(\boldsymbol{x}(t),t)}{\partial \boldsymbol{x}} f_2(\boldsymbol{x}(t),t) > 0 \\ \dfrac{\partial \sigma(\boldsymbol{x}(t),t)}{\partial \boldsymbol{x}} f_1(\boldsymbol{x}(t),t) > 0 \end{cases}$$

across switching surface

L1 - SM in VSS

When considering Variable Structure Systems, the system dynamics can be represented by a discontinuous right-hand side differential equation

$$\dot{\mathbf{x}}(t) = f_k(\mathbf{x}(t), \mathbf{u}(t), t) \qquad \forall \mathbf{x} \in \mathcal{S}_k$$

A discontinuous right-hand side differential equation can also be represented by a differential inclusion

$$\dot{\mathbf{x}}(t) \in \mathcal{F}$$

$$\{f_1, f_2, \ldots, f_m\} \subset \mathcal{F} \;, \quad f_k(\mathbf{x}, \mathbf{u}, t) : \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^+ \to \mathbb{R}^n$$

In the Sliding Mode a specific solution of the differential inclusion satisfying $\boldsymbol{\sigma} = \mathbf{0}$ is “selected”

L1 – VSC, problem statement

Variable Structure Control of dynamical systems is a nonlinear control technique in which the control variable is usually chosen so that a sliding mode behavior on a proper surface of the state plane is enforced

PRO	CONS
✔ Robustness with respect to matching disturbances ✔ Robustness with respect to system uncertainty ✔ Simple implemen-tation and tuning	• Theoretically, infinite frequency switching is required • Unpredictable oscillations can appear in real implementations (chattering)

$L1$ – VSC, problem statement

$$\ddot{y} - a_1\dot{y} + a_0 y = 0$$

$$a_1 > 0 \quad a_0 = \pm\alpha$$

$a_0 = \alpha$ $a_0 = -\alpha$

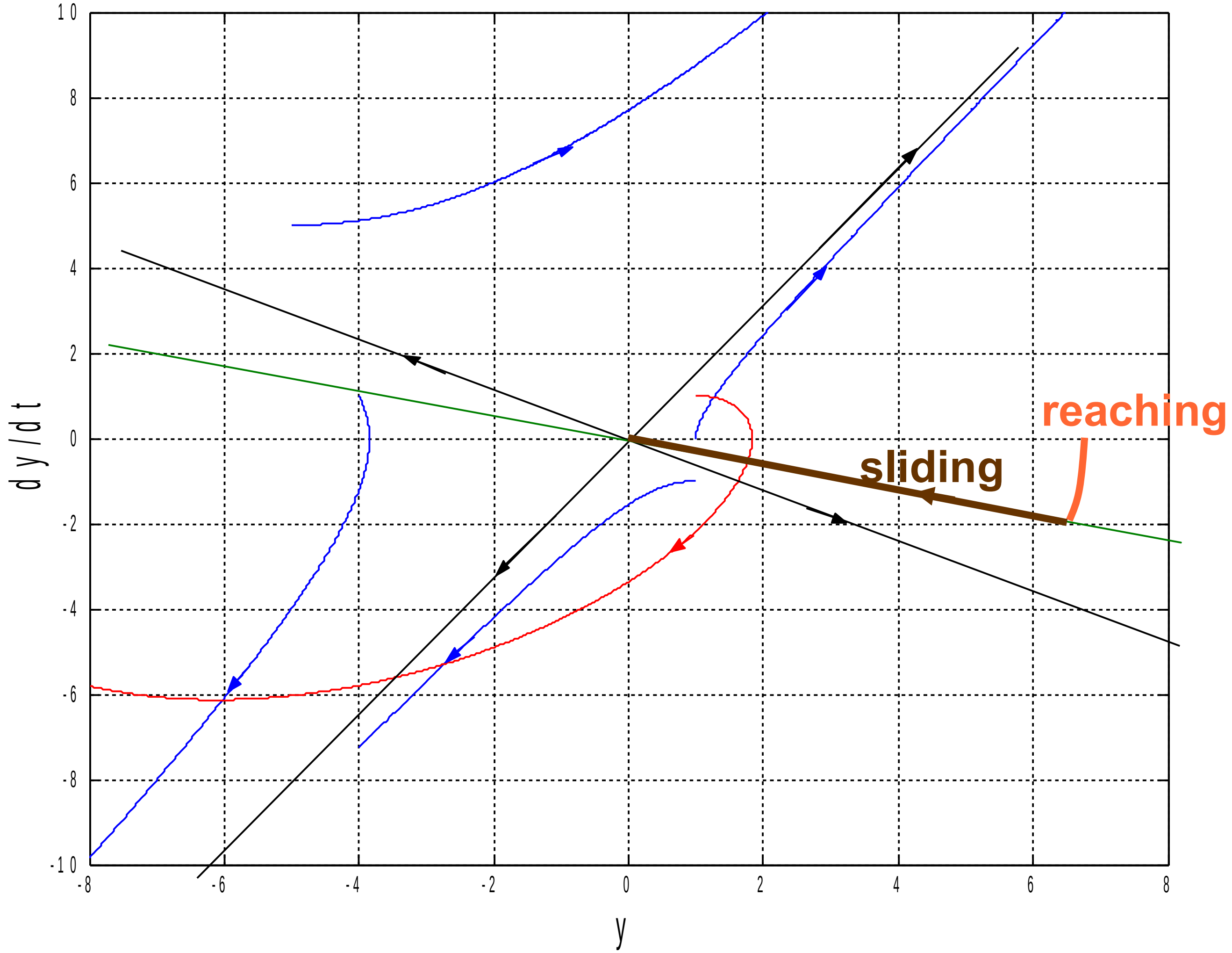
$\alpha > 0$

$$\ddot{y} - a_1\dot{y} + a_0 y \cdot u = 0$$

$$a_1 = 1, \quad a_0 = 1, \quad c = 0.2,$$

$$u = \mathrm{sgn}(\sigma)$$

$$\sigma = \dot{y} + cy$$



$L1$ – VSC, problem statement

In Variable Structure Control the switching between dynamics is enforced by a proper control variable

$$\begin{aligned} \dot{\mathbf{x}}(t) &= f_k\left(\mathbf{x}(t), \mathbf{u}(t), t\right) \\ f_k\left(\mathbf{x}(t), \mathbf{u}(t), t\right) &= f\left(\mathbf{x}(t), \mathbf{u}_k(t), t\right) \end{aligned} \qquad \forall \mathbf{x} \in S_k$$

$$\Downarrow$$

$$\dot{\mathbf{x}}(t) = f\left(\mathbf{x}(t), \mathbf{u}_k(t), t\right) \qquad \forall \mathbf{x} \in S_k$$

The problem is to define a suitable switching control such that the Sliding Mode is established, possibly in a finite time, and the resulting controlled dynamics fulfill the requirements

L1 – VSC, problem statement

Theorem.

Consider the system dynamics $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t), t)$

If it is possible to define the control variable

$$\mathbf{u}(t) = \mathbf{u}_k(t) \qquad \forall \mathbf{x} \in \mathsf{S}_k$$

in any ε-vicinity of the switching surface $\boldsymbol{\sigma}(\mathbf{x})=\mathbf{0}$

$$\mathsf{V}_\varepsilon = \left\{\mathbf{x} : \left\|\boldsymbol{\sigma}(\mathbf{x})\right\|_1 < \varepsilon, \varepsilon > 0\right\}$$

such that

$$\operatorname{sgn}\left(\frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}} \cdot f(\mathbf{x}(t), \mathbf{u}(t), t)\right) = -\operatorname{sgn}(\boldsymbol{\sigma}) \qquad \forall \mathbf{x} \in \mathsf{V}_k$$

then the surface $\boldsymbol{\sigma}(\mathbf{x})=\mathbf{0}$ is an invariant set in the state space and a sliding mode occurs on it.

L1 – VSC, problem statement

Proof.
Consider the positive definite function

$$V(\boldsymbol{\sigma}) = \frac{1}{2}\boldsymbol{\sigma}^T \cdot \boldsymbol{\sigma}$$

Its time derivative is

$$\begin{aligned}\dot{V} &= \boldsymbol{\sigma}^T \cdot \dot{\boldsymbol{\sigma}} = \boldsymbol{\sigma}^T \cdot diag(\mathrm{sgn}(\dot{\boldsymbol{\sigma}}))|\dot{\boldsymbol{\sigma}}| \\ &= -\boldsymbol{\sigma}^T diag(\mathrm{sgn}(\boldsymbol{\sigma}))|\dot{\boldsymbol{\sigma}}| = -|\boldsymbol{\sigma}^T| \cdot |\dot{\boldsymbol{\sigma}}| < 0\end{aligned}$$

Therefore $V(\boldsymbol{\sigma})$ is a Lyapunov function and the origin of the p-dimensional space of variables $\boldsymbol{\sigma}$ is, at least, an asymptotically stable equilibrium point.

L1 – VSC, problem statement

Condition $$\operatorname{sgn}\left(\frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}} \cdot f\left(\mathbf{x}(t), \mathbf{u}(t), t\right)\right) = -\operatorname{sgn}(\boldsymbol{\sigma}) \qquad \forall \mathbf{x} \in \mathsf{V}_k$$

implies that in a vicinity of the sliding surface the vector field is always directed towards the surface itself

If the control vector **u**(*t*) is such that $$\left|\dot{\sigma}_i\right| > \eta \qquad (i = 1,2,\ldots,p)$$

the time derivative of the Lyapunov function is such that

$$\dot{V} = -\left|\boldsymbol{\sigma}^T\right| \cdot \left|\dot{\boldsymbol{\sigma}}\right| < -\eta\sqrt{V}$$

⇓

the sliding mode behavior is reached in a finite time

$$\tau_\infty \le t_0 + \frac{\left\|\boldsymbol{\sigma}(t_0)\right\|}{\eta}$$

L1 – VSC, problem statement

The vector field is discontinuous on the sliding surface and the set of switching time instants $T=\{\tau_1, \tau_2, \ldots, \tau_\infty\}$ is a zero-measure set

⇓

$$\dot{\mathbf{x}}(t) = f_k(\mathbf{x}(t), \mathbf{u}(t), t) \qquad \forall \mathbf{x} \in S_k$$

is Lebeasgue integrable on time and a continuous solution **x**(*t*) exists in the Filippov sense

$L1$ – VSC, problem statement

Usually the dimension of variable $\boldsymbol{\sigma}$ defining the sliding surface is the same of the control vector

p=q

↓

Each control variable u_i is usually defined as a discontinuous function of a single sliding variable σ_i, even if it affects all components of $\boldsymbol{\sigma}$

L1 – VSC, problem statement

The Sliding Mode Control problem cannot be reduced to find a control law such that the previous Theorem is satisfied since control specifications has to be fulfilled

Sliding Mode Control of dynamical systems is a two step design approach:

- Define a proper sliding surface such that once the system is constrained onto it the control specifications are fulfilled
- Define a feedback control logic such that the system state is constrained onto the sliding surface

A switching control is compulsory when uncertainty in the system dynamics is dealt with

L1 – Internal and input-output dynamics

Consider the sliding variable as the output of a dynamical system

$$\begin{cases} \dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t), t) \\ \mathbf{y}(t) = \boldsymbol{\sigma}(\mathbf{x}(t)) \end{cases} \qquad \mathbf{x} \in \mathrm{R}^n, \quad \mathbf{u} \in \mathrm{R}^q, \quad \boldsymbol{\sigma} \in \mathrm{R}^q$$

In classical Sliding Mode Control the sliding surface is chosen so that

$$rank\left\{\frac{\partial \dot{\boldsymbol{\sigma}}}{\partial \mathbf{u}}\right\} = q \qquad \forall \mathbf{x} \in \mathsf{V}_\varepsilon$$

⇒

All the output variables have well defined relative degree 1

L1 – Internal and input-output dynamics

The system state can be represented by a combination of output and internal variables

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{w} \end{bmatrix} = \Phi(\mathbf{x}) \qquad \mathbf{x} \in \mathrm{R}^{n}, \quad \mathbf{y} \in \mathrm{R}^{q}, \quad \mathbf{w} \in \mathrm{R}^{n-q}$$

w: internal variables
y: output variables

$$\Phi : \mathrm{R}^{n} \to \mathrm{R}^{q} \times \mathrm{R}^{n-q}$$
$$\mathbf{0} = \Phi(\mathbf{0})$$

Diffeomorphism preserving the origin

The output variables are the sliding variables that we want to nullify

L1 – Internal and input-output dynamics

$$\begin{cases} \dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t), t) \\ \mathbf{y}(t) = \boldsymbol{\sigma}(\mathbf{x}(t)) \end{cases} \qquad \mathbf{x} \in \mathrm{R}^n, \quad \mathbf{u} \in \mathrm{R}^q, \quad \boldsymbol{\sigma} \in \mathrm{R}^q$$

Is equivalent to

$$\begin{cases} \dot{\mathbf{y}}(t) = \varphi(\mathbf{y}(t), \mathbf{w}(t), \mathbf{u}(t), t) \\ \dot{\mathbf{w}}(t) = \psi(\mathbf{y}(t), \mathbf{w}(t), t) \end{cases} \qquad \mathbf{y} \in \mathrm{R}^q, \quad \mathbf{w} \in \mathrm{R}^{n-q}, \quad \mathbf{u} \in \mathrm{R}^q$$

$$\frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}} \cdot f(\mathbf{x}(t), \mathbf{u}(t), t) = \varphi(\mathbf{y}(t), \mathbf{w}(t), \mathbf{u}(t), t)$$

$\varphi(\mathbf{y},\mathbf{w},\mathbf{u},t)$ is the input-output dynamics

$\psi(\mathbf{y},\mathbf{w},t)$ is the internal dynamics

L1 – Internal and input-output dynamics

The internal dynamics is strictly related on the choice of the sliding variables as functions of the state

The internal dynamics strongly affects the performance of the system and therefore the fulfillment of the control specifications depends on how the sliding variables are defined

The internal dynamics is needed to be Input-to-State Stable (ISS), at least

The system state is stabilizable if its input-output dynamics is controllable (stability can be enforced by the control) and its zero dynamics $\dot{\mathbf{w}}(t)=\psi(\mathbf{0},\mathbf{w}(t),t)$ is stable

L1 – Internal and input-output dynamics

Example

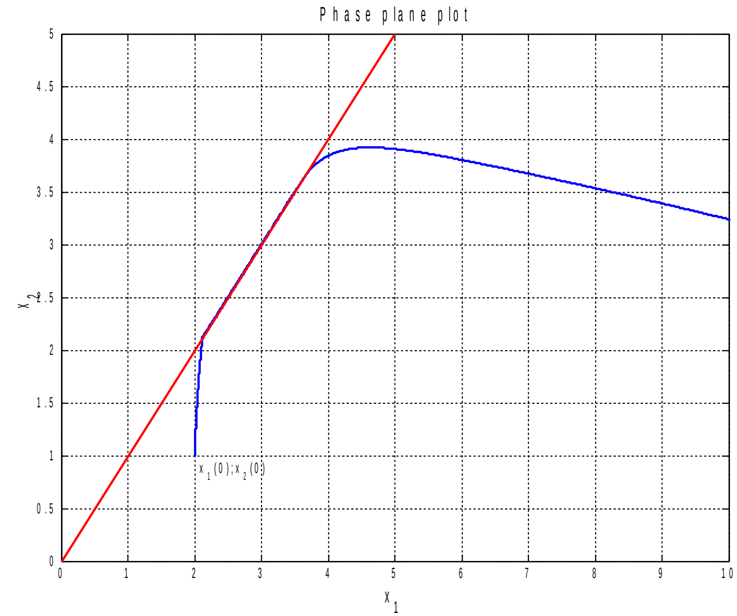
$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - x_2|x_2| + u \\ \sigma = x_2 - x_1 \end{cases}$$

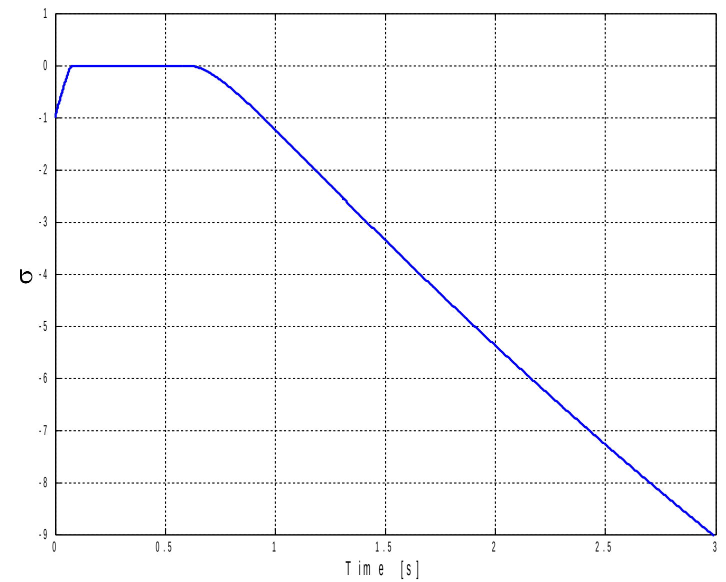
$$\begin{bmatrix} \sigma \\ w \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \mathbf{x}$$

$$\begin{cases} \dot{\sigma} = -(\sigma + w)(1 + |\sigma + w|) - w + u \\ \dot{w} = w \end{cases}$$

$$x_1(0) = 2; x_2(0) = 1$$

$$u = -20\,\mathrm{sgn}\,\sigma$$





The unstable zero-dynamics causes the loss of the sliding mode behaviour

L1 – Classic SMC

Finding a control law such that the stability condition for the sliding mode can be fulfilled for a generic nonlinear system is not easy.

⬇

Nonlinear systems affine in the control law

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{x}(t), t) + \mathbf{B}(\mathbf{x}(t), t) \cdot \mathbf{u}(t)$$

$$\mathbf{A} : \mathrm{R}^n \times \mathrm{R}^+ \rightarrow \mathrm{R}^n$$

$$b_{i,j} : \mathrm{R}^n \times \mathrm{R}^+ \rightarrow \mathrm{R}, \quad (\mathrm{i} = 1, \dots, \mathrm{n}), \quad (\mathrm{j} = 1, \dots, q)$$

Vector **A** and matrix **B** can be uncertain but some knowledge will be required to design the control law and assure the system stability

$L1$ – Classic SMC

The control affine assumption can be considered not much strong since any system can be reduced to such a form if a proper augmented dynamics is considered

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{u}} \\ \dot{\hat{\mathbf{x}}} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{x}} \\ \mathbf{0} \\ \dfrac{\partial f(\mathbf{x},\mathbf{u},t)}{\partial \mathbf{x}} \cdot \hat{\mathbf{x}} + \dfrac{\partial f(\mathbf{x},\mathbf{u},t)}{\partial t} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_q \\ \dfrac{\partial f(\mathbf{x},\mathbf{u},t)}{\partial \mathbf{u}} \end{bmatrix} \mathbf{v}$$

This generalization is relatively simple for Single-Input systems but give rise to very large and complicated systems in the multi-input case

$L1$ – Classic SMC

Theorem.
Consider the system dynamics $\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{x}(t),t) + \mathbf{B}(\mathbf{x}(t),t) \cdot \mathbf{u}(t)$

Chose the sliding variable set $\boldsymbol{\sigma}(\mathbf{x})$ such that the corresponding internal dynamics is ISS stable.

Assume that the uncertain matrix A is bounded by a known function

$$\left\| \mathbf{A}(\mathbf{x}(t),t) \right\| \le F(\mathbf{x})$$

Assume that the known square matrix $\dfrac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}} \cdot \mathbf{B}(\mathbf{x}(t),t)$ is non singular

The following state feedback control law assures the finite time stability of the sliding surface $\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{0}$

$$\mathbf{u} = -\left(F(\mathbf{x}) \left\| \frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}} \right\| + \eta \right) \left[\frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}} \cdot \mathbf{B}(\mathbf{x},t) \right]^{-1} \cdot \operatorname{sgn}(\boldsymbol{\sigma}) \qquad \eta > 0$$

L1 – Classic SMC

Proof.

Sliding variable dynamics

$$\dot{\boldsymbol{\sigma}}(t) = \frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}} \cdot \mathbf{A}(\mathbf{x}(t), t) + \frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}} \cdot \mathbf{B}(\mathbf{x}(t), t) \cdot \mathbf{u}(t)$$
$$= \frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}} \cdot \mathbf{A}(\mathbf{x}(t), t) - \left(F(\mathbf{x}) \left\| \frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}} \right\| + \eta \right) \operatorname{sgn}(\boldsymbol{\sigma})$$

Lyapunov function

$$V(\boldsymbol{\sigma}) = \frac{1}{2} \boldsymbol{\sigma}^T \cdot \boldsymbol{\sigma}$$

$$\dot{V} = \boldsymbol{\sigma}^T \cdot \dot{\boldsymbol{\sigma}}$$
$$= \boldsymbol{\sigma}^T \cdot \left[\frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}} \cdot \mathbf{A}(\mathbf{x}(t), t) - \left(F(\mathbf{x}) \left\| \frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}} \right\| + \eta \right) \operatorname{sgn}(\boldsymbol{\sigma}) \right]$$
$$\le -\eta \boldsymbol{\sigma}^T \cdot \operatorname{sgn}(\boldsymbol{\sigma}) = -\eta \|\boldsymbol{\sigma}\|_1 \le -\eta \|\boldsymbol{\sigma}\|_2$$
$$< -\eta \sqrt{V} < 0$$

L1 – Classic SMC

Perfect knowledge of the gain matrix **B** is problematic in practice

A switching control law which is able to enforce a sliding mode behavior can be defined if the gain matrix **B** is uncertain but has some properties

It must be possible to define the sliding variable vector $\boldsymbol{\sigma}(\mathbf{x})$ such that the gain matrix of the sliding variables dynamics:

- never vanishes
- it is positive definite
- a lower bound for its eigenvalues is known

L1 – Classic SMC

Theorem.
Consider the system dynamics $\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{x}(t),t) + \mathbf{B}(\mathbf{x}(t),t) \cdot \mathbf{u}(t)$

Chose the sliding variable set $\boldsymbol{\sigma}(\mathbf{x})$ such that the corresponding internal dynamics is ISS stable.

Assume that the uncertain matrix A is bounded by a known function

$$\left\| \mathbf{A}(\mathbf{x}(t),t) \right\| \leq F(\mathbf{x})$$

Assume that the square matrix $\frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}} \cdot \mathbf{B}(\mathbf{x}(t),t)$ is positive definite

Assume that $\Lambda_m = \min\left\{ eig\left\{ \frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}} \cdot \mathbf{B}(\mathbf{x}(t),t) \right\} \right\}$

The state feedback control law assures the finite time stability of the sliding surface $\boldsymbol{\sigma}(\mathbf{x})= \mathbf{0}$

$$\mathbf{u} = -\frac{F(\mathbf{x})\left\| \frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}} \right\| + \eta}{\Lambda_m} \frac{\boldsymbol{\sigma}}{\left\| \boldsymbol{\sigma} \right\|_2} \qquad \eta > 0$$

L1 – Classic SMC

Proof.

Sliding variable dynamics

$$\dot{\boldsymbol{\sigma}}(t)=\frac{\partial\boldsymbol{\sigma}}{\partial\mathbf{x}}\cdot\mathbf{A}(\mathbf{x}(t),t)+\frac{\partial\boldsymbol{\sigma}}{\partial\mathbf{x}}\cdot\mathbf{B}(\mathbf{x}(t),t)\cdot\mathbf{u}(t)$$

$$=\frac{\partial\boldsymbol{\sigma}}{\partial\mathbf{x}}\cdot\mathbf{A}(\mathbf{x}(t),t)-\frac{F(\mathbf{x})\left\|\frac{\partial\boldsymbol{\sigma}}{\partial\mathbf{x}}\right\|+\eta}{\Lambda_m}\frac{\boldsymbol{\sigma}}{\|\boldsymbol{\sigma}\|_2}$$

Lyapunov function

$$V(\boldsymbol{\sigma})=\frac{1}{2}\boldsymbol{\sigma}^T\cdot\boldsymbol{\sigma}$$

$$\dot{V}=\boldsymbol{\sigma}^T\cdot\dot{\boldsymbol{\sigma}}$$

$$=\boldsymbol{\sigma}^T\cdot\left[\frac{\partial\boldsymbol{\sigma}}{\partial\mathbf{x}}\cdot\mathbf{A}(\mathbf{x}(t),t)-\frac{F(\mathbf{x})\left\|\frac{\partial\boldsymbol{\sigma}}{\partial\mathbf{x}}\right\|+\eta}{\Lambda_m}\frac{\partial\boldsymbol{\sigma}}{\partial\mathbf{x}}\cdot\mathbf{B}(\mathbf{x}(t),t)\cdot\frac{\boldsymbol{\sigma}}{\|\boldsymbol{\sigma}\|_2}\right]$$

$$\leq-\frac{\eta}{\Lambda_m\|\boldsymbol{\sigma}\|_2}\boldsymbol{\sigma}^T\cdot\left[\frac{\partial\boldsymbol{\sigma}}{\partial\mathbf{x}}\cdot\mathbf{B}(\mathbf{x}(t),t)\right]\cdot\boldsymbol{\sigma}\leq-\eta\|\boldsymbol{\sigma}\|_2<-\eta\sqrt{V}<0$$

L1 – Classic SMC

If a local stability suffices, the magnitude of the switching control can be set to a sufficiently large value

$$\mathbf{u} = -U \cdot \operatorname{sgn}(\boldsymbol{\sigma}) \qquad U > \frac{F(\mathbf{x})\left\|\dfrac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}}\right\| + \eta}{\Lambda_m}$$

If it is necessary to limit the magnitude of the switching control, some knowledge about the system dynamics can be exploited

$$\dot{\boldsymbol{\sigma}}(t) = \varphi(\mathbf{x}(t),t) + \tilde{\varphi}(\mathbf{x}(t),t) + \boldsymbol{\Gamma}(\mathbf{x}(t),t) \cdot \mathbf{u}(t)$$
$$\|\tilde{\varphi}(\mathbf{x}(t),t)\| \le \Phi$$

$$\mathbf{u} = -\boldsymbol{\Gamma}^{-1}(\mathbf{x}(t),t) \cdot \left(\varphi(\mathbf{x}(t),t) + \Phi \operatorname{sgn}(\boldsymbol{\sigma})\right)$$

L1 – Invariance in SMC

A n^{th} order dynamical system constrained onto a q-dimensional surface of the state space presents a reduced order dynamics when in sliding mode

n state variables, q constraints → n-q "free" motions

↓

The zero dynamics

$$\dot{\mathbf{w}}(t) = \psi(\mathbf{0}, \mathbf{w}(t), t)$$

is the reduced order dynamics

L1 – Invariance in SMC

All the matching uncertainties (model mismatching and external disturbances) are compensated for by the control and do not affect the zero dynamics

⬇

The zero dynamics of a Variable Structure System with a Sliding Mode is invariant

⬇

All different systems having the same zero dynamics behave the same way

$L1$ – Invariance in SMC

Example: the Single-Input case of a system in the Brunowsky canonical form

$$\dot{x}_i = x_{i+1} \quad i = 1,2,\ldots n-1$$
$$\dot{x}_n = f(\mathbf{x},t) + b(\mathbf{x},t)u$$

The system is uncertain with known bounds

$$|f(\mathbf{x},t)| \le F(\mathbf{x}), \qquad 0 < b_m(\mathbf{x}) \le b(\mathbf{x},t)$$

c_i are chosen such that the corresponding polynomial is Hurwitz

$$\sigma = x_n + \sum_{i=1}^{n-1} c_i x_i$$
$$\dot{\sigma} = f(\mathbf{x},t) + b(\mathbf{x},t)u + \sum_{i=1}^{n-1} c_i x_{i+1}$$

Finite time convergence to the sliding manifold is assured

$$u = -\frac{F(\mathbf{x}) + \left|\sum_{i=1}^{n-1} c_i x_{i+1}\right| + k^2}{b_m(\mathbf{x})}\operatorname{sgn}(\sigma) \quad \Rightarrow \quad \sigma\dot{\sigma} \le -k^2|\sigma|$$

L1 – Invariance in SMC

The system is invariant when constrained on the sliding manifold σ

$$\dot{x}_i = x_{i+1} \quad i = 1,2,\ldots n-2$$
$$\dot{x}_{n-1} = -\sum_{i=1}^{n-1} c_i x_i + \sigma$$
$$x_n = -\sum_{i=1}^{n-1} c_i x_i + \sigma$$

The system behaves as a reduced order system with prescribed eigenvalues

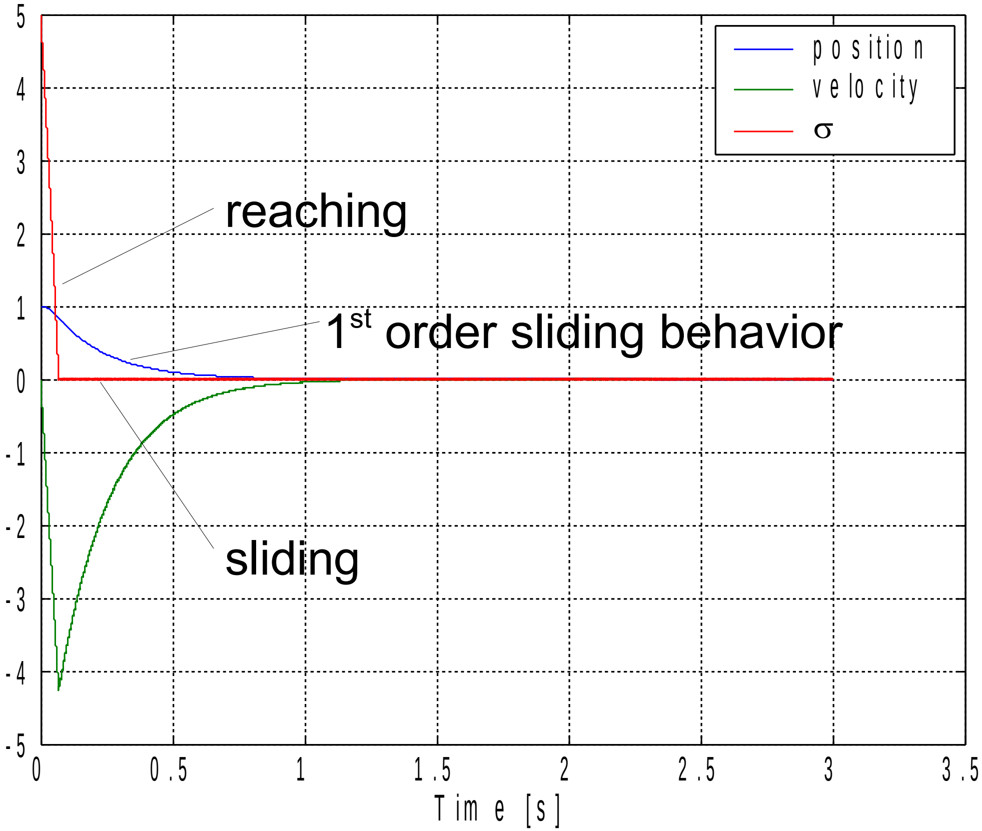
Matching uncertainties, included in the uncertain function f, are completely rejected

In the sliding mode it is not possible to “recover” the original system dynamics (*semi-group property*)

L1 – Invariance in SMC

$$m(t)\ddot{y} + \left(b_1 + b_2\left|\dot{y}\right|\right)\dot{y} + b_3\,\mathrm{sgn}(\dot{y}) + \left(k_1 + k_3 y^2\right)y = u - \sin(\pi t)$$

$$\sigma = \dot{y} + cy$$



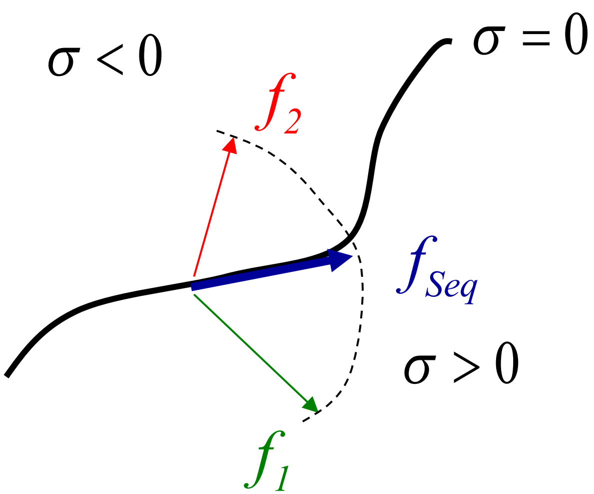
$$u = -U\,\mathrm{sgn}(\sigma)$$

L1 – Equivalent Control

Equivalent dynamics method

It is used when the discontinuity is due to switching of an independent variable, the “control”, $\boldsymbol{u}(t)$

$$\dot{\mathbf{x}}(t)=\begin{cases} f\left(\mathbf{x}(t),t,\mathbf{u}_1(t)\right) & \sigma\left(\mathbf{x}(t),t\right)<0 \\ f\left(\mathbf{x}(t),t,\mathbf{u}_{eq}(t)\right) & \sigma\left(\mathbf{x}(t),t\right)=0 \\ f\left(\mathbf{x}(t),t,\mathbf{u}_2(t)\right) & \sigma\left(\mathbf{x}(t),t\right)>0 \end{cases}$$



$$f_{s\,eq}\left(\mathbf{x}(t),t,\mathbf{u}(t)\right)=f\left(\mathbf{x}(t),t,\mathbf{u}_{eq}(t)\right)$$

$$\mathbf{u}_{eq}:\frac{\partial\sigma\left(\mathbf{x}(t),t\right)}{\partial\mathbf{x}}f_{s\,eq}\left(\mathbf{x}(t),t,\mathbf{u}(t)\right)=0$$

L1 – Equivalent Control

The equivalent control can be derived by looking for the continuous control that makes the time derivative of the sliding variable to zero

$$\dot{\boldsymbol{\sigma}}(t) = \frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}} \cdot \mathbf{A}(\mathbf{x}(t), t) + \frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}} \cdot \mathbf{B}(\mathbf{x}(t), t) \cdot \mathbf{u}_{eq}(t) = 0$$

⇓

$$\mathbf{u}_{eq}(t) = -\left[\frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}} \cdot \mathbf{B}(\mathbf{x}(t), t)\right]^{-1} \cdot \frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}} \cdot \mathbf{A}(\mathbf{x}(t), t)$$

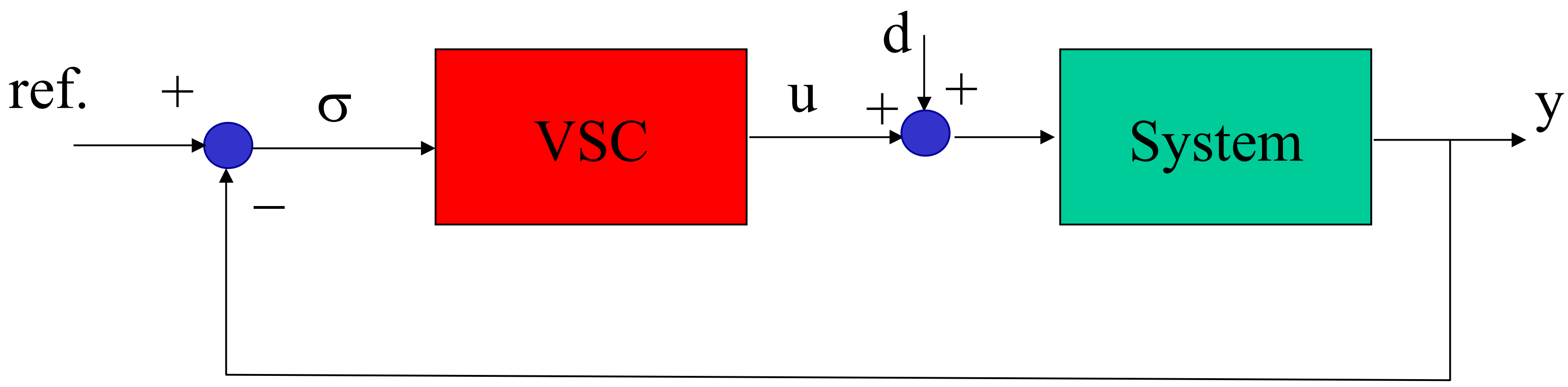
Filippov's continuation method and the equivalent control method can give different solutions in nonlinear systems

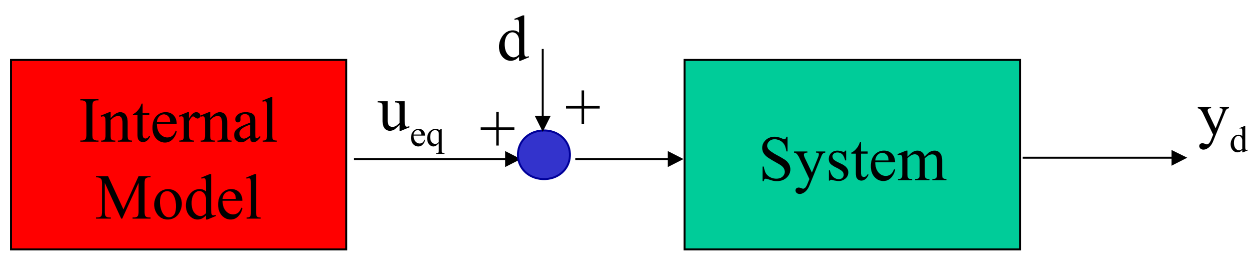
⇓

Not uniqueness problems in finding the solution of the sliding mode dynamics

L1 – Equivalent Control

The invariance property during the sliding mode means that the “**Internal Model Principle**” is fulfilled





The equivalent control compensates for uncertainties and generates the right input to the system to achieve the desired performance

L1 – Equivalent Control

The controlled system dynamics belongs to a differential inclusion

$$\dot{\mathbf{x}} \in \mathsf{F} \;\; = \mathbf{A}(\mathbf{x},t) + \mathbf{B}(\mathbf{x},t)[-U,+U]$$

The sliding variable $\boldsymbol{\sigma}$ can be considered as a performance index to be nullified to find the "right" solution

$$\dot{\mathbf{x}}^* = \mathbf{A}(\mathbf{x}^*,t) + \mathbf{B}(\mathbf{x}^*,t)\mathbf{u}_{eq} \in \mathsf{F}$$

The equivalent control is the continuous control corresponding to the "right" solution

L1 – Equivalent Control

The equivalent control is generated by infinite frequency switchings

$$U(j\omega) = U_{eq}(j\omega) + \overline{U}(j\omega)\Big|_{\omega=\infty}$$

The spectrum of the discontinuous control contains that of the equivalent control and can be recovered by low-pass filtering

$$\tau u_{av} + u_{av} = u$$

The equivalent control contains information about uncertainties

$$\mathbf{u}_{eq}(t) = -\left[\frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}} \cdot \mathbf{B}(\mathbf{x}(t), t)\right]^{-1} \cdot \frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}} \cdot \mathbf{A}(\mathbf{x}(t), t)$$

L1 – Equivalent Control

$$\tau u_{av} + u_{av} = u$$

If u_{eq} is bounded with its time derivative then

$$\lim_{\substack{\tau \to 0 \\ \frac{\Delta}{\tau} \to 0}} u_{av} = u_{eq} \qquad \left|\sigma\right| \leq \Delta$$

The *cut-off frequency* of the low-pass filter must be

- Greater than the bandwidth of the equivalent control
- Lower than the real switching frequency

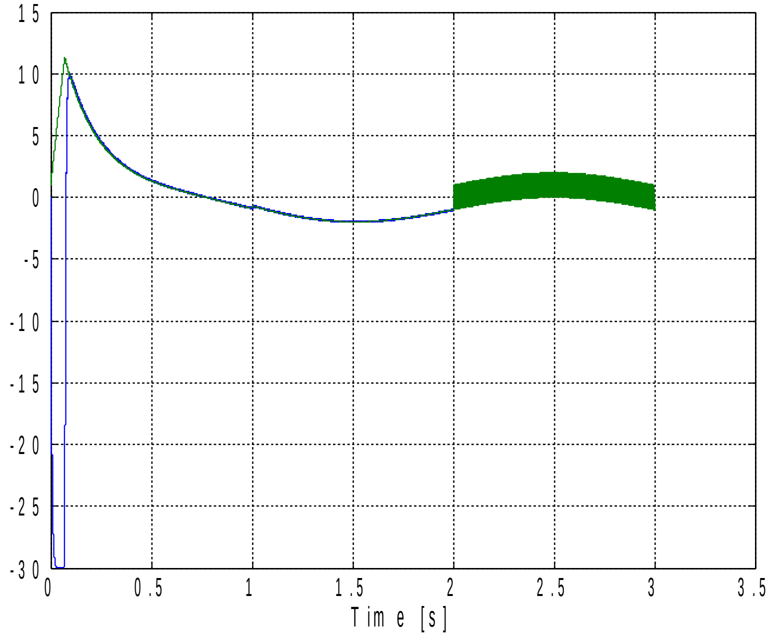
In practice only an estimate of u_{eq} can be evaluated

$L1$ – Equivalent Control

$$m(t)\ddot{y}+\left(b_1+b_2\left|\dot{y}\right|\right)\dot{y}+b_3\,\mathrm{sgn}\left(\dot{y}\right)+\left(k_1+k_3y^2\right)y=u-\sin\left(\pi t\right)$$

$$\sigma=\dot{y}+cy$$

$$u_{eq}=+\left(b_1+b_2\left|\dot{y}\right|\right)\dot{y}+b_3\,\mathrm{sgn}\left(\dot{y}\right)+\left(k_1+k_3y^2\right)y+\sin\left(\pi t\right)+cm(t)\dot{y}$$



L1 – Filippov solution

The controlled system dynamics belongs to a differential inclusion

$$\dot{\mathbf{x}} \in F = \left[\varphi^-, \varphi^+\right]$$

$$\varphi^- = \varphi\left(\mathbf{x}, t, u^-\right)$$

$$\varphi^+ = \varphi\left(\mathbf{x}, t, u^+\right)$$

The solution $\mathbf{x}^*$ such that state trajectory is tangent to the sliding manifold $\sigma=0$ belong to a convex set

$$\dot{\mathbf{x}} = \mu\varphi^+ + (1-\mu)\varphi^-, \quad \mu \in [0,1], \quad grad(\sigma)\cdot\dot{\mathbf{x}} = 0$$

L1 – Filippov solution

The average velocity is defined by

$$\mu = \frac{grad(\sigma)\cdot\varphi^{-}}{grad(\sigma)\cdot\left(\varphi^{-}-\varphi^{+}\right)}, \quad 1-\mu = \frac{grad(\sigma)\cdot\varphi^{+}}{grad(\sigma)\cdot\left(\varphi^{-}-\varphi^{+}\right)},$$

If the system dynamics is affine in the control variable the equivalent control and the Filippov solution agree

If the system dynamics is nonlinear the Filippov solution may be not unique

L1 – References

- V.I. Utkin, Sliding Modes In Control And Optimization, Springer Verlag, Berlin, 1992.
- C. Edwards, S.K. Spurgeon, Sliding mode control: theory and applications, Taylor and Francis Ltd, London, 1998.
- Y. Shtessel, C. Edwards, L. Fridman, A. Levant, Sliding Mode Control and Observation, Birkhäuser, New York, 2014
- A.F. Filippov, Differential Equations with Discontinuous Right–Hand Side, Kluwer, Dordrecht, The Netherlands, 1988.
- B. Draženović, “The invariance conditions in variable structure systems”, Automatica, 5, pp. 287-295, 1969.

L1 – References

- Pisano A., Usai E., "Sliding Mode Control: a Survey with Applications in Math", Mathematics and Computers in Simulation, vol.81, pp. 954-979, 2011
- R.A. De Carlo, S.H. Zak, G.P. Matthews, "Variable Structure Control of Nonlinear Multivariable System: A Tutorial, Proceedings of IEEE, vol. 76, no. 3, pp. 212-232, 1988
- G. Bartolini, T. Zolezzi, "Variable Structure Systems Nonlinear in the Control Law", IEEE Transactions on Automatic Control, vol.30, pp. 681-684, 1985
- K.H. Johansson, J. Lygeiros, S. Sastry, M. Egerstedt, "Simulation of Zeno Hybrid Automata, IEEE-CDC 1999, 3538-3543, Phenix, Arizona, USA, December 1999