NOTA 1. EQUAZIONI DI CONGRUENZA INTERNA.

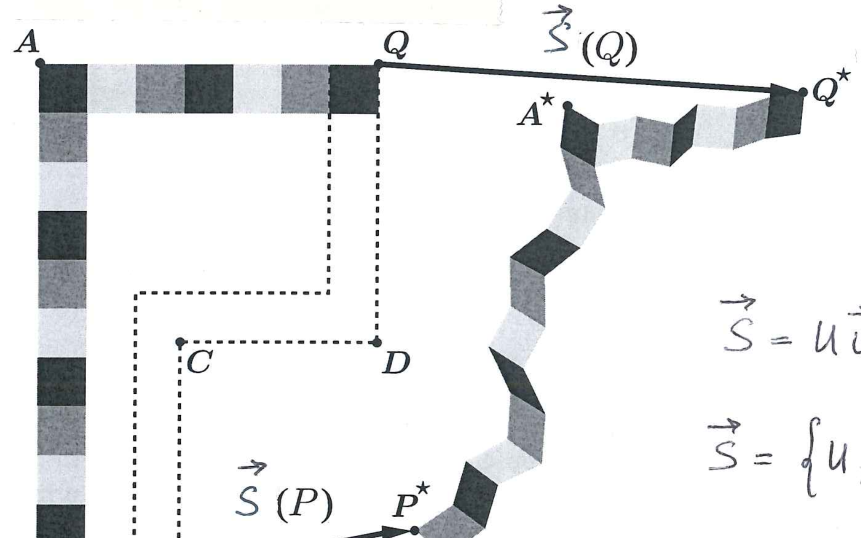
SI OSSERVA CHE SE LE 6 COMPONENTI DI DEFORMAZIONE $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}$ SONO ASSEGNATE COME FUNZIONI DELLE COORDINATE NON È A PRIORI GARANTITO CHE IL CAMPO DI SPOSTAMENTI CORRISPONDENTE RISULTI CONTINUO E CONGRUENTE CON LA CONTINUITA' INTERNA DELLA MATERIA, CHE ESCLUDE LA POSSIBILITA' DI LACERAZIONI E/O COMPENETRAZIONI.

DI QUESTO CI SI PUÒ RENDERE CONTO SE SI PENSA CHE LE QUANTITA' INCOGNITE DA DETERMINARE SONO 3 (u, v, w) MENTRE SI DISPONE DI 6 EQUAZIONI, IL CHE RENDE IL PROBLEMA SOVRA-DETERMINATO, E CHE LA SOLUZIONE POSSA ESISTERE SOLO SE SONO SODDISFATTE DELLE CONDIZIONI ADDIZIONALI PER LE COMPONENTI DI DEFORMAZIONE.

DAL PUNTO DI VISTA FISICO SI PUÒ GIUSTIFICARE LA NECESSITA' DI QUESTE CONDIZIONI CON QUESTO RAGIONAMENTO: SE SI SUDDIVIDE IL VOLUME V IN VOLUMETTI INFINITESIMI MEDIANTE UNA TRIPLICE FAMIGLIA DI PIANI PARALLELI A QUELLI COORDINATI, DANDO LUOGO A PARALLELEPIPEDI RETTI DI LATO INFINITESIMO, SE SI SUPPONE DI ATTRIBUIRE A CIASCUN VOLUMETTO COSÌ OTTENUTO LE DEFORMAZIONI CORRISPONDENTI AL RISPETTIVO BARICENTRO, SI TROVA CHE I PARALLELEPIPEDI A LATI RETTI SI TRASFORMANO IN PARALLELEPIPEDI OBLIQUI CON LATI DI LUNGHEZZA VARIATA.
IN GENERALE IN QUESTE CONDIZIONI NON RISULTA PIÙ POSSIBILE RICOMPORRE IL CORPO CONTINUO DI PARTENZA CON SOLI MOTI RIGIDI DEI PARALLELEPIPEDI DEFORMATI, A MENO CHE LE DEFORMAZIONI IMPOSTE NON SODDISFINO LE CONDIZIONI DI CONGRUENZA INTERNA.

LA SITUAZIONE È ILLUSTRATA IN FIGURA PER UN CORPO INIZIALMENTE PIANO.

LE CONDIZIONI DI CONGRUENZA ESPRIMONO IL FATTO CHE SCELTI 2 PUNTI ARBITRARI, P E Q ALL'INTERNO DEL CORPO LE COMPONENTI DI SPOSTAMENTO u, v, w E DI ROTAZIONE INFINITESIMA $\omega_x, \omega_y, \omega_z$ NECESSARIE A RICOMPORRE LA CONTINUITA' DEL CORPO PER QUALSIASI PERCORSO CONGIUNGENTE P CON Q NON DIPENDANO DAL PERCORSO STESSO, MA SOLTANTO DALLA POSIZIONE DEI PUNTI ESTREMI, P E Q.



$$\vec{S} = u\vec{i} + v\vec{j} + w\vec{k}$$

$$\vec{S} = \{u, v, w\}$$

QUESTO RICHIEDE CHE RISULTI:

$$u_Q = u_P + \int_P^Q du \quad ; \quad v_Q = v_P + \int_P^Q dv \quad ; \quad w_Q = w_P + \int_P^Q dw$$

$$\omega_{xQ} = \omega_{xP} + \int_P^Q d\omega_x \quad ; \quad \omega_{yQ} = \omega_{yP} + \int_P^Q d\omega_y \quad ; \quad \omega_{zQ} = \omega_{zP} + \int_P^Q d\omega_z$$

E CHE LE QUANTITA' INTEGRANDE SIANO DEI <u>DIFFERENZIALI ESATTI</u>

SI INIZIA RICORDANDO CHE $\underline{\underline{e}} = \underline{\underline{\omega}} + \underline{\underline{\varepsilon}}$, $\quad \underline{\underline{\omega}} = \frac{1}{2}(\underline{\underline{e}} - \underline{\underline{e}}^T)$; $\underline{\underline{\varepsilon}} = \frac{1}{2}(\underline{\underline{e}} + \underline{\underline{e}}^T)$

$$\underline{\underline{\omega}} = \begin{bmatrix} 0 & -\omega_z & +\omega_y \\ +\omega_z & 0 & -\omega_x \\ -\omega_y & +\omega_x & 0 \end{bmatrix} \qquad \underline{\underline{\varepsilon}} = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_z \end{bmatrix}$$

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \qquad \varepsilon_x = \frac{\partial u}{\partial x} \qquad \frac{1}{2}\gamma_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \frac{1}{2}\gamma_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad \varepsilon_z = \frac{\partial w}{\partial z} \qquad \frac{1}{2}\gamma_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

PERTANTO

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz$$

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz$$

E DUNQUE:

$$du = \varepsilon_x\,dx + \underbrace{\left(\frac{1}{2}\gamma_{xy} - \omega_z\right)}_{\frac{1}{2}\left(\frac{\partial u}{\partial y} + \cancel{\frac{\partial v}{\partial x}}\right) - \frac{1}{2}\left(\cancel{\frac{\partial v}{\partial x}} - \frac{\partial u}{\partial y}\right)} dy + \underbrace{\left(\frac{1}{2}\gamma_{xz} + \omega_y\right)}_{\frac{1}{2}\left(\frac{\partial u}{\partial z} + \cancel{\frac{\partial w}{\partial x}}\right) + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \cancel{\frac{\partial w}{\partial x}}\right)} dz \qquad [1]$$

$$dv = \underbrace{\left(\frac{1}{2}\gamma_{xy} + \omega_z\right)}_{\frac{1}{2}\left(\cancel{\frac{\partial u}{\partial y}} + \frac{\partial v}{\partial x}\right) + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \cancel{\frac{\partial u}{\partial y}}\right)} dx + \varepsilon_y\,dy + \underbrace{\left(\frac{1}{2}\gamma_{yz} - \omega_x\right)}_{\frac{1}{2}\left(\frac{\partial v}{\partial z} + \cancel{\frac{\partial w}{\partial y}}\right) - \frac{1}{2}\left(\cancel{\frac{\partial w}{\partial y}} - \frac{\partial v}{\partial z}\right)} dz \qquad [2]$$

$$dw = \underbrace{\left(\frac{1}{2}\gamma_{xz} - \omega_y\right)}_{\frac{1}{2}\left(\cancel{\frac{\partial u}{\partial z}}+\frac{\partial w}{\partial x}\right)-\frac{1}{2}\left(\cancel{\frac{\partial u}{\partial z}}-\frac{\partial w}{\partial x}\right)} dx + \underbrace{\left(\frac{1}{2}\gamma_{yz} + \omega_x\right)}_{\frac{1}{2}\left(\cancel{\frac{\partial v}{\partial z}}+\frac{\partial w}{\partial y}\right)+\frac{1}{2}\left(\frac{\partial w}{\partial y}-\cancel{\frac{\partial v}{\partial z}}\right)} dy + \varepsilon_z\, dz \qquad [3]$$

SE dG È UN DIFFERENZIALE ESATTO, ALLORA

$$dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}dy + \frac{\partial G}{\partial z}dz \qquad [4]$$

MA SE dG HA LA FORMA:

$$dG = F_x\, dx + F_y\, dy + F_z\, dz \qquad [5]$$

ALLORA È RICHIESTO CHE SIA:

$$\begin{cases} F_x = \dfrac{\partial G}{\partial x} \\[2ex] F_y = \dfrac{\partial G}{\partial y} \\[2ex] F_z = \dfrac{\partial G}{\partial z} \end{cases} \iff \begin{aligned} \frac{\partial^2 G}{\partial x \partial y} &= \frac{\partial^2 G}{\partial y \partial x} \\ \frac{\partial^2 G}{\partial x \partial z} &= \frac{\partial^2 G}{\partial z \partial x} \\ \frac{\partial^2 G}{\partial y \partial z} &= \frac{\partial^2 G}{\partial z \partial y} \end{aligned} \iff \begin{aligned} \frac{\partial F_y}{\partial x} &= \frac{\partial F_x}{\partial y} \qquad [6] \\ \frac{\partial F_z}{\partial x} &= \frac{\partial F_x}{\partial z} \qquad [7] \\ \frac{\partial F_z}{\partial y} &= \frac{\partial F_y}{\partial z} \qquad [8] \end{aligned}$$

DOVE IL PRIMO PASSAGGIO È CONSEGUENZA DEL TEOREMA DI SCHWARTZ. (PER UNA FUNZIONE CONTINUA E DERIVABILE LE DERIVATE PARZIALI SECONDE MISTE SONO EGUALI)

APPLICANDO LE [6]-[8] ALLA [1] SI TROVA:

$(x-y)$ $$\frac{\partial \varepsilon_x}{\partial y} = \frac{1}{2}\frac{\partial \gamma_{xy}}{\partial x} - \frac{\partial \omega_z}{\partial x} \Rightarrow \frac{\partial \omega_z}{\partial x} = \frac{1}{2}\frac{\partial \gamma_{xy}}{\partial x} - \frac{\partial \varepsilon_x}{\partial y} \qquad [zx]$$

$(x-z)$ $$\frac{\partial \varepsilon_x}{\partial z} = \frac{1}{2}\frac{\partial \gamma_{xz}}{\partial x} + \frac{\partial \omega_y}{\partial x} \Rightarrow \frac{\partial \omega_y}{\partial x} = \frac{\partial \varepsilon_x}{\partial z} - \frac{1}{2}\frac{\partial \gamma_{xz}}{\partial x} \qquad [yx]$$

$(y-z)$ $$\frac{1}{2}\frac{\partial \gamma_{xy}}{\partial z} - \frac{\partial \omega_z}{\partial z} = \frac{1}{2}\frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \omega_y}{\partial y} \Rightarrow \frac{\partial \omega_y}{\partial y} + \frac{\partial \omega_z}{\partial z} = \frac{1}{2}\frac{\partial \gamma_{xy}}{\partial z} - \frac{1}{2}\frac{\partial \gamma_{xz}}{\partial y} \qquad [yy, zz]$$

APPLICANDO LE [6]-[8] ALLA [2] SI OTTIENE:

$(x-y)$ $$\frac{1}{2}\frac{\partial \gamma_{xy}}{\partial y} + \frac{\partial \omega_z}{\partial y} = \frac{\partial \varepsilon_y}{\partial x} \Rightarrow \frac{\partial \omega_z}{\partial y} = \frac{\partial \varepsilon_y}{\partial x} - \frac{1}{2}\frac{\partial \gamma_{xy}}{\partial y} \qquad [zy]$$

$(x-z)$ $$\frac{1}{2}\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \omega_z}{\partial z} = \frac{1}{2}\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \omega_x}{\partial x} \Rightarrow \frac{\partial \omega_x}{\partial x} + \frac{\partial \omega_z}{\partial z} = \frac{1}{2}\frac{\partial \gamma_{yz}}{\partial x} - \frac{1}{2}\frac{\partial \gamma_{xy}}{\partial z} \qquad [xx, zz]$$

$(y-z)$ $$\frac{\partial \varepsilon_y}{\partial z} = \frac{1}{2}\frac{\partial \gamma_{yz}}{\partial y} - \frac{\partial \omega_x}{\partial y} \Rightarrow \frac{\partial \omega_x}{\partial y} = \frac{1}{2}\frac{\partial \gamma_{yz}}{\partial y} - \frac{\partial \varepsilon_y}{\partial z} \qquad [xy]$$

APPLICANDO LE [6]-[8] ALLA [3] SI OTTIENE INFINE:

$$(x-y)\quad \frac{1}{2}\frac{\partial \gamma_{xz}}{\partial y}-\frac{\partial \omega_y}{\partial y}=\frac{1}{2}\frac{\partial \gamma_{yz}}{\partial x}+\frac{\partial \omega_x}{\partial x}\;\Rightarrow\;\frac{\partial \omega_x}{\partial x}+\frac{\partial \omega_y}{\partial y}=\frac{1}{2}\frac{\partial \gamma_{xz}}{\partial y}-\frac{1}{2}\frac{\partial \gamma_{yz}}{\partial x}\qquad [xx,yy]$$

$$(x-z)\quad \frac{1}{2}\frac{\partial \gamma_{xz}}{\partial z}-\frac{\partial \omega_y}{\partial z}=\frac{\partial \varepsilon_z}{\partial x}\;\Rightarrow\;\frac{\partial \omega_y}{\partial z}=+\frac{1}{2}\frac{\partial \gamma_{xz}}{\partial y}-\frac{\partial \varepsilon_z}{\partial x}\qquad [yz]$$

$$(y-z)\quad \frac{1}{2}\frac{\partial \gamma_{yz}}{\partial z}+\frac{\partial \omega_x}{\partial z}=\frac{\partial \varepsilon_z}{\partial y}\;\Rightarrow\;\frac{\partial \omega_x}{\partial z}=\frac{\partial \varepsilon_z}{\partial y}-\frac{1}{2}\frac{\partial \gamma_{yz}}{\partial z}\qquad [xz]$$

DALLE [yy,zz], [xx,zz] E [xx,yy] SEGUE POI:

$$\frac{\partial \omega_y}{\partial y}+\frac{\partial \omega_z}{\partial z}=\frac{1}{2}\frac{\partial \gamma_{xy}}{\partial z}-\frac{1}{2}\frac{\partial \gamma_{xz}}{\partial y}\qquad [11]$$

$$\frac{\partial \omega_x}{\partial x}+\frac{\partial \omega_z}{\partial z}=-\frac{1}{2}\frac{\partial \gamma_{xy}}{\partial z}+\frac{1}{2}\frac{\partial \gamma_{yz}}{\partial x}\qquad [12]$$

$$\frac{\partial \omega_x}{\partial x}+\frac{\partial \omega_y}{\partial y}=\frac{1}{2}\frac{\partial \gamma_{xz}}{\partial y}-\frac{1}{2}\frac{\partial \gamma_{yz}}{\partial x}\qquad [13]$$

SOMMANDO LE [11]-[13] SI TROVA:

$$2\left(\frac{\partial \omega_x}{\partial x}+\frac{\partial \omega_y}{\partial y}+\frac{\partial \omega_z}{\partial z}\right)=0\;\Rightarrow\;\frac{\partial \omega_x}{\partial x}+\frac{\partial \omega_y}{\partial y}+\frac{\partial \omega_z}{\partial z}=0$$

COMBINANDO ORA LA [12] E LA [13] SI OTTIENE:

$$\frac{\partial \omega_x}{\partial x}+\underbrace{\left(\frac{\partial \omega_x}{\partial x}+\frac{\partial \omega_y}{\partial y}+\frac{\partial \omega_z}{\partial z}\right)}_{0}=-\frac{1}{2}\frac{\partial \gamma_{xy}}{\partial z}+\frac{1}{2}\frac{\partial \gamma_{xz}}{\partial y}\;\Rightarrow\;\frac{\partial \omega_x}{\partial x}=\frac{1}{2}\frac{\partial \gamma_{xz}}{\partial y}-\frac{1}{2}\frac{\partial \gamma_{xy}}{\partial z}\qquad [xx]$$

MENTRE COMBINANDO LA [11] E LA [13] SI TROVA:

$$\underbrace{\left(\frac{\partial \omega_x}{\partial x}+\frac{\partial \omega_y}{\partial y}+\frac{\partial \omega_z}{\partial z}\right)}_{0}+\frac{\partial \omega_y}{\partial y}=\frac{1}{2}\frac{\partial \gamma_{xy}}{\partial z}-\frac{1}{2}\frac{\partial \gamma_{yz}}{\partial x}\;\Rightarrow\;\frac{\partial \omega_y}{\partial y}=\frac{1}{2}\frac{\partial \gamma_{xy}}{\partial z}-\frac{1}{2}\frac{\partial \gamma_{yz}}{\partial x}\qquad [yy]$$

INFINE, COMBINANDO LA [11] E LA [12] SI HA:

$$\underbrace{\left(\frac{\partial \omega_x}{\partial x}+\frac{\partial \omega_y}{\partial y}+\frac{\partial \omega_z}{\partial z}\right)}_{0}+\frac{\partial \omega_z}{\partial z}=-\frac{1}{2}\frac{\partial \gamma_{xz}}{\partial y}+\frac{1}{2}\frac{\partial \gamma_{yz}}{\partial x}\;\Rightarrow\;\frac{\partial \omega_z}{\partial z}=\frac{1}{2}\frac{\partial \gamma_{yz}}{\partial x}-\frac{1}{2}\frac{\partial \gamma_{xz}}{\partial y}\qquad [zz]$$

ORA SI IMPONE CHE ANCHE $d\omega_x$, $d\omega_y$, $d\omega_z$ RISULTINO DIFFERENZIALI ESATTI.
PER IL PRIMO SI HA: $d\omega_x=\frac{\partial \omega_x}{\partial x}dx+\frac{\partial \omega_x}{\partial y}dy+\frac{\partial \omega_z}{\partial z}dz$

E TENENDO CONTO DELLE [xx], [xy], [xz] SI SCRIVE:

$$d\omega_x = \left(\frac{1}{2}\frac{\partial \gamma_{xz}}{\partial y} - \frac{1}{2}\frac{\partial \gamma_{xy}}{\partial z}\right)dx + \left(\frac{1}{2}\frac{\partial \gamma_{yz}}{\partial y} - \frac{\partial \varepsilon_y}{\partial z}\right)dy + \left(\frac{\partial \varepsilon_z}{\partial y} - \frac{1}{2}\frac{\partial \gamma_{yz}}{\partial z}\right)dz$$

DI QUI SI OTTIENE:

(x-y)
$$\frac{1}{2}\frac{\partial^2 \gamma_{xz}}{\partial y^2} - \frac{1}{2}\frac{\partial^2 \gamma_{xy}}{\partial y \partial z} = \frac{1}{2}\frac{\partial^2 \gamma_{yz}}{\partial x \partial y} - \frac{\partial^2 \varepsilon_y}{\partial x \partial z}$$

$$\Rightarrow \boxed{\frac{\partial^2 \varepsilon_y}{\partial x \partial z} = \frac{1}{2}\frac{\partial}{\partial y}\left[\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xy}}{\partial z} - \frac{\partial \gamma_{xz}}{\partial y}\right]} \quad [14]$$

(x-z)
$$\frac{1}{2}\frac{\partial^2 \gamma_{xz}}{\partial z \partial y} - \frac{1}{2}\frac{\partial^2 \gamma_{xy}}{\partial z^2} = \frac{\partial^2 \varepsilon_z}{\partial x \partial y} - \frac{1}{2}\frac{\partial^2 \gamma_{yz}}{\partial x \partial z}$$

$$\Rightarrow \boxed{\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{1}{2}\frac{\partial}{\partial z}\left[\frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xy}}{\partial z}\right]} \quad [15]$$

(y-z)
$$\frac{1}{2}\frac{\partial^2 \gamma_{yz}}{\partial z \partial y} - \frac{\partial^2 \varepsilon_y}{\partial z^2} = \frac{\partial^2 \varepsilon_z}{\partial y^2} - \frac{1}{2}\frac{\partial^2 \gamma_{yz}}{\partial y \partial z}$$

$$\Rightarrow \boxed{\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z}} \quad [16]$$

SIMILMENTE DA $d\omega_y = \frac{\partial \omega_y}{\partial x}dx + \frac{\partial \omega_y}{\partial y}dy + \frac{\partial \omega_y}{\partial z}dz$ SEGUE, TENENDO CONTO DELLE [yx], [yy], [yz]:

$$d\omega_y = \left(\frac{\partial \varepsilon_x}{\partial z} - \frac{1}{2}\frac{\partial \gamma_{xz}}{\partial x}\right)dx + \left(\frac{1}{2}\frac{\partial \gamma_{xy}}{\partial z} - \frac{1}{2}\frac{\partial \gamma_{yz}}{\partial x}\right)dy + \left(\frac{1}{2}\frac{\partial \gamma_{xz}}{\partial z} - \frac{\partial \varepsilon_z}{\partial x}\right)dz$$

E DUNQUE

(x-y)
$$\frac{\partial^2 \varepsilon_x}{\partial y \partial z} - \frac{1}{2}\frac{\partial^2 \gamma_{xz}}{\partial y \partial x} = \frac{1}{2}\frac{\partial \gamma_{xy}}{\partial x \partial z} - \frac{1}{2}\frac{\partial^2 \gamma_{yz}}{\partial x^2}$$

$$\Rightarrow \boxed{\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{1}{2}\frac{\partial}{\partial x}\left[\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{xz}}{\partial y} - \frac{\partial \gamma_{yz}}{\partial x}\right]} \quad [17]$$

(x-z)
$$\frac{\partial^2 \varepsilon_x}{\partial z^2} - \frac{1}{2}\frac{\partial^2 \gamma_{xz}}{\partial z \partial x} = \frac{1}{2}\frac{\partial^2 \gamma_{xz}}{\partial x \partial z} - \frac{\partial^2 \varepsilon_z}{\partial x^2}$$

$$\Rightarrow \boxed{\frac{\partial^2 \varepsilon_x}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial x^2} = \frac{\partial^2 \gamma_{xz}}{\partial x \partial z}} \quad [18]$$

(y-z)
$$\frac{1}{2}\frac{\partial^2 \gamma_{xy}}{\partial z^2} - \frac{1}{2}\frac{\partial^2 \gamma_{yz}}{\partial z \partial x} = \frac{1}{2}\frac{\partial^2 \gamma_{xz}}{\partial y \partial z} - \frac{\partial^2 \varepsilon_z}{\partial y \partial x}$$

$$\Rightarrow \boxed{\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{1}{2}\frac{\partial}{\partial z}\left[\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right]} \qquad [19]$$

INFINE DA $d\omega_z = \frac{\partial \omega_z}{\partial x} dx + \frac{\partial \omega_z}{\partial y} dy + \frac{\partial \omega_z}{\partial z} dz$ SEGUE, TENENDO CONTO DELLE [zx], [zy], [zz]:

$$d\omega_z = \left(\frac{1}{2}\frac{\partial \gamma_{xy}}{\partial x} - \frac{\partial \varepsilon_x}{\partial y}\right) dx + \left(\frac{\partial \varepsilon_y}{\partial x} - \frac{1}{2}\frac{\partial \gamma_{xy}}{\partial y}\right) dy + \left(\frac{1}{2}\frac{\partial \gamma_{yz}}{\partial x} - \frac{1}{2}\frac{\partial \gamma_{xz}}{\partial y}\right) dz$$

PERTANTO:

(x-y)
$$\frac{1}{2}\frac{\partial^2 \gamma_{xy}}{\partial y \partial x} - \frac{\partial^2 \varepsilon_x}{\partial y^2} = \frac{\partial^2 \varepsilon_y}{\partial x^2} - \frac{1}{2}\frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

$$\Rightarrow \boxed{\frac{\partial^2 \varepsilon_y}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial y^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}} \qquad [20]$$

(x-z)
$$\frac{1}{2}\frac{\partial^2 \gamma_{xy}}{\partial z \partial x} - \frac{\partial^2 \varepsilon_x}{\partial z \partial y} = \frac{1}{2}\frac{\partial^2 \gamma_{yz}}{\partial x^2} - \frac{1}{2}\frac{\partial^2 \gamma_{xz}}{\partial x \partial y}$$

$$\Rightarrow \boxed{\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{1}{2}\frac{\partial}{\partial x}\left[\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{xz}}{\partial y} - \frac{\partial \gamma_{yz}}{\partial x}\right]} \qquad [21]$$

(y-z)
$$\frac{\partial^2 \varepsilon_y}{\partial z \partial x} - \frac{1}{2}\frac{\partial^2 \gamma_{xy}}{\partial z \partial y} = \frac{1}{2}\frac{\partial^2 \gamma_{yz}}{\partial y \partial x} - \frac{1}{2}\frac{\partial^2 \gamma_{xz}}{\partial y^2}$$

$$\Rightarrow \boxed{\frac{\partial^2 \varepsilon_y}{\partial x \partial z} = \frac{1}{2}\frac{\partial}{\partial y}\left[\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xz}}{\partial y}\right]} \qquad [22].$$

SI NOTI CHE FRA LE [14] - [22] SOLO 6 SONO INDIPENDENTI (LE [14] E [22]; [15] E [19]; [17] E [21] COINCIDONO) E RAPPRESENTANO LE CONDIZIONI CHE GARANTISCONO (IN UN CORPO MONOCONNESSO) L'ESISTENZA E L'UNICITA' (A MENO DI UN MOTO RIGIDO) DI UN CAMPO DI SPOSTAMENTI CONTINUO CHE PRODUCA LE DEFORMAZIONI ASSEGNATE.

SI OSSERVI, PER ESEMPIO, CHE DALLA [14] SEGUE:

$$\frac{\partial^2 \varepsilon_y}{\partial x \partial z} = \frac{1}{2}\frac{\partial}{\partial y}\left[\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xy}}{\partial z} - \frac{\partial \gamma_{xz}}{\partial y}\right] \Rightarrow \frac{\partial^2}{\partial x \partial z}\left[\frac{\partial v}{\partial y}\right] = \frac{1}{2}\frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right]$$

$$\Rightarrow \frac{\partial^3 v}{\partial x \partial y \partial z} = \frac{1}{2}\frac{\partial}{\partial y}\left[\frac{\partial^2 v}{\partial x \partial z} + \cancel{\frac{\partial^2 w}{\partial x \partial y}} + \cancel{\frac{\partial^2 u}{\partial z \partial y}} + \frac{\partial^2 v}{\partial x \partial z} - \cancel{\frac{\partial^2 u}{\partial y \partial z}} - \cancel{\frac{\partial^2 w}{\partial x \partial y}}\right] = 2 \cdot \frac{1}{2}\frac{\partial^3 v}{\partial x \partial y \partial z}.$$

SIMILMENTE DALLA [16] SI HA:

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \Rightarrow \frac{\partial^2}{\partial z^2}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial w}{\partial z}\right) = \frac{\partial^2}{\partial y \partial z}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \frac{\partial^3 v}{\partial y \partial z^2} + \frac{\partial^3 w}{\partial y \partial z}.$$

$\square$