Financial Derivatives

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Preliminary requirements

- Please email TODAY your CV with passport-like photo at

  merella@unica.it

- In the email text, rate your knowledge of the following topics

  [1-10, 1 = awful, 10 = excellent]

  - mathematics (calculus)
  - statistics and econometrics
  - computational methods

Preliminary information

- Course material

unica.it/unica/page/it/financial_derivatives_and_risk_management

- Office hours

Fridays, 11-12, meet.google.com/imo-jxka-qvn

  - please email me in advance

Examination format

- The exam features 12 on-screen written questions, answered orally
- The questions are divided in 4 groups, each covering a different topic
  - the 3 questions in each group are in an increasing order of difficulty
  - each question yield up to three points
    - you can still score 27/30 by failing to reply to one whole topic!
    - and still 24/30 by failing to reply to the most difficult questions!
- The duration of the exam is 12 minutes per candidate
  - this gives you one minute per question on average
  - practising the course exercise is of paramount importance!
- A mock exam will be provided at the end of week 5

Objective

- The market of *financial derivatives* is huge

  en.wikipedia.org/wiki/Derivative_(finance)

  - its value is about \$640T when measured in terms of underlying assets
  - more than four times as large as the world GDP (\$142T)
  - more than ten times as large as the value of stock traded (\$60T)
- The reason is that derivatives play a key role in transferring a wide range of risks in the economy from one entity to another
- Derivatives have become so important that who works in finance *needs* to understand how they work, are used and priced
- Our goal is to try and understand exactly that!

Paradigm

- One simple question conveys the key issue of financial economics

what is the price of financial risk?

- In asset pricing, we take a macroeconomic stance and find that the price of systematic risk is ultimately unknown
  - quantifying and pricing risk would help avoiding financial crises, but...
  - we see the failure of *absolute pricing* models (CAPM, CCAPM, ...)
  - we observe the inconsistency of *relative pricing* models (APT, ...)
- In risk management, we study the microeconomic perspective and show that the price of risk for specific asset can be identified
  - macroeconomic inconsistency of relative pricing is due to the failure of identifying the reference asset (market portfolio composition)
  - from a microeconomic point of view, the reference asset is perfectly identified for financial derivatives: it is the **underlying asset**
  - hence we can manage individual investors' exposure to financial risk

Approach

- We conceive risk as an *opportunity cost*

  en.wikipedia.org/wiki/Opportunity_cost

- If my capital is invested in bonds, the risk (opportunity cost) is the inability to devote those resources to a more profitable investment
- If my capital is invested in a more profitable investment, that might (in fact, *must*) come at the risk (opportunity cost) of making losses
- If I have a *short position* in an investment, I face the exactly opposite risk (opportunity cost)
  - things may go too well (for the investment in question)
  - I may have to pay a lot of money to close my position

  en.wikipedia.org/wiki/Short_(finance)

Preview of findings

- We show that derivatives can *completely* offset financial risk
- We also show that derivatives can be *exactly* priced
- We develop out intuitions and the resulting model in discrete time but we also lay the foundations for continuous-time pricing
- Our applications focus on **options**, by far the most interesting nontrivial type of derivative

en.wikipedia.org/wiki/Option_(finance)

- We study classical hedging strategies based on the so-called **Greeks**

en.wikipedia.org/wiki/Greeks_(finance)

Overview

- Week 1: Basic risk management strategies
- Week 2: Binomial model
- Week 3: Multiperiod binomial model
- Week 4: Discrete-time options
- Week 5: Classical hedging strategies
- Week 6: Continuous-time options

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Financial risk management

- We begin by studying a typical example of opportunity cost handling
  - consider a stock, for instance Intel Corporation

    finance.yahoo.com/quote/INTC

  - suppose that we have a *long position* (we own the stock)

    en.wikipedia.org/wiki/Long_(finance)

- The risk is measured by the opportunity cost of keeping our capital invested in the stock we own
- In turn, the opportunity cost equals the cash flow that obtains by immediately dismissing the stock
- We assess financial strategies by comparing *potential revenues* and opportunity cost

Strategy 0: sell the stock

en.wikipedia.org/wiki/Spot_date

- The most elemental strategy is to dismiss the stock immediately
  - we may store the revenue in the bank, yielding interest at the rate R
  - if $R = 0$, the revenue exactly equals the opportunity cost
  - the *potential* profit is, quite obviously, *deterministically* null
- Imagine to stick to the asset under consideration in isolation
- We have an alternative option: keep the stock in our portfolio for some time (say 6 months)
- Comparatively, what is the best strategy?
  - the answer is straightforward: we don't know!
  - to see this, let's compare profit opportunities between the strategy **buy-and-hold** and the (zero-profit!) strategy **spot-sale**

Strategy 1: hold the stock

- If we hold the stock, we need to simulate the future stock value in order to assess our profit opportunities
  - the cash flow obtains at a future date, so it should be discounted
  - for simplicity, also here we assume that the discount rate is $R = 0$
- We use MATLAB codes designed to forecast daily stock prices
- The code is simple and requires only a few data on our stock, namely **spot** (current) **price**, (annual) **trend** and (annual) **volatility**
- We easily obtain these values online, for instance once again at

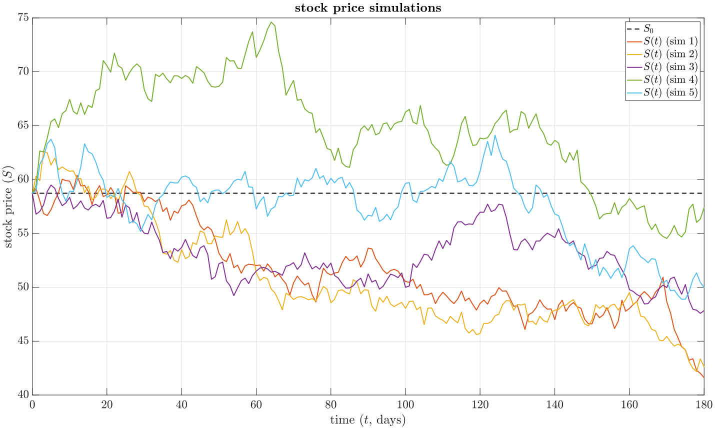
finance.yahoo.com/quote/INTC

  - we click on the 'chart' tab, select '1Y', then 'Historical Volatility' from the 'indicators' menu
  - we take note of current price, price a year ago and volatility and add this information to the MATLAB codes

Strategy 1: hold the stock

Possible evolution of the stock price

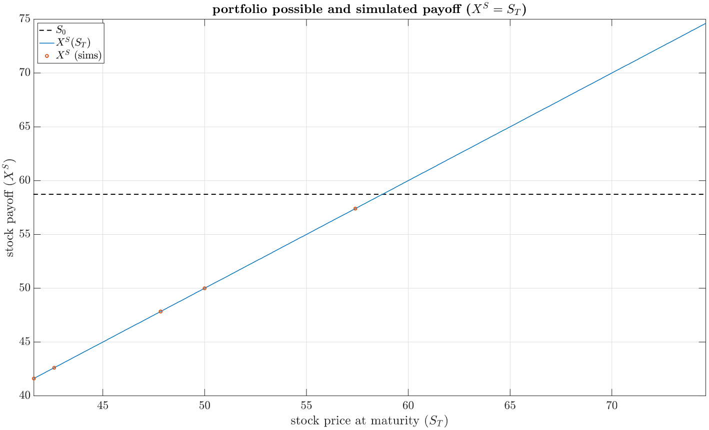
- MATLAB code: `AE_GraphStockSim.m`



Strategy 1: hold the stock

Possible payoff (revenue) when holding the stock at maturity

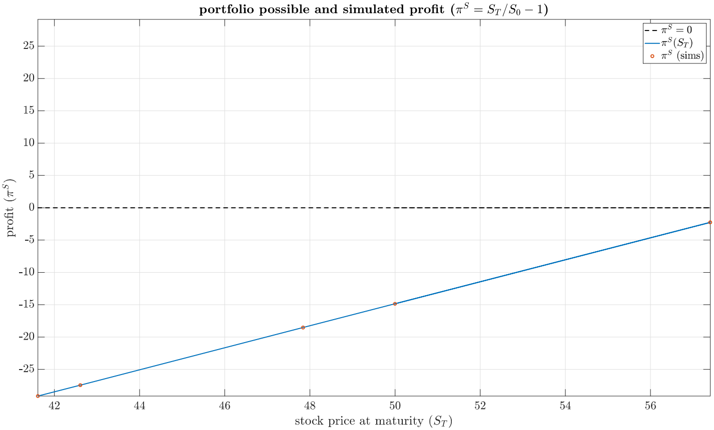
- MATLAB code: `AE_GraphStockPayoff.m`



Strategy 1: hold the stock

Possible profit when holding the stock at maturity

- MATLAB code: `AE_GraphStockProfit.m`



Strategy 2: add a long European put option

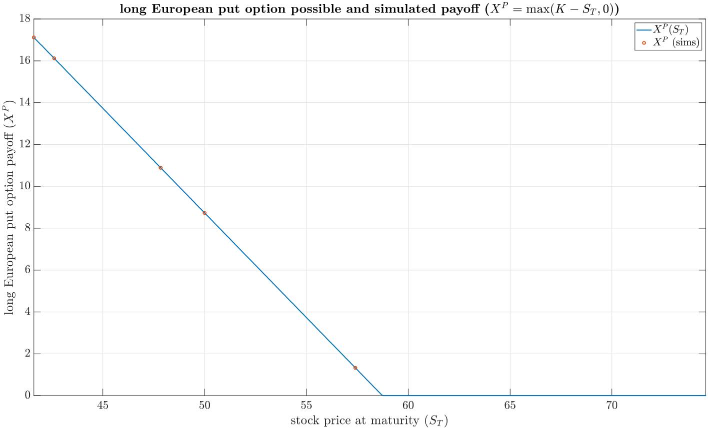
- From our simulations, we learn that relatively heavy losses could result from holding the stock
- So it would seem that dismissing the stock today is our best option
- Yet, is that really so?
  - we have only made a few simulations, can we trust them?
  - we have only used to one asset in isolation, what about adding some?
- There is another possibility: buy a European put option

en.wikipedia.org/wiki/Put_option

Strategy 2: add a long a European put option

Possible payoff of the option at maturity

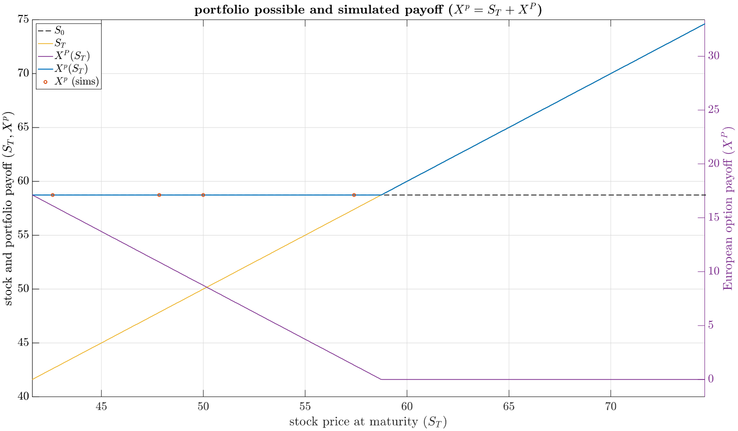
- MATLAB code: `AE_GraphEurPutPayoff.m`



Strategy 2: add a long a European put option

Possible payoff of the portfolio (stock,option) at maturity

- MATLAB code: `AE_GraphPortfolioPayoff.m`



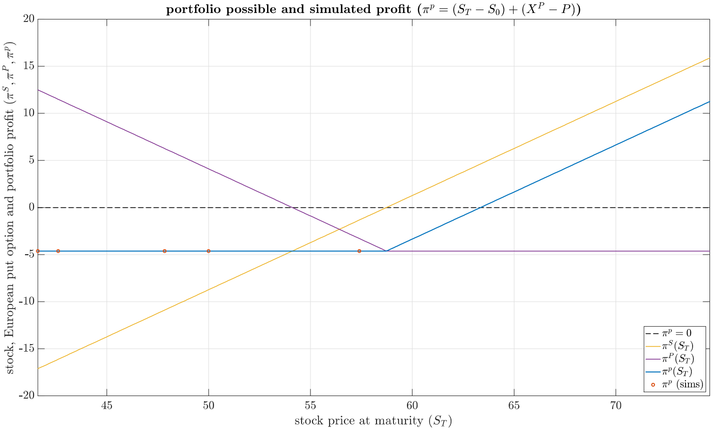
Strategy 2: add a long a European put option

- From our analysis we learn that the payoff structure of the option
  - is positive and complements the revenue structure of the stock when $S_T \leq K$, so $X^p = 0$ (left-hand side of the graph)
  - equals zero otherwise, so $X^p > 0$ (right-hand side of the graph)
- The payoff structure change dramatically when compared to Strategy 1
  - there are no more losses due to stock price fall
  - therefore, only gains are allowed
- Why would anyone take on those potential losses?
  - the **option price** is the *reward* for the counterpart taking on those potential losses
  - it is the price of (that part of) risk!

Strategy 2: add a long a European put option

Possible profit of the portfolio (stock,option) at maturity

- MATLAB code: `AE_GraphPortfolioProfit.m`



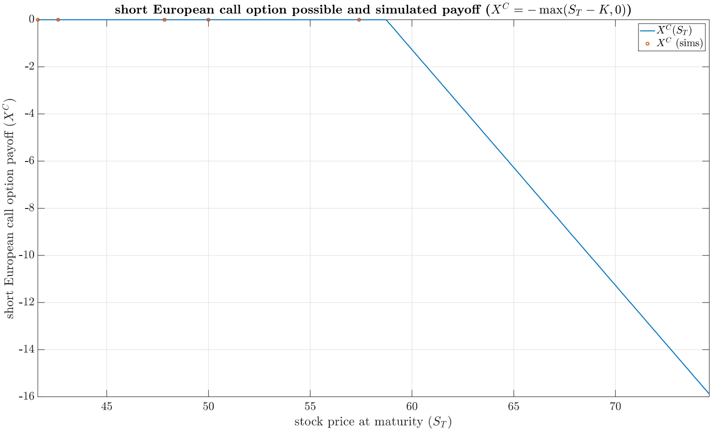
Strategy 3: add a short European call option

- In our scenario, the put option price P evaluates the downward risk
- But risk stems from any uncertain value (also positive ones!)
- To obtain the (overall) price of risk, we can use the following logic:
  - we wish to synthesize a portfolio payoff that is exactly *deterministic*
  - this way, the cost we pay to implement this strategy measures the price of the entire risk that we are trading on the market
- In order to do so, we need to
  - buy a put option (neutralizing the potentially negative payoff of a low stock price at maturity)
  - *sell* a **call option** (neutralizing the potentially positive payoff of a high stock price at maturity)

Strategy 3: add a short European call option

Possible payoff of the option at maturity

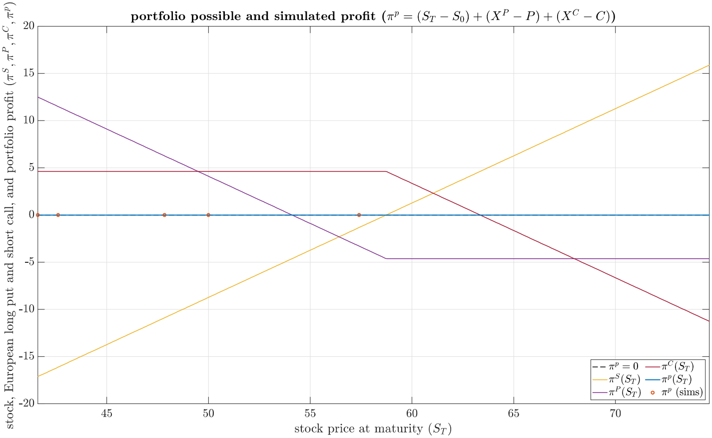
- MATLAB code: `AE_GraphEurCallPayoff.m`



Strategy 3: add a short European call option

Possible profit of the portfolio (stock,long put,short call) at maturity

- MATLAB code: `AE_GraphPortfolio2Profit.m`



Strategy 3: add a short European call option

- Neutralizing "positive" and "negative" risk we obtain Strategy 0
  - the portfolio with a short European call and long stock and European put has the same profit as a bank deposit
  - by definition, the bank's interest rate is zero
  - as a result, the profit is deterministically *null*!
- This concept is known as the **law of one price**

  en.wikipedia.org/wiki/Law_of_one_price

- More generally, this logic follows the **arbitrage-free** principle

  en.wikipedia.org/wiki/Arbitrage

Understanding financial strategies

- Consider again the portfolio given in Strategy 2 above
- We now wish to use this strategy to understand how we can analyze a given financial portfolio
- This is a critical task to assess a financial position
- We pose ourselves a number of questions, which will drive the analysis of the main aspects our financial strategies

Market outlook

- What belief regarding the evolution of the underlying asset can we infer by observing the portfolio composition?
- The answer is: *cautiously bullish*
  - *bullish*: optimistic on the future price evolution of the underlying
    - the stock price will stay well above the strike price
    - for this reason, we wish to have a long position in the stock
  - *cautiously*: although optimistic, the position is also prudent
    - risk arising from potential falls in the stock price kept to a minimum
    - hence the long position in a European put option

Market outlook

- In fact, the portfolio is a **synthetic call**

  en.wikipedia.org/wiki/Synthetic_position

  - an investor may hold it to receive the benefits of stock ownership
  - insurance policy that protects against depreciation in the stock price
- The degree of optimism also matters
  - the more bullish the view, the further **out-of-the-money** we can buy the option in order to create maximum leverage

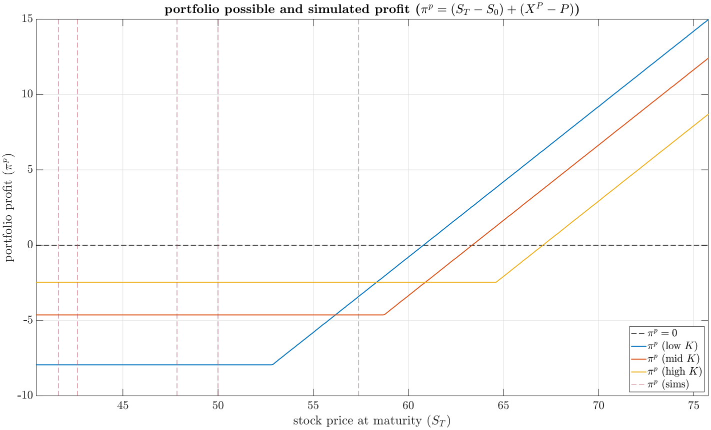
    en.wikipedia.org/wiki/Moneyness

  - the more unlikely one believes a fall in the stock price will be, the lower one can choose the strike price
    - the *net* profit will increase (*in absolute value*) at both ends of the potential stock price distribution
    - the initial cost of the portfolio will fall: hence the rising leverage

Market outlook

Possible portfolio profits for different strike prices

- MATLAB code: `AE_GraphPortfolioProfitK.m`



Market outlook

- Both larger losses and larger profits obtains with a lower K
  - P declines, boosting the potential π^p when S_T is sufficiently high
  - but what is cashed in $(K - S_T)$ lowers when the option is exercised, thereby increasing the potential loss
- A few other features of Strategy 2 emerge
  - π^p increases, boundlessly, as S_T rises
  - the break-even point obtains when S_t equals the sum of P and S_0
    - as a result, it moves rightward as K increases, rising with P
  - the maximum loss is the sum between P and $S_T - S_0$
    - the higher K, the larger the first component and the smaller the second: the net result is, anyway, a lower loss

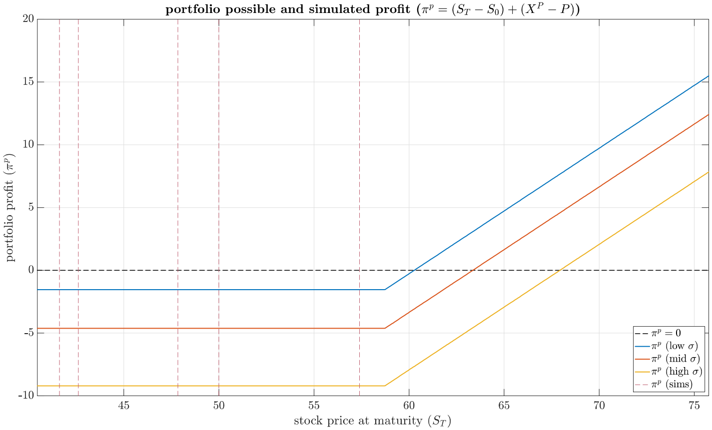
Volatility

- What is the performance of our portfolio at maturity when the evolution of the stock is more uncertain?
- The answer is: profit lowers at all levels of S_T
- The option value, in fact, increases with volatility
  - higher volatility raises the probability of a sharp fall in the stock price
  - in turn, the probability that the option is in the money increases
  - the initial cost grows without changing the structure of the payoff
  - hence the uniform fall in the portfolio profitability
- Expected profit, however, does not necessarily decline, since higher volatility also raises the probability of a sharp *rise* in S_T

Volatility

Possible portfolio profits for different stock prices variabilities

- MATLAB code: `AE_GraphPortfolioProfitSigma.m`



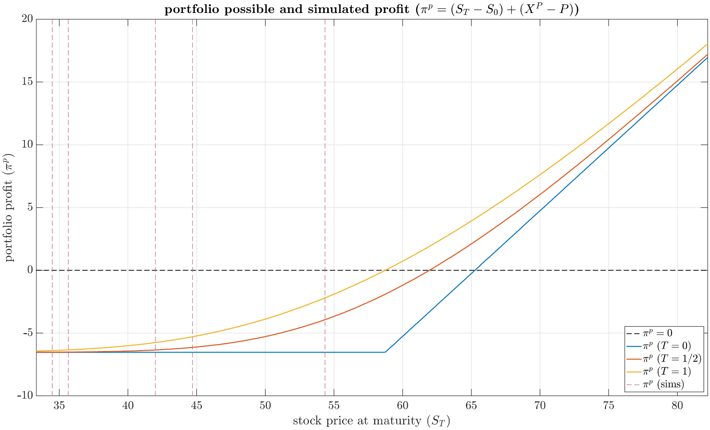
Time decay

- What is the price evolution of our portfolio as time passes?
- The answer is: the value of the portfolio declines over time
  - the likelihood that S_T will move away from the value at present falls
  - in the range of S_T where the option is in the money, it approaches K
  - otherwise, it approaches S_T itself
- Why from above?
  - higher T raises the likelihood of larger strike price movements
  - portfolio value is lower-bounded by K but unboundedly raise with S_T
  - so, in probability, the portfolio value is always larger before maturity
- To see this, we must analyze the portfolio profit as t approaches T

Time decay

Possible portfolio profits at different maturities

- MATLAB code: `AE_GraphPortfolioProfitT.m`



Option-based financial strategies

en.wikipedia.org/wiki/Options_strategy

- Risk management often relies on options, due to their versatility
- It is thus important to familiarize with the leading available strategies

Exercise Analyze the following financial strategies

1. Butterfly en.wikipedia.org/wiki/Butterfly_(options)
2. Straddle en.wikipedia.org/wiki/Straddle
3. Strangle en.wikipedia.org/wiki/Strangle_(options)
4. Collar en.wikipedia.org/wiki/Collar_(finance)
5. Fence en.wikipedia.org/wiki/Fence_(finance)

Theoretical take-away

- We price risk by using derivatives on a given asset
- Our benchmark is the price of *that* asset
- We therefore price the risk of that asset, not the systematic risk
  - recall the notion of relative pricing: this is a primary example of that
  - we are *not* identifying the stochastic discount factor!
- The reason why risk is priced correctly stems from the fact that the "market" composed by the asset and the derivative is *complete*
  - our line of reasoning can be extended to a finite number of "markets"
  - however, it cannot be extended to the entire universe of assets
    - we do not have knowledge of all possible assets (recall the discussion over APT last year)
    - hence, we fail in correctly identifying the market portfolio

Theoretical take-away

- In micro terms, the interesting part is that we can partition the risk of a given position, and price those risk segments separately
  - this is the role played by financial derivatives
  - that is why it is so important to be able to price them!
- Our goal is therefore twofold
  - we need to master strategies to manage financial risk
  - we need to price complex derivatives, even those for which we do not have an exact pricing formula

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Introduction

- We develop a simple model to price any financial derivative
  - the model building block is the **no-arbitrage condition**

    en.wikipedia.org/wiki/Arbitrage

  - **risk-neutral probabilities** stem from this condition
  - we use these probabilities to build a **risk-neutral valuation formula**
- The one-period two-asset structure of the model comprises
  - two dates, $t = 0$ (current) and $t = 1$ (future)
  - a risk-free asset (**bond**), with price B_t at time t
  - a risky asset (**stock**), with price S_t at time t

    [we refer to a stock for simplicity, but the risky asset category spans virtually every financial instrument]

Bond

- The **price process** B is

$$B_0 = 1$$
$$B_1 = 1 + R$$

- R is the (known) deterministic return rate between $t = 0$ and $t = 1$
- The bond can be interpreted as a bank deposit with **interest rate** R

Stock

- The **price process** S is

$$
\begin{aligned}
S_0 &= s \\
S_1 &= \begin{cases} s \cdot u, & \text{with probability } p_u \\ s \cdot d, & \text{with probability } p_d \end{cases}
\end{aligned}
$$

- The process can also be written as

$$
\begin{aligned}
S_0 &= s \\
S_1 &= s \cdot Z
\end{aligned}
$$

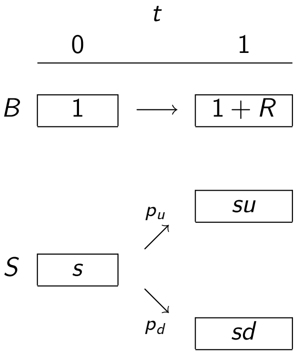
- The random variable Z is then defined by

$$
Z = \begin{cases} u, & \text{with probability } p_u \\ d, & \text{with probability } p_d \end{cases}
$$

Basic assumptions

- The following quantities are known at time $t = 0$
  - the interest rate, R
  - the current stock price, s
  - the stock move coefficients, u and d
  - the probabilities, p_u and p_d
- We assume stock move coefficients are such that $d < u$
- Naturally, probabilities are *exhaustive*, hence $p_u + p_d = 1$

Tree diagram



Portfolio

- We want to form portfolios of the two assets featured in the model
- A **portfolio** is defined as the vector $h = (x, y)$, where
  - x is the number of bonds we hold in our portfolio
  - y is the number of units of the stock
- x and y may be positive as well as negative, for example
  - $x = 3$ means that we hold three bonds at $t = 0$
  - $y = -2$ means that we owe two shares of the stock at $t = 0$
  - we have a *long* position in the bond and a *short* position in the stock

Assumptions on portfolio formation

- *Short positions*, as well as *fractional holdings*, are allowed

$$\forall h \in R^2 \quad \text{is a feasible portfolio}$$

- There is *no bid-ask spread*: selling price equals buying price
- There are *no transactions costs*
- The market is *completely liquid*
  - it is always possible to buy/sell unlimitedly at the market price
  - it is possible to borrow unlimitedly from the bank (sell bonds short)

Value process of a portfolio

- The portfolio value is simply the sum of the values of the assets held
  - it has a deterministic market value at $t = 0$
  - is has a stochastic value at $t = 1$ (unless $y = 0$)
- Formally, the value process for h is

$$V_t^h = xB_t + yS_t, \quad t = 0, 1$$

- More explicitly

$$\begin{aligned} V_0^h &= x + ys \\ V_1^h &= x(1 + R) + ysZ \end{aligned}$$

Arbitrage portfolio

- Can one make positive profit for sure by trading on the market?
- We call a portfolio that guarantees just that an **arbitrage portfolio**
  - Formally, h is an arbitrage portfolio if:

$$\begin{aligned} V_0^h &= 0 \\ V_1^h &> 0, \quad \text{with probability } 1 \end{aligned}$$

  - An arbitrage portfolio is thus a deterministic money making machine
- We interpret the existence of an arbitrage portfolio as equivalent to a serious case of mispricing on the market

No-arbitrage condition

- When is a given market model arbitrage free?
- That is to say, when are there *no arbitrage portfolios*?
- The binomial model is **free of arbitrage** if and only if

$$d \leq 1 + R \leq u \tag{1}$$

- This inequality implies that
  - the bond payoff is not allowed to *dominate* the stock payoff
  - and vice versa!

Violation of the condition

- To see that our claim is correct
  - we first show that the no arbitrage condition implies inequality (1)
  - if the inequality *does not* hold, then we have an arbitrage opportunity
- Suppose that

$$(1 + R) > u$$

  - this inequality implies $s\,(1 + R) > su$
  - since $u > d$, we also have $su > sd$
- As a result, it is always more profitable to invest in the bond

$$s\,(1 + R) > su > sd$$

- An arbitrage strategy is then formed by the portfolio $h = (s, -1)$
  - we sell the stock short
  - we invest all the money in the bond

Violation of the condition

- At $t = 0$, the value of this portfolio is obviously zero

$$V_0^h = x + ys = s + (-1)\,s = 0$$

- At $t = 1$ the value of this portfolio is always positive

$$V_1^h = x\,(1+R) + ysZ = \begin{cases} s\,(1+R) - su > 0 \\ s\,(1+R) - sd > 0 \end{cases}$$

- To avoid this arbitrage opportunity, it must then be that

$$1 + R \leq u$$

Exercise Repeat the argument after assuming that $(1+R) < d$

Arbitrage-free market

- We now turn to show that inequality (1) implies no arbitrage
- If the inequality *does* hold, then we have no arbitrage opportunity
- Consider an arbitrary portfolio such that $V_0^h = 0$
- We have $x + ys = 0$, which in turn yields

$$x = -ys$$

- The value of the portfolio at $t = 1$ is then

$$V_1^h = \begin{cases} -ys\,(1+R) + ysu = ys\,[u - (1+R)]\,, & \text{if } Z = u \\ -ys\,(1+R) + ysd = ys\,[d - (1+R)]\,, & \text{if } Z = d \end{cases}$$

Arbitrage-free market

- Suppose that $y > 0$
  - Then h is an arbitrage strategy if and only if we have

$$\begin{aligned} s\left[u-(1+R)\right] &> 0 \\ s\left[d-(1+R)\right] &> 0 \end{aligned}$$

  - This implies

$$\begin{aligned} u &> 1+R \\ d &> 1+R \end{aligned}$$

- But this violates inequality (1), since it requires $d \leq 1+R$

  Exercise Repeat the argument after assuming that $y < 0$

Interpretation of the no-arbitrage condition

- How can we interpret this result?
- If inequality (1) holds, then a pair (q_u, q_d) exists, with $q_u \geq 0$, $q_d \geq 0$, and $q_u + q_d = 1$, such that

$$1 + R = q_u \cdot u + q_d \cdot d$$

Interpretation (q_u, q_d) represent a new probability measure Q, with

$$\begin{aligned} Q(Z = u) &= q_u \\ Q(Z = d) &= q_d \end{aligned}$$

Risk-neutral valuation formula

- We may compute the discounted (at rate R) expected value of S_1 using the probability measure Q (denoted by E^Q)

$$\frac{1}{1+R}E^Q[S_1] = \frac{1}{1+R}(q_u su + q_d sd) = \frac{s}{1+R}\overbrace{(q_u u + q_d d)}^{1+R} = s$$

- We thus obtain the **risk-neutral valuation formula**

$$s = \frac{1}{1+R}E^Q[S_1]$$

  - it gives today's stock price as the discounted expected value of tomorrow's stock price
  - we do *not* assume that agents are risk neutral
  - merely, if we use the Q-probabilities instead of the objective ones, then we get a risk neutral valuation of the stock

Risk-neutral probability

- The probability measure Q is called a risk-neutral measure, or alternatively a risk-adjusted measure or a martingale measure
- Formally, Q is a martingale measure if

$$S_0 = \frac{1}{1+R} E^Q \left[S_1 \right]$$

- We can also re-state the no arbitrage condition: the binomial model is arbitrage free if and only if there exists a martingale measure Q
- For the binomial model, the martingale probabilities are

$$\begin{cases} q_u = \dfrac{(1+R)-d}{u-d} \\ q_d = \dfrac{u-(1+R)}{u-d} \end{cases}$$

Risk-neutral probability

- Obtaining them is easy; we begin by the definition equations

$$\begin{cases} 1 + R = q_u u + q_d d \\ q_u + q_d = 1 \end{cases}$$

- Obtain from the second $q_d = 1 - q_u$; then substitute into the first

$$\begin{aligned} 1 + R &= q_u u + (1 - q_u) d \\ &= q_u (u - d) + d \end{aligned}$$

  - Isolating q_u, we obtain the first probability
  - Replacing it into $q_d = 1 - q_u$, we obtain the second

Definition of financial derivative

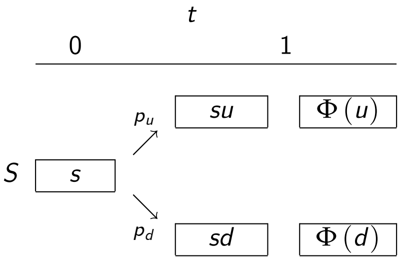
- We define a financial derivative (contingent claim) as a stochastic variable X of the form

$$X = \Phi(Z)$$

where, as before,

$$Z = \begin{cases} u, & \text{with probability } p_u \\ d, & \text{with probability } p_d \end{cases}$$

Tree diagram



Example of financial derivative

- Consider a European call option on S with exercise price K
- The example is interesting only if $sd < K < su$
  - when $S_1 = su > K$
    - we use the option, paying K to get the stock
    - we sell the stock on the market for su
    - we make a net profit of $su - K$
  - when $S_1 = sd < K$, then the option is obviously worthless
- We therefore have the payoff structure

$$X = \begin{cases} \Phi(u) = su - K, & \text{if } Z = u \\ \Phi(d) = 0, & \text{if } Z = d \end{cases}$$

Price of a financial derivative

- We now turn to determine the “fair” price of a financial derivative
- We denote the price of X at time t by $\Pi(t; X)$
- At $t = 1$, in order to avoid arbitrage, we must have

$$\Pi(1; X) = X$$

  - If $\Pi(1; X) < X$, then one can obtain an instant profit by buying the derivative for $\Pi(1; X)$ and cash in its payoff X
  - If $\Pi(1; X) > X$, one would sell short the derivative for $\Pi(1; X)$ and pay it back using part of its payoff X

Price of a financial derivative

- We now tackle the harder issue of determining $\Pi(0; X)$
- We first define a hedging portfolio
- We then characterize a complete market
- We show that a arbitrage-free binomial model is complete
- We finally work out the price process for the derivative

Hedging portfolio

- A given derivative X is **reachable** if there exists a portfolio h such that, with probability one

$$V_1^h = X$$

- In this case, h is a **hedging portfolio** or a **replicating portfolio**
- If a certain derivative X is reachable with replicating portfolio h, then
  - there is no difference between holding the derivative and holding the portfolio
  - no matter what happens on the stock market, the derivative value at $t = 1$ will be exactly equal to the portfolio value at $t = 1$
- *The derivative price equals the market value of the portfolio*

Hedging portfolio and no-arbitrage condition

- Formally, if X is reachable with a replicating portfolio h, then the only *reasonable* price process for X is

$$\Pi(t; X) = V_t^h, \quad t = 0, 1$$

- Any $\Pi(0; X)$ other than V_0^h will lead to an arbitrage opportunity

Hedging portfolio and no-arbitrage condition

- Suppose that $V_0^h < \Pi(0; X)$
- At $t = 0$
  - We short sell X
  - We buy h
  - We use the difference to purchase bonds
- At $t = 1$
  - the payoff of X and h coincide by definition
  - hence the net worth of this portion of the portfolio is zero
  - the bonds yield $\left(\Pi(0; X) - V_0^h\right)(1 + R) > 0$, an arbitrage profit

Exercise Repeat the argument after assuming that $V_0^h > \Pi(0; X)$

Complete market

- A **market** is **complete** if all derivatives can be replicated
- This implies that in a complete market we can price all derivatives
- Thus, it is crucial to investigate when a given market is complete
- For the binomial model, we have that the market is complete if the model is arbitrage-free

Hedging portfolio for a generic derivative

- Consider an arbitrary derivative $X = \Phi(Z)$
- We want to show that there exists a portfolio $h = (x, y)$ such that

$$\begin{aligned} V_1^h &= \Phi(u), \quad \text{if } Z = u \\ V_1^h &= \Phi(d), \quad \text{if } Z = d \end{aligned}$$

- Specifically, we seek a solution (x, y) to the system of equations

$$\begin{aligned} (1+R)\,x + suy &= \Phi(u) \\ (1+R)\,x + sdy &= \Phi(d) \end{aligned}$$

Hedging portfolio for a generic derivative

- We isolate x in the first equation

$$x = \frac{\Phi\left(u\right) - suy}{1+R}$$

- We replace it into the second, obtaining

$$\begin{aligned} x &= \frac{1}{1+R}\frac{u\Phi\left(d\right) - d\Phi\left(u\right)}{u-d} \\ y &= \frac{1}{s}\frac{\Phi\left(u\right) - \Phi\left(d\right)}{u-d} \end{aligned}$$

- these two expressions give us the number of bonds and stock shares, respectively, that we need to hold in the portfolio to replicate X

Risk Neutral Valuation

- Since the hedging portfolio exists for any arbitrary derivative X, then the binomial model is shown to be complete
- We can thus price the derivative using the expression

$$\Pi(0; X) = V_0^h$$

- Recall that the value of a portfolio h at $t = 0$ is

$$V_0^h = x + ys$$

- Recall also that the martingales probabilities are

$$\begin{cases} q_u = \dfrac{(1+R)-d}{u-d} \\ q_d = \dfrac{u-(1+R)}{u-d} \end{cases}$$

Risk Neutral Valuation

- Then, using the hedging portfolio holdings, we have

$$
\begin{aligned}
\Pi(0;X) &= x + sy \\
&= \frac{1}{1+R}\frac{u\Phi(d) - d\Phi(u)}{u-d} + \frac{1}{s}\frac{\Phi(u)-\Phi(d)}{u-d}\cdot s \\
&= \frac{1}{1+R}\left[\frac{u\Phi(d) - d\Phi(u)}{u-d} + (1+R)\frac{\Phi(u)-\Phi(d)}{u-d}\right] \\
&= \frac{1}{1+R}\left[\underbrace{\frac{(1+R)-d}{u-d}}_{q_u}\Phi(u) + \underbrace{\frac{u-(1+R)}{u-d}}_{q_d}\Phi(d)\right]
\end{aligned}
$$

Risk Neutral Valuation

- Since the binomial model is arbitrage-free, then $d \leq (1+R) \leq u$
- Hence (q_u, q_d) are *probabilities* (non negative, sum up to one) and

$$\Pi(0;X) = \frac{1}{1+R}\left[\Phi\left(u\right)\cdot q_u + \Phi\left(d\right)\cdot q_d\right] = \frac{1}{1+R}E^Q\left(\Phi\left(Z\right)\right)$$

  - the right-hand side can now be interpreted as an expected value under the martingale probability measure Q
  - the price process of a derivative is just the *discounted expected value of its payoff, computed using risk neutral probabilities*

Binomial model recap

- If the binomial model is free of arbitrage, then the arbitrage-free price of a derivative X is given by

$$\Pi(0; X) = \frac{1}{1+R} E^Q[X]$$

- The martingale measure Q is uniquely determined by the relation

$$S_0 = \frac{1}{1+R} E^Q[S_1]$$

- The explicit expression for the probabilities are given by

$$\begin{cases} q_u = \dfrac{(1+R)-d}{u-d} \\ q_d = \dfrac{u-(1+R)}{u-d} \end{cases}$$

Binomial model recap

- The derivative can be replicated by a portfolio $h = (x, y)$ such that

$$
\begin{aligned}
x &= \frac{1}{1+R}\frac{u\Phi(d) - d\Phi(u)}{u-d} \\
y &= \frac{1}{s}\frac{\Phi(u) - \Phi(d)}{u-d}
\end{aligned}
$$

- The objective probabilities determine which events are possible
- The risk neutral probabilities simplify the computations of the arbitrage-free price of a financial derivative
- This does not mean that we live in a risk-neutral world
- The formula holds for all investors, regardless of their attitude towards risk, as long as they prefer more deterministic money to less

An illustrative example

- We set
  - $s = 100$
  - $u = 1.2$ and $d = 0.8$
  - $p_u = 0.6$, hence $p_d = 0.4$
  - $R = 0$
- Thus we have the price dynamics

$$S_0 = 100$$
$$S_1 = \begin{cases} 120, & \text{with probability } 0.6 \\ 80, & \text{with probability } 0.4 \end{cases}$$

- $d \leq (1 + R) \leq u$ holds, hence the market is arbitrage-free

$$d = 0.8 < 1 = 1 + R < 1.2 = u$$

An illustrative example

- We consider a **European call option** with strike price $K = 110$

  en.wikipedia.org/wiki/Call_option

- So (the payoff of) the derivative X is given by:

$$X = \max(S_1 - K, 0) = \begin{cases} 10, & \text{with probability } 0.6 \\ 0, & \text{with probability } 0.4 \end{cases}$$

- Using the (*naive*) method of computing the price as the discounted expected values under the objective probabilities, we would get

$$\Pi^P(0; X) = \frac{1}{1+0}[10 \cdot 0.6 + 0 \cdot 0.4] = 6$$

Price of the European call option

- Using the theory above it is easily seen that the martingale probabilities are given by

$$\begin{cases} q_u = \dfrac{(1+R)-d}{u-d} = \dfrac{(1+0)-0.8}{1.2-0.8} = 0.5 \\ q_d = \dfrac{u-(1+R)}{u-d} = \dfrac{1.2-(1+0)}{1.2-0.8} = 0.5 \end{cases}$$

- Thus the theoretical price is

$$\Pi(0;X) = \frac{1}{1+0}\left[10 \cdot 0.5 + 0 \cdot 0.5\right] = 5$$

- The theoretical price differs (is lower than) its *naive* counterpart

Hedging portfolio of the European call option

- To verify our finding, we replicate the option
- Using the hedging portfolio holding stated above, we obtain

$$
\begin{aligned}
x &= \frac{1}{1+R}\frac{u\Phi(d) - d\Phi(u)}{u-d} = \frac{1}{1+0}\frac{1.2\cdot 0 - 0.8\cdot 10}{1.2-0.8} = -20 \\
y &= \frac{1}{s}\frac{\Phi(u) - \Phi(d)}{u-d} = \frac{1}{100}\frac{10-0}{1.2-0.8} = \frac{1}{4}
\end{aligned}
$$

- The replicating portfolio is then formed by
  - borrowing 20 from the bank
  - investing this money to (partially) fund the purchase of a quarter of a share in the stock

Replication of the European call option

- Thus the net value of the portfolio at $t = 0$ is

$$V_0^h = x + ys = -20 + \frac{1}{4} \cdot 100 = 5$$

- At $t = 1$ it replicates the option payoff, as it is given by

$$\begin{aligned} V_1^h &= x(1+R) + yS_1 = -20(1+0) + \frac{1}{4} \cdot 120 = 10, \quad \text{if } S_1 = 120 \\ V_1^h &= x(1+R) + yS_1 = -20(1+0) + \frac{1}{4} \cdot 80 = 0, \quad \text{if } S_1 = 80 \end{aligned}$$

Illustrative example wrap-up

- So we see that we have indeed replicated the option
- We also see that if anyone is foolish enough to buy the option from us for 6, then we can make a riskless profit
- We sell the option, thereby obtaining 6
- Out of these 6 we invest five in the replicating portfolio and invest the remaining one in the bank
- At $t = 1$, the claims of the buyer of the option are completely balanced by the value of the replicating portfolio, and we still have one dollar (plus interest) invested in the bank
- We have thus made an arbitrage profit
- If someone is willing to sell the option to us at a price lower than five dollars, we can also make an arbitrage profit by selling the portfolio short

Theoretical take-away

- We have seen that in a complete market, like the binomial model above, there is indeed a unique price for any derivative
- The price is given by the value of the replicating portfolio
- In fact, there exists a theoretical price for the derivative precisely because, strictly speaking, the derivative is superfluous
- *The derivative can equally well be replaced by its hedging portfolio*

Theoretical take-away

- We see that the structural reason for the completeness of the binomial model is that
  - we have two financial instruments at our disposal (bond and stock)
  - to solve two equations (one for each possible outcome)
- This fact can be generalized
- *A model is complete if the number of underlying assets (including the bank account) equals the number of outcomes*

Theoretical take-away

- If we would like to make a more realistic multiperiod model of the stock market, then the last remark above seems discouraging
- If we make a (non-recombining) tree with 20 time steps this means that we have $2^{20} \sim 10^6$ elementary outcomes
- This number exceeds by a large margin the number of assets on any existing stock market
- It would *seem* that it is impossible to construct an interesting complete model with a reasonably large number of time steps

Theoretical take-away

- Fortunately the situation is not at all as bad as that
- In a multiperiod model we also have the possibility of considering intermediary trading
- We can thus allow for portfolios which are rebalanced over time
- This will give us many more degrees of freedom

Theoretical take-away

- Finally, note that the source of risk is clearly identified, and (ex-post) observed, in the underlying risky asset of the derivative
- Unfortunately, this is not the case for the entire market
- This is the reason why this line of reasoning does not apply to a macroeconomic asset pricing approach

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Market structure

- The time index is $t = 0, ..., T$, where the horizon T is fixed
- As before, we have two assets
  - a risk free asset (bond), with price B_t at time t
  - a risky asset (stock), with price S_t at time t
- The price process B is redefined as

$$\begin{aligned} B_0 &= 1 \\ B_1 &= 1 + R \end{aligned} \quad \longrightarrow \quad \begin{aligned} B_0 &= 1 \\ B_{t+1} &= (1 + R)\, B_t \end{aligned}$$

- R is the (*known*) *constant* deterministic return rate for each period
  - the bond is still interpreted as a bank deposit with interest rate R
  - the first period process is unchanged, since setting $t = 0$ yields

$$B_1 = (1 + R)\, B_0 = 1 + R$$

Market structure

- The price process S is redefined as

$$\begin{aligned} S_0 &= s \\ S_1 &= s \cdot Z \end{aligned} \quad \longrightarrow \quad \begin{aligned} S_0 &= s \\ S_{t+1} &= S_t \cdot Z_t \end{aligned}$$

- Each Z_t is still the random variable

$$Z_t = \begin{cases} u, & \text{with probability } \; p_u \\ d, & \text{with probability } \; p_d \end{cases}$$

  - $Z_0, ..., Z_{T-1}$ are i.i.d. (independent and identically distributed)
  - the first period process is unchanged, since setting $t = 0$ yields

$$S_1 = S_0 \cdot Z_0 = s \cdot Z$$

  - we purposely drop the time subscript (possible given the i.i.d. assumption) to stress the parallel to the one-period model

Tree diagram

$t = 0$		$t = 1$		$t = 2$
			p_u $\nearrow$	$B_2 = (1+R)^2$, $S_2 = su^2$
	p_u $\nearrow$	$B_1 = 1 + R$, $S_1 = su$		
$B_0 = 1$, $S_0 = s$			$\searrow$ p_d / p_u $\nearrow$	$B_2 = (1+R)^2$, $S_2 = sdu$
	$\searrow$ p_d	$B_1 = 1 + R$, $S_1 = sd$		
			$\searrow$ p_d	$B_2 = (1+R)^2$, $S_2 = sd^2$

Portfolio

- We want to form portfolios of the two assets featured in the model
- We redefine a portfolio as the stochastic process

$$h = (x, y) \quad \rightarrow \quad h_t = (x_t, y_t)\,,\; t = 1, ..., T$$

- The portfolio's value process becomes

$$V_t^h = xB_t + yS_t,\; t = 0, 1 \quad \rightarrow \quad V_t^h = x_t\,(1 + R) + y_t S_t,\; t = 1, ..., T$$

- Understanding the model's timing is crucially important here
  - x_t is the amount we invest in bonds at $t - 1$ and keep until t
  - y_t is the number of shares that we buy at $t - 1$ and keep until t
  - V_t^h is therefore the market value of the portfolio (x_t, y_t) at time t (which we held since $t - 1$)

Assumptions on portfolio formation

- *Short positions*, as well as *fractional holdings*, are allowed

$$\forall h \in R^2 \quad \text{is a feasible portfolio}$$

- There is *no bid-ask spread*: selling price equals buying price
- There are *no transactions costs*
- The market is *completely liquid*
  - it is always possible to buy/sell unlimitedly at the market price
  - it is possible to borrow unlimitedly from the bank (sell bonds short)

Self-financing portfolio

- We are primarily interested in **self-financing portfolios**

  en.wikipedia.org/wiki/Self-financing_portfolio

  - modified over time without exogenous money infusion nor withdrawal
  - the accession of new assets must be financed through selling others

- Formally, a portfolio h is self financing if $V_t^h = V_{t+1}^h$ for all $t = 0, ..., T-1$
  - in turn, this implies

$$x_t (1 + R) + y_t S_t = x_{t+1} + y_{t+1} S_t$$

  - the expression simply represents a budget equation, so that
  - at each t, the market value of (x_t, y_t), created at $t-1$, equals the purchase value of (x_{t+1}, y_{t+1}), formed at t (and held until $t+1$)

Arbitrage portfolio

- Redefine the arbitrage portfolio a *self-financing* portfolio h such that

$$
\begin{aligned}
&V_0^h = 0 \\
&V_1^h > 0, \quad \text{with probability } 1
\end{aligned}
\quad \longrightarrow \quad
\begin{aligned}
&V_0^h = 0 \\
&P\left(V_T^h \geq 0\right) = 1 \\
&P\left(V_T^h > 0\right) > 0
\end{aligned}
$$

- The arbitrage portfolio need *not* be strictly positive *in every state*
- An arbitrage portfolio can nonetheless still be interpreted as a money making machine, just not *deterministic*

Arbitrage-free market

- The binomial model is free of arbitrage if and only if

$$d \leq 1 + R \leq u$$

- The risk-neutral valuation formula is redefined as

$$s = \frac{1}{1+R} E^Q \left[S_1 \right] \quad \rightarrow \quad s = \frac{1}{1+R} E^Q \left[S_{t+1} | S_t = s \right]$$

  - the expectation is now conditioned on the current stock value S_t
  - S_t is of course time-dependent (on the realizations of S up to time t)

- The martingale probabilities are still given by

$$\begin{cases} q_u = \dfrac{(1+R) - d}{u - d} \\ q_d = \dfrac{u - (1+R)}{u - d} \end{cases}$$

Definition of financial derivative

- We redefine a financial derivative as

$$X = \Phi(Z) \quad \rightarrow \quad X = \Phi(S_T)$$

  - the reason is that now an entire set of random variables $\{Z\}$ jointly determine the value of the stock at maturity
  - hence, it is more convenient to refer to S_T rather than writing $X = \Phi(Z_0, Z_1, ..., Z_{T-1})$, equally correct but more cumbersome
- We still denote the price of X at time t by $\Pi(t; X)$
  - formally, we have

$$\Pi(t; X), \text{ for } t = 0, ..., T$$

  - the only difference is that now t indexes more than two dates

Hedging portfolio

- As before, a given derivative X is **reachable** if there exists a portfolio h such that, with probability one

$$V_T^h = X$$

- The difference is that h is now a **self-financing** hedging portfolio
- Once again, from a financial point of view there is no difference between holding the derivative or the portfolio
  - suppose that X is reachable using the self-financing portfolio h
  - fix t and suppose that at time t we have access to the amount V_t^h
  - then we can invest this money in the portfolio h
  - rebalance h over time without extra cost so as to have V_T^h at time T
  - by definition $V_T^h = X$ with probability 1
  - thus, from a financial point of view, h and X are equivalent so they should fetch the same price

Price of a financial derivative

- Formally, if X is reachable with a replicating portfolio h, then the only *reasonable* price process for X is

$$\Pi\left(t;X\right)=V_{t}^{h},\quad t=0,1,..,T$$

- Any price other than V_{τ}^{h} at time τ leads to an arbitrage opportunity
  - suppose that $V_{\tau}^{h}<\Pi\left(\tau;X\right)$
  - at τ, we short sell X, buy h and purchase bonds with the difference
  - at T the payoffs of X and h coincide by definition
  - selling the bonds yields $\left(\Pi\left(\tau;X\right)-V_{\tau}^{h}\right)\left(1+R\right)^{T-\tau}>0$
  - we have an arbitrage profit

Exercise Repeat the argument after assuming that $V_{\tau}^{h}>\Pi\left(\tau;X\right)$

Complete market

- **As before**, we define the **market** as **complete** if all derivatives can be replicated
- This implies that in a complete market we can price all derivatives
- Therefore, it is crucial to investigate when a given market is complete
- It is possible, and not very hard, to give a formal proof which, however, look rather messy with lots of indices
- We therefore prove completeness for a concrete example instead

Example setup

- Let $T = 3$, $S_0 = 80$, $u = 1.5$, $d = 0.5$, $p_u = 0.6$, $p_d = 0.4$, $R = 0$

			$B_3 = 1$ $S_3 = 270$
		$B_2 = 1$ $S_2 = 180$	
	$B_1 = 1$ $S_1 = 120$		$B_3 = 1$ $S_3 = 90$
$B_0 = 1$ $S_0 = 80$		$B_2 = 1$ $S_2 = 60$	
	$B_1 = 1$ $S_1 = 40$		$B_3 = 1$ $S_3 = 30$
		$B_2 = 1$ $S_2 = 20$	
			$B_3 = 1$ $S_3 = 10$

Financial derivative

- Consider a European call option on the underlying stock
  - the date of expiration of the option is $T = 3$
  - the strike price is chosen to be $K = 80$
  - Formally this claim can be described as

$$X = \max\left[S_T - K, 0\right]$$

- We will now show that this particular claim can be replicated
  - it will be obvious from the argument that the result can be generalized to any binomial model and any claim
  - the idea is to use induction on the time variable and to work backwards in the tree from the leaves at $t = T$ to the root at $t = 0$
- We start by computing the price of the option at the expiration date
  - this is easily done since obviously we must have, for any claim X

$$\Pi\left(T; X\right) = X$$

Tree diagram for the underlying and derivative

$S_0 = 80$
$\Pi_0 = ?$

$S_1 = 120$
$\Pi_1 = ?$

$S_1 = 40$
$\Pi_1 = ?$

$S_2 = 180$
$\Pi_2 = ?$

$S_2 = 60$
$\Pi_2 = ?$

$S_2 = 20$
$\Pi_2 = ?$

$S_3 = 270$
$\Pi_3 = 190$

$S_3 = 90$
$\Pi_3 = 10$

$S_3 = 30$
$\Pi_3 = 0$

$S_3 = 10$
$\Pi_3 = 0$

Partitioning the tree diagram

- Our problem is now to replicate the payoff structure at $t = 3$
  - consider some node at $t = 2$, for instance the node $S_2 = 180$
  - we face the diagram of a simple one-period binomial model

$$S_3 = 270$$
$$\Pi_3 = 190$$

$$S_2 = 180$$
$$\Pi_2 = \ ?$$

$$S_3 = 90$$
$$\Pi_3 = 10$$

Pricing the derivative within the sub-diagram

- It follows from the one-period theory that the payoff structure can indeed be replicated from the node $S_2 = 180$
- If that is the case, then we can compute the price of the replicated claim by risk-neutral valuation
  - we need to calculate the martingale probabilities

$$\begin{cases} q_u = \dfrac{(1+R)-d}{u-d} = \dfrac{(1+0)-0.5}{1.5-0.5} = 0.5 \\ q_d = \dfrac{u-(1+R)}{u-d} = \dfrac{1.5-(1+0)}{1.5-0.5} = 0.5 \end{cases}$$

  - and apply the risk-neutral valuation formula

$$\Pi(2; X) = \frac{1}{1+R} E^{Q}[X] = \frac{1}{1+0}[190 \cdot 0.5 + 10 \cdot 0.5] = 100$$

Pricing the derivative within the other sub-diagrams

			$S_3 = 270$ $\Pi_3 = 190$
		$S_2 = 180$ $\Pi_2 = 100$	
	$S_1 = 120$ $\Pi_1 = \ ?$		$S_3 = 90$ $\Pi_3 = 10$
$S_0 = 80$ $\Pi_0 = \ ?$		$S_2 = 60$ $\Pi_2 = 5$	
	$S_1 = 40$ $\Pi_1 = \ ?$		$S_3 = 30$ $\Pi_3 = 0$
		$S_2 = 20$ $\Pi_2 = 0$	
			$S_3 = 10$ $\Pi_3 = 0$

Pricing the derivative in the rest of the diagram

- If a self-financing portfolio replicates the claim at $t = 2$
  - then it is possible to price at $t = 2$ the claim's payoff at $t = 3$
  - we have thus reduced the problem in the time variable
- We thus simply need to reproduce the same procedure at $t = 1$
  - take, for example, the node $S_1 = 40$
  - from the point of view of this node, we have

$$
\begin{array}{ccc}
 & & S_2 = 60 \\
 & & \Pi_2 = 5 \\
S_1 = 40 & & \\
\Pi_1 = \ ? & & \\
 & & S_2 = 20 \\
 & & \Pi_2 = 0
\end{array}
$$

- By risk neutral valuation we can price this payoff structure

$$\Pi(1; X) = \frac{1}{1+R} E^Q[X] = \frac{1}{1+0}[5 \cdot 0.5 + 0 \cdot 0.5] = 2.5$$

Pricing the derivative in the rest of the diagram

- In this manner we fill the nodes at $t = 1$ with the claim prices, then apply the procedure again at $t = 0$

			$S_3 = 270$ $\Pi_3 = 190$
		$S_2 = 180$ $\Pi_2 = 100$	
	$S_1 = 120$ $\Pi_1 = 52.5$		$S_3 = 90$ $\Pi_3 = 10$
$S_0 = 80$ $\Pi_0 = 27.5$		$S_2 = 60$ $\Pi_2 = 5$	
	$S_1 = 40$ $\Pi_1 = 2.5$		$S_3 = 30$ $\Pi_3 = 0$
		$S_2 = 20$ $\Pi_2 = 0$	
			$S_3 = 10$ $\Pi_3 = 0$

Hedging portfolio at $t = 0$

- For now, we have just shown how to price *backward* an European call option *if* a self-financing portfolio replicates the claim
- To prove market completeness, we need to show that a self-financing portfolio *does reach* the option
- We do so by following a possible price path *forward* through the tree
- We start at $t = 0$ and reproduce the claim $(52.5, 2.5)$ at $t = 1$

$$V_1^h = \begin{cases} x_1 (1+0) + y_1 \cdot 80 \cdot 1.5 = 52.5 \\ x_1 (1+0) + y_1 \cdot 80 \cdot 0.5 = 2.5 \end{cases}$$

  - we isolate x_1

$$\begin{cases} x_1 = 52.5 - 120 y_1 \\ x_1 = 2.5 - 40 y_1 \end{cases}$$

  - we merge the two equations and solve for y_1 then for x_1

$$\begin{aligned} (120 - 40)\, y_1 &= 52.5 - 2.5 \rightarrow y_1 = 0.625 \\ x_1 &= 2.5 - 40 \cdot 0.625 = -22.5 \end{aligned}$$

Hedging portfolio at $t = 0$

- The hedging portfolio's value at $t = 0$ is the same as the claim's

$$V_0^h = -22.5 + 80\frac{5}{8} = 27.5 = \Pi_0$$

- If the price now moves to $S_1 = 120$, the portfolio is worth

$$V_1^h = -22.5 \cdot (1 + 0) + \frac{5}{8} \cdot 120 = 52.5$$

- Otherwise, with $S_1 = 40$, we have

$$V_1^h = -22.5 \cdot (1 + 0) + \frac{5}{8} \cdot 40 = 2.5$$

- Portfolio h has the same payoff structure (and value!) as X

Hedging portfolio at $t = 1$

- Suppose to face the claim $(100, 5)$ at $t = 2$, so we impose

$$V_2^h = \begin{cases} x_2\,(1+0) + y_2 \cdot 120 \cdot 1.5 = 100 \\ x_2\,(1+0) + y_2 \cdot 120 \cdot 0.5 = 5 \end{cases}$$

- We isolate x_2, merge the two equations and solve for y_2 then for x_2

$$\begin{aligned} 100 - 180y_2 &= 5 - 60y_2 \rightarrow y_2 = 95/120 \approx 0.792 \\ x_2 &= 5 - 60 \cdot 95/120 = -42.5 \end{aligned}$$

- The cost of this portfolio equals the value of our old portfolio

$$V_1^h = -42.5 + 120\frac{95}{120} = 52.5$$

- Thus, the replicating portfolio is also self-financing

Hedging portfolio at $t = 2$

- If the price falls to $S_2 = 60$, then our portfolio is worth

$$V_2^h = -42.5 \cdot (1+0) + \frac{95}{120} \cdot 60 = 5$$

- Facing the claim $(10, 0)$ at $t = 3$, we impose

$$V_3^h = \begin{cases} x_3\,(1+0) + y_3 \cdot 60 \cdot 1.5 = 10 \\ x_3\,(1+0) + y_3 \cdot 60 \cdot 0.5 = 0 \end{cases}$$

- We isolate x_3, merge the two equations and solve for y_3 then for x_3

$$\begin{aligned} 10 - 90y_3 &= -30y_3 \rightarrow y_3 = 1/6 \approx 0.167 \\ x_3 &= -30/6 = -5 \end{aligned}$$

- Once again, the cost equals the old portfolio's value

$$V_2^h = -5 + 60\,(1/6) = 5$$

Hedging portfolio at $t = 3$

- If the price rises to $S_3 = 90$, then our portfolio is exactly valued as the option payoff at that node in the tree

$$V_3^h = -5 \cdot (1+0) + \frac{1}{6} \cdot 90 = 10$$

- We can thus compute the hedging portfolio at each node

$$\begin{array}{ccc}
 & & x_3 = -80 \\
 & & y_3 = 1 \\
 & x_2 = -85/2 & \\
 & y_2 = 95/120 & \\
x_1 = -55/2 & & x_3 = -5 \\
y_1 = 5/8 & & y_3 = 1/6 \\
 & x_2 = -5/2 & \\
 & y_2 = 1/8 & \\
 & & x_3 = 0 \\
 & & y_3 = 0
\end{array}$$

General algorithm

- To develop a general binomial algorithm, we introduce some more notation to help us keep track of the price evolution
- From the procedure above, the value of the price process at time t is

$$S_t = su^k d^{t-k}, \quad k = 0, ..., t$$

- The index k denotes the number of up-moves that have occurred
- Each node in the binomial tree is a pair (t, k) with $k = 0, ..., t$
- If $V_t(k)$ denotes the value of the portfolio at the node (t, k), then $V_t(k)$ can be computed recursively by the scheme

$$\begin{aligned} V_t(k) &= \frac{1}{1+R}\left\{q_u V_{t+1}(k+1) + q_d V_{t+1}(k)\right\} \\ V_T(k) &= \Phi\left(su^k d^{T-k}\right) \end{aligned}$$

General algorithm

- The hedging portfolio is given by

$$x_t(k) = \frac{1}{1+R}\frac{uV_t(k) - dV_t(k+1)}{u-d}$$

$$y_t(k) = \frac{1}{S_{t-1}}\frac{V_t(k+1) - V_t(k)}{u-d}$$

- The arbitrage-free price at $t = 0$ of a T-claim X is redefined as

$$\Pi(0; X) = \tfrac{1}{1+R}\left[\Phi(u)\cdot q_u + \Phi(d)\cdot q_d\right]$$

$$\downarrow$$

$$\Pi(0; X) = \tfrac{1}{(1+R)^T}\textstyle\sum_{k=0}^{T}\binom{T}{k} q_u^k q_d^{T-k}\Phi\left(su^k d^{T-k}\right)$$

- In the last formula, k has binomial distribution with coefficient

$$\binom{T}{k} = \frac{T!}{k!\,(T-k)!}$$

General algorithm

- In our case, for $T = 3$:
  - $k = 0$ occurs only in one way (always go down in the tree)

$$\binom{3}{0} = \frac{3!}{0!\,(3-0)!} = \frac{3 \cdot 2}{1 \cdot (3 \cdot 2)} = 1$$

  - $k = 1$ occurs in three ways (go up in $t = 1$, $t = 2$, or $t = 3$)

$$\binom{3}{1} = \frac{3!}{1!\,(3-1)!} = \frac{3 \cdot 2}{1 \cdot (2)} = 3$$

  - $k = 2$ occurs in three ways (go down in $t = 1$, $t = 2$, or $t = 3$)

$$\binom{3}{2} = \frac{3!}{2!\,(3-2)!} = \frac{3 \cdot 2}{2 \cdot (1)} = 3$$

  - $k = 3$ occurs only in one way (always go up in the tree)

$$\binom{3}{3} = \frac{3!}{3!\,(3-3)!} = \frac{3 \cdot 2}{3 \cdot 2 \cdot (1)} = 1$$

Tree diagram in practice

- First task: create this matrix, using **time** and **stock moves** as inputs

		row col *val* (row, col)
	row col *val* (row, col)	
row col *val* (row, col)		row col *val* (row, col)
	row col *val* (row, col)	
		row col *val* (row, col)

Time

- N is the number of periods, so $N+1$ is the number of dates
- $t = 0, ..., N$ indexes dates
- example: $N = 2$ periods; $N + 1 = 3$ dates

$$\overbrace{t = 0 \quad t = 1}^{\text{first period}} \quad t = 2$$

$$t = 0 \quad \overbrace{t = 1 \quad t = 2}^{\text{second period}}$$

Time

- N governs both the total number of columns and rows

$N + 1$ (columns)

$2N + 1$ (rows)

		row col $val\,(\text{row}, \text{col})$
	row col $val\,(\text{row}, \text{col})$	
row col $val\,(\text{row}, \text{col})$		row col $val\,(\text{row}, \text{col})$
	row col $val\,(\text{row}, \text{col})$	
		row col $val\,(\text{row}, \text{col})$

Stock moves

- $t = 0, ..., N$ is also the number of total moves at date t ($+1$ per period)
- example: no moves at $t = 0$, one move at $t = 1$, two moves at $t = 2$...
- $k = 0, ..., t$ is the number of (possible) up-moves
- $t - k$ is the number of (possible) down-moves

Stock moves

	2. 2nd column = "2nd" date $\text{col} = t+1$ $\downarrow$	row col $val(\text{row}, \text{col})$
1. first column = "first" date $\text{col} = t+1$ $\downarrow$	row col $val(\text{row}, \text{col})$	$\longleftarrow$ 5. top filled row ($k = t$) k moves up from bottom $\text{row} = N+1+t-2k$
row col $val(\text{row}, \text{col})$	$\longleftarrow$ 3. central row ($2k = t$) $\text{row} = N+1$	row col $val(\text{row}, \text{col})$
	row col $val(\text{row}, \text{col})$	$\longleftarrow$ 4. bottom filled row ($k = 0$) t moves down from central $\text{row} = N+1+t$
		row col $val(\text{row}, \text{col})$

Columns

- Columns represent dates

$$\text{col} = \overbrace{t+1}^{\text{0 not recognized as column}}$$

- example 1: $t = 0$ is the 1^{st} date, so: $\text{col} = t + 1 = 1$, as in "1^{st}"
- example 2: $t = 1$ is the 2^{nd} date, so: $\text{col} = t + 1 = 2$, as in "2^{nd}"

Rows

- Rows represent moves

$$\text{row} = \overbrace{\overbrace{N+1}^{\text{central row}} + \overbrace{t}^{\text{adds total moves}}}^{\text{finds bottom cell to fill in}} - \overbrace{\overbrace{2k}^{\text{links cell to up-moves}}}^{2\times \text{ to leave blank cell}}$$

Rows

- example: $N = 2$, so $t = 0, 1, 2$
- example 1: if $t = 0$, then $k = 0$, so

$$\text{row} = N + 1 + t - 2k = 2 + 1 + 0 - 2 \cdot 0 = 3$$

- example 2: if $t = 1$, then $k = 0, 1$
  - example 2a: if $k = 0$, then $t - k = 1$ (down-move)

$$\text{row} = N + 1 + t - 2k = 2 + 1 + 1 - 2 \cdot 0 = 4$$

  - example 2b: if $k = 1$, then $t - k = 0$ (up-move)

$$\text{row} = N + 1 + t - 2k = 2 + 1 + 1 - 2 \cdot 1 = 2$$

- example 3: if $t = 2$...

Tree diagram filled-in

		$N+1+t-2k$ 1 $t+1$ 3 $val(1,3)$
	$N+1+t-2k$ 2 $t+1$ 2 $val(2,2)$	
$N+1+t-2k$ 3 $t+1$ 1 $val(3,1)$		$N+1+t-2k$ 3 $t+1$ 3 $val(3,3)$
	$N+1+t-2k$ 4 $t+1$ 2 $val(4,2)$	
		$N+1+t-2k$ 5 $t+1$ 3 $val(5,3)$

Stock value

- The value of the stock depends on the direction of the moves

$$val = \overbrace{s}^{\text{initial value}} \cdot \overbrace{u^k}^{\#\text{ of up-moves}} \cdot \overbrace{d^{t-k}}^{\#\text{ of down-moves}}$$

Stock value

- example: $N = 2$, so $t = 0, 1, 2$
- example 1: if $t = 0$, then $k = 0$ and $t - k = 0$, so

$$val = s \cdot u^k \cdot d^{t-k} = s \cdot u^0 \cdot d^{0-0} = s$$

- example 2: if $t = 1$, then $k = 0, 1$
  - example 2a: if $k = 0$, then $t - k = 1$ (down-move)

$$val = s \cdot u^k \cdot d^{t-k} = s \cdot u^0 \cdot d^{1-0} = sd$$

  - example 2b: if $k = 1$, then $t - k = 0$ (up-move)

$$val = s \cdot u^k \cdot d^{t-k} = s \cdot u^1 \cdot d^{1-1} = su$$

- example 3: if $t = 2$...

Stock value

		$N+1+t-2k$ 1 $t+1$ 3 $s \cdot u^k \cdot d^{t-k}$ su^2
	$N+1+t-2k$ 2 $t+1$ 2 $s \cdot u^k \cdot d^{t-k}$ su	
$N+1+t-2k$ 3 $t+1$ 1 $s \cdot u^k \cdot d^{t-k}$ s		$N+1+t-2k$ 3 $t+1$ 3 $s \cdot u^k \cdot d^{t-k}$ sdu
	$N+1+t-2k$ 4 $t+1$ 2 $s \cdot u^k \cdot d^{t-k}$ sd	
		$N+1+t-2k$ 5 $t+1$ 3 $s \cdot u^k \cdot d^{t-k}$ sd^2

Hedging portfolio

- The hedging portfolio is last composed at the last-but-one date, so the columns are $N+1-1=N$
- We need two rows for each cell at that date, so the rows are $2\,(2N+1-2)=2\,(2N-1)$

HEDGING PORTFOLIO: VALUES (POSITION)

$$
\begin{aligned}
\text{row} &= \overbrace{\overbrace{2N-1}^{\text{central row}} + \overbrace{2t}^{\text{adds total moves}}}^{\text{finds bottom cell to fill in}} - \overbrace{\overbrace{4k}^{\text{links cell to up-moves}}}^{2\times\text{ to leave blank cell}} \\
&= 2\,(N+t-2k)-1
\end{aligned}
$$

Hedging portfolio

HEDGING PORTFOLIO: VALUES (LINKS)

hedging portfolio		option value
		$(N+t-2k, t+2)$
$x: \quad (2(N+t-2k)-1, t+1)$ $y: \quad (2(N+t-2k), t+1)$	$\longleftrightarrow$	
		$(N+2+t-2k, t+2)$

- We write the values for x and y as in the model

MATLAB codes

- `BT_StockTree.m`
- `BT_EurCallTree.m`
- `BT_EurCallHP.m`

Exercise Create the MATLAB codes for a European put option

Table of contents

Calibration

- For practical use, the binomial model needs *calibration*
- Calibration expresses the model's parameter as functions of observables
- Practitioners only observe a limited amount of information
  - e.g. current value of S_t, its trend and volatility, interest rate R
  - they do not observe u and d, or q_u and q_d
- We thus need to infer some values from the available information
- The CRR (Cox, Ross and Rubinstein) is a widely used calibration

$$u = e^{\sigma\sqrt{\delta t}}, \quad d = e^{-\sigma\sqrt{\delta t}}, \quad q_u = \frac{e^{r\delta t} - d}{u - d}$$

  - σ indicates the variability of the underlying stock
  - δt is the length of the time interval considered
  - r is the constant deterministic interest rate

Calibration

- The parameterization uses the exponential version of interest rate
  - this makes no difference since we can freely switch between the two versions by using the approximation

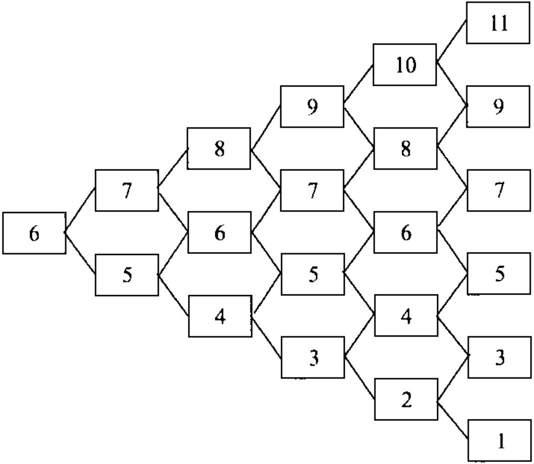
$$1 + r \simeq e^r$$

  - note also that $d = 1/u$, thus the resulting tree is recombining
- We also need increasing the number of state that we consider
  - the model is of no practical use if only a few states are allowed
  - to do so, we let trade occur at intermediate dates before maturity
  - this is not only practical, but also essential at times (e.g. when we want to price American options)
- MATLAB code (function): `BTC_EurCallHP.m` (`BT_Calib`)

Faster codes

- The implementation of binomial tree can be improved
  - no need to repeat calculation of discounted probabilities in the '`for`' loop; we can multiply discount factor and probabilities once
  - with CRR calibration ($ud = 1$), we may save memory by storing the underlying asset prices in a vector rather than a matrix
- Considering N time-steps, we store $2N + 1$ elements rather than N^2
  - for five periods, only eleven different values are used for the underlying asset price, down from 25 elements
  - with greater accuracy, e.g. $N = 1000$, the contrast is between (one-million-element) matrix and (2001-entry) vector becomes huge

Tree-diagram computational strategy



Tree-diagram computational strategy

- The numbers shown there are locations in the vector
  - element 1 stores the lowest value (down-steps only)
  - odd-numbered entries correspond to the last time-layer
  - even-numbered entries correspond to the second-to-last time-layer
  - the tree's root may be even-numbered or odd-numbered, depending on the number of time steps
- The same scheme may be adopted to store option values
- In principle, we should use two vectors corresponding to two consecutive time layers
- However, we may use one vector of $2N + 1$ elements since
  - even-numbered elements belong to a layer
  - odd-numbered elements belong to another one

Computational strategies' outcome comparison

MATLAB codes

- BTC_EurOptions.m
- BTS_EurOptions.m
- BT_EurOptions_Cfr.m

Pricing derivatives with the binomial model

I. **Pay-later options**

II. American options

III. Spread options

Pricing a pay-later option

- Consider a pay-later call option on a non-dividend-paying stock
  - the key feature of the pay-later option is that no premium is paid up-front, when the contract is entered
  - under certain conditions, it will be paid later instead
- If the option is in the money at expiration, then the option *must* be exercised and a *premium* is paid to the writer
  - otherwise, the option expires worthless and no premium is due
  - the net payoff for the option holder can be negative:
    - if the option is not deeply in the money
    - then the payoff may be smaller than the premium
- By no-arbitrage argument, if the net payoff were always non-negative, we could not have a zero-value contract at $t = 0$

Pay-later premium

- How can we find the fair premium value?
- Given a premium M, the payoff will be

$$\Pi(S_T, M) = \begin{cases} S_T - K - M & \text{if } S_T \geq K \\ 0 & \text{if } S_T < K \end{cases}$$

- Given M, we may find the value of the option using a binomial model
- Now we must find a value M such that the risk-neutral expectation of the payoff, with respect to S_T, is zero

$$E^Q\left[\Pi(S_T, P)\right] = 0$$

- Provided that interest rate is constant over time, the discount factor does not play any role

Pay-later option price

MATLAB codes

- To solve this equation for M, we couple the binomial tree with the numerical method to solve non-linear equations
- We first prepare a function returning the discounted payoff of the pay-later option, for a given M
- Then, we solve for M using the pre-compiled function '`fzero`' and a starting premium for the search

- `BTS_PayLaterCall.m` (`BTS_PLC`)

Exercise Create MATLAB codes for a European pay-later put

Pricing derivatives with the binomial model

I. Pay-later options

II. American options

III. Spread options

Pricing an American put option

- Pricing an American option by the binomial tree technique that we have illustrated in the last section is fairly easy
- The only critical point is how we should account for early exercise
- We deal here with a vanilla American-style *put* option on a non-dividend paying stock
- What about pricing an American *call* option?
  - if the underlying security does not pay any dividend
  - and the risk-free interest rate is positive, i.e. $B_t < 1 = B_T$, $\forall t < T$
  - then you should never exercise an American call option in advance
  - this means American and European call options have the same value

$$C_t^A = C_t^E,\ \forall t < T$$

Why an American call is never exercised

A three-step argument

i. The American call written on a financial asset that does not pay dividends provides for the additional right of early exercise

- Hence, it cannot be valued less than its European counterpart

$$C_t^A \geq C_t^E, \ \forall t < T$$

ii. The European call cannot be worth less than the difference between the current values of the financial asset and the strike price

$$C_t^E \geq S_t - K\,B_t, \ \forall t < T$$

Why an American call is never exercised

A three-step argument

- Suppose

$$C_t^E < S_t - K\,B_t$$

- We can buy the call, sell the underlying, $K\,B_t$ goes into the bank and the profit generated is

$$S_t - \left(C_t^E + K\,B_t\right) > 0$$

- At maturity, our position cannot be negative
  - if the call is exercised, the exercise price is covered by the bonds and our short position is neutralized

$$K - K\,B_T = 0 \text{ and } S_T - S_T = 0$$

  - otherwise, the long position in bonds has greater value than the short position in the underlying, with the additional arbitrage profit

$$K - S_T > 0$$

Why an American call is never exercised

A three-step argument

- Jointly considering the two condition, it must then be

$$C_t^A \geq S_t - K\,B_t,\ \forall t < T$$

iii. Since by definition $B_t < 1$, early exercise would produce

$$S_t - K < S_t - K\,B_t,\ \forall t < T$$

$\rightarrow$ An active American call is worth more than its early exercise

$$C_t^A \geq S_t - K,\ \forall t < T$$

- this holds true even if one believes that the value of the stock will fall below the value of K
- in this case, the option should be sold rather than exercised

American put at maturity

- Consider a node (i, N) on the last time layer of the tree diagram
- If the put is in the money, it is obviously optimal to exercise it
- Hence, at expiration we have

$$\Pi_{i,N} = \max\{K - S_{i,N}, 0\}$$

- $S_{i,N} = Su^i d^{N-i}$ is the underlying asset price on node (i, N)

American put before maturity

- Now consider a point in the second-to-last time layer
  - if the option is not in the money ($S_{i,N-1} > K$), we do not exercise
  - if the option is in the money, we should wonder about the opportunity of taking an immediate profit $K - S_{i,N-1}$
- In other words, we have to solve an **optimal stopping** problem

en.wikipedia.org/wiki/Optimal_stopping

  - at each time step, we must observe the state of a dynamic system
  - decide whether we should stop the game and just grab the money
  - or we should go onto the next step

Optimal stopping

- We solve the problem in a simple way, comparing the immediate payoff (*intrinsic value*) against the *continuation value*
- Naturally, the intrinsic value is

$$\Pi^{x}_{i,N-1} = \max\left\{K - S_{i,N-1}, 0\right\}$$

- We exercise if the intrinsic value exceeds the continuation value

$$\Pi^{c}_{i,N-1} = e^{-r\,\delta t}\left[q\,\Pi_{i+1,N} + (1-q)\,\Pi_{i-1,N}\right]$$

- Hence, the option value in each second-to-last time layer node is

$$\Pi_{i,N-1} = \max\left\{K - S_{i,N-1}, e^{-r\,\delta t}\left[q\,\Pi_{i+1,N} + (1-q)\,\Pi_{i-1,N}\right]\right\}$$

American put price

MATLAB codes

- The same argument may be repeated recursively for any time layer
  - we start from the last time layer, where the option value is the payoff
  - we proceed backward in time using the slight modification of the usual discounted expectation scheme given above
  - specifically, we first compute the continuation value, then compare it against the intrinsic value
- MATLAB Financial Toolbox provides the function `binprice.m`
  - prices vanilla American puts and calls (allowing for dividends)
  - we may compare our result with that produced by this function
- `BTS_AmericanPut.m` (`BTS_AP`)

Pricing derivatives with the binomial model

I. Pay-later options

II. American options

III. Spread options

Pricing a spread option

- What happens if the financial derivatives are written on more than one underlying asset?
- We extend the binomial model technique to multidimensional options and consider an American spread option on two assets
- The payoff (intrinsic value) of this option at (before) maturity is

$$\max\{S_1 - S_2 - K, 0\}$$

- The basic approach can be extended to more general options
  - we generalize it by considering continuous dividend yields (e_1, e_2)
  - this is easily done as we need only adjust the risk-neutral dynamics

Spread option calibration

- Calibration yields the following move factors and probabilities

$$
\begin{aligned}
u_1 &= e^{\sigma_1\sqrt{\delta t}} \quad d_1 = 1/u_1 \\
u_2 &= e^{\sigma_2\sqrt{\delta t}} \quad d_2 = 1/u_2 \\
p_{uu} &= \frac{1}{4}\left[1+\sqrt{\delta t}\left(\frac{\mu_1}{\sigma_1}+\frac{\mu_2}{\sigma_2}\right)+\rho\right] \\
p_{ud} &= \frac{1}{4}\left[1+\sqrt{\delta t}\left(\frac{\mu_1}{\sigma_1}-\frac{\mu_2}{\sigma_2}\right)-\rho\right] \\
p_{du} &= \frac{1}{4}\left[1-\sqrt{\delta t}\left(\frac{\mu_1}{\sigma_1}-\frac{\mu_2}{\sigma_2}\right)-\rho\right] \\
p_{dd} &= \frac{1}{4}\left[1-\sqrt{\delta t}\left(\frac{\mu_1}{\sigma_1}+\frac{\mu_2}{\sigma_2}\right)+\rho\right]
\end{aligned}
$$

- The first (second) probability subscript refers to an asset 1 (2) move
- μ and σ capture the asset price trend and volatility
- ρ is the price correlation between the two assets

Probability interpretation

- These conditions have an intuitive interpretation
- Two up-moves probability is large when the two drifts are large (relative to the respective volatilities) and correlation is positive
- In the probability of a S_1-up-move and a S_2-down-move
  - the drift μ_2 has a negative impact (the larger the drift, the less likely a down-move)
  - while negative correlation makes this joint movement more likely
- A similar reasoning applies to a S_1-down-move and a S_2-up-move
- Two up-moves probability is smaller when drifts are large and is larger when correlation is positive

Spread option price

MATLAB codes

- The implementation of this bidimensional model really requires careful memory management
- Since each asset's move are the same in absolute value, we may once again resort to vectors rather than matrices
- The prices of the options are stored in a matrix instead, and initialized with the option payoffs
- Since the options are American, we compute the continuation values as a risk-neutral expectation
- We then compare them against the respective intrinsic values

- `BTS_AmericanSpread.m` (`BTS_AS`)

Table of contents

Numerical simulation

- A great deal of financial concepts builds on *continuous-time* pricing
- Continuous-time computations involve complex **definite integrals**: some can be algebraically solved, others cannot

en.wikipedia.org/wiki/Integral

- Numerical simulation helps in addressing this issue when integrals cannot be solved algebraically in any obvious way
- A function's definite integral is a number: calculating this number is a deterministic problem that has nothing to do with stochasticity
- However, the involved computations can be perceived as part of a stochastic context by interpreting the integral as an *expected value*

Integrals and the law of large numbers

- For example, consider an integral defined on the unit interval $[0, 1]$

$$I = \int_0^1 g(x)\, dx$$

- We can interpret this integral as the expected value $E[g(x)]$, with $x \sim U(0, 1)$ a uniform random variable defined on the interval $(0, 1)$
- We may estimate the expected value (a number) with a sample mean (a random variable)

Integrals and the law of large numbers

- We generate a sequence $\{x_i\}$ of independent random draws from the uniform distribution, then compute the sample mean

$$\hat{I}_m = \frac{1}{m}\sum_{i=1}^{m} g\left(x_i\right)$$

- The **law of large numbers** then implies, with certainty

en.wikipedia.org/wiki/Law_of_large_numbers

$$\lim_{m\to\infty}\hat{I}_m = I$$

Monte Carlo simulation

- A well-established simulation approach is the **Monte Carlo method**

  en.wikipedia.org/wiki/Monte_Carlo_method

  - the name "Monte Carlo" is attributed to Nicholas C. Metropolis
  - he linked the random sampling to the activity of the well-known casino located in the Principality of Monaco
- The method *mimics* random sampling
  - it is factually impossible to perform random sampling on computers
  - but we can use a sequence of **pseudo-random** numbers, created by special *generators* as provided by most programming languages

    en.wikipedia.org/wiki/Pseudo-random_number_sampling

Monte Carlo simulation

MATLAB commands

- For instance, we may consider the integral

$$I = \int_0^1 e^x dx = [e^x]_0^1 = e - 1 \cong 1.7183$$

- In order to generate random sampling from a uniform distribution, we can use the MATLAB `rand` function
  - the `rand(m,n)` command produces an $m \times n$ matrix of random extractions from a uniform distribution
  - it should be noted that the parameters m and n have nothing to do with the distribution, which merely is $U(0,1)$
- The estimates reliability varies according to the sample size
  - to have a reproducible experiment: `rand('state',0)`
  - we average over 10 extractions: `mean(exp(rand(1,10)))`
  - and over one million extractions: `mean(exp(rand(1,1000000)))`
  - the estimate is not very reliable for a few extractions, but gets closer to the correct number with more extractions

European call option continuous-time pricing

- Specifically, we are interested in pricing derivatives, for instance a European call option
- Continuous-time pricing also involves the calculation of a risk-neutral valuation formula of the derivative's payoff

$$F(t, s) = e^{-r(T-t)} E^{Q}_{t,s} \left[\Phi\left(S\left(T\right)\right)\right]$$

- Depending on the type of derivative, we may need to generate
  - the entire sample paths of the underlying value
  - simply the price of the asset at maturity
- The generation of entire paths will be covered later
- A European call option requires sampling the payoff only

$$\Phi\left(S\left(T\right)\right) = \max\left\{S\left(T\right) - K, 0\right\}$$

European call option continuous-time pricing

- In the continuous-time model, the underlying's price at maturity is

$$S(T) = s\, e^{\left(r-\sigma^2/2\right)(T-t)+\sigma\left[\bar{W}(T)-\bar{W}(t)\right]}$$

  - $\bar{W}(T) - \bar{W}(t)$ is the **Wiener process** innovation between t and T
  - Wiener process innovations are normally distributed with zero mean and variance equal to the time interval, $T - t$
  - the innovation sampling equals an extraction from a standardized normal random variable, $\varepsilon \sim N(0,1)$ multiplied by the value $\sqrt{T-t}$

en.wikipedia.org/wiki/Wiener_process

- Our pricing equation therefore becomes

$$F(t,s) = e^{-r(T-t)} E^{Q}_{t,s}\left[\max\left\{s\, e^{\left(r-\sigma^2/2\right)(T-t)+\sigma\,\varepsilon\sqrt{T-t}} - K, 0\right\}\right]$$

- MATLAB code (function): `AE_EurCall.m` (`AE_EC`)

Precision of continuous-time pricing

- Clearly, to obtain sufficiently robust results, we should not refer to a simple point estimate
- We should also calculate some confidence intervals for the estimate
- Ideally, we should understand how many extractions are needed to achieve a certain precision
- In general, many extractions seem to be needed, especially if
  - the underlying asset is very volatile
  - the option value depends on the entire underlying price process
- The number of draws could then become prohibitive to manage
- In sum, we need clever ways to reduce the variance of the estimator

Simulation of the underlying price process

- The application of Monte Carlo methods to derivative pricing builds on generating sample paths of the underlying price process
- We've just seen that pricing European options needs only computing the price of the underlying asset at maturity
- But if the derivative depends on the time path of the underlying price process, then we theoretically need to know the whole path
- In practical terms, however, we may simply generate a sequence of values that the underlying takes at certain instants of time
- After all, numerical simulation software works with a discrete number of elements
- We must therefore carry out a *time discretization* of the model

Time discretization and price process

- The simplest approach to time discretization is the **Euler method**

en.wikipedia.org/wiki/Euler_method

- We choose a discrete interval of time, denoted by δ
- We generate a draw $\varepsilon \sim N(0,1)$ at each instant of time which define these intervals
- We write the process that we need to discretize as a function of these draws
- For financial assets, the relevant process is the **geometric Brownian motion**, here expressed in integral form on the interval $(t, t+\delta)$

$$S(t+\delta) = S(t)\; e^{\left(r-\sigma^2/2\right)\delta+\sigma\left[\bar{W}(t+\delta)-\bar{W}(t)\right]}$$

en.wikipedia.org/wiki/Geometric_Brownian_motion

The geometric Brownian motion

- Recall that a Wiener process innovation is a normal random variable with zero mean and variance equal to the time interval length

$$\bar{W}(t+\delta) - \bar{W}(t) \sim N(0,\delta) \equiv \sqrt{\delta} \cdot N(0,1)$$

- We may build on our draw-generating process to obtain

$$\sqrt{\delta}\,\varepsilon \sim N\left(0, \sqrt{\delta}\right)$$

- Therefore, we can rewrite the geometric Brownian motion as

$$S(t+\delta) = S(t)\; e^{\left(r-\sigma^2/2\right)\delta+\sigma\sqrt{\delta}\,\varepsilon}$$

- MATLAB code (function): `AE_GraphStockSim.m` (`AE`)

Dynamic risk management

- We now turn to consider dynamic risk-management strategies based on a portfolio's **Greeks**

en.wikipedia.org/wiki/Greeks_(finance)

- The most popular Greeks are *delta* (sensitivity to underlying's price) and *gamma* (sensitivity to underlying's price volatility)
- We use **delta-hedging** to exemplify the strategy, but analogous schemes can be based upon all the other Greeks

en.wikipedia.org/wiki/Delta_neutral

- A strategy may also involve more than one Greek
- Once the strategy is implemented, the portfolio is said to be *neutral* with respect to the relevant Greek(s)

Definition of delta hedging

- Delta hedging consists of composing a portfolio whose value remains unchanged when small changes occur in the value of the underlying
- The delta of a portfolio is the derivative of its value with respect to the price of the underlying

$$\Delta^p = \frac{\partial V_t^p}{\partial s}$$

- If $\Delta^p = 0$, then the portfolio is *delta-neutral* (around a given underlying's value s)
- If $\Delta^p \neq 0$, then we may obtain neutrality:
  - by selling the portfolio and investing the resulting resounrces in bonds
  - by modifying the portfolio composition (buying or selling untis of underlying/derivative)

Implementation of delta hedging

- The value of any security based on a given underlying is correlated with the price of that underlying
- We can therefore *rebalance* the portfolio composition so that the *resulting* portfolio is delta-neutral
  - let $V^z(t,s)$ denote the value of the generic security z at time t, expressed as a function of the price s of the underlying
  - let x denote the units of the securiy added to the portfolio
- The value of the rebalanced portfolio p' is

$$V^{p'}(t,s) = V^p(t,s) + x \cdot V^z(t,s) \tag{2}$$

- For p' to be delta-neutral, we choose x such that $\partial V^{p'}/\partial s = 0$

$$\frac{\partial V^p}{\partial s} + x\frac{\partial V^z}{\partial s} = 0$$

- Utilizzando the definition of delta, the solution is

$$x = -\frac{\Delta_p}{\Delta_z} \tag{3}$$

Example: delta hedging of a derivative

- Suppose to have a short position in a derivative g with price process $V^g(t,s)$
- We want to hedge the risk of this transaction using the underlying
- In order to apply (2), we substitute $V^p = -V^g$ and $z = s$

$$\frac{\partial}{\partial s}\left(-V^g + xs\right) = 0$$

- The solution is

$$x = \frac{\partial V^g(t,s)}{\partial s} = \Delta_g$$

- The delta of a derivative gives us the units of underlying necessary to hedge the derivative

Delta hedging in practice

- Hedging the derivatives amounts to dismissing the risk that the assumed position involves
- It should then come as no surprise that it is possible to show an option price is essentially the cost of a delta hedging strategy
- We can write a MATLAB code to estimate the average cost of a delta hedging strategy
- We need to produce an array of sample paths, generated by a specific function (for example, `AE`)
- In this case, unlike the evaluation of the option, the actual underlying's trand must be used in the simulation
- To purchase units of underlying, we may resort to a loan
  - since the interest rate is deterministic and constant, we simply record the cash flows necessary for the negotiations
  - we then discount the cash flows at time $t = 0$

Delta hedging in practice

- The number of underlying units to be held in the portfolio is given by the option delta
- We calculate the delta with the `blsdelta` function for each point of the sample path
- We then compare compare the cost of the delta hedging strategy with the price of the option
- To simplify matter, we compute the option price using the built-in MATLAB function (`blsprice`)

- MATLAB code (function): `AE_DeltaHedging.m` (`AE_DH`)

Definition of stop-loss

- A strategy providing limar coverage as the delta hedging but easier to implement is the so-called *stop-loss*
- With reference to a European call option, the strategy consists of the following elements
  - a hedged position (hold a unit of underlying) when the price of the underlying is greater than the strike price (the option is in the money)
  - otherwise (the option is out of the money) one should opt for an uncovered position (hold no underlying unit)
- We should therefore buy a unit (if we do not already own it) when the price of the underlying rises above the strike price
- We sell the unit (if we own it) when it falls below the strike price

Implementation of stop-loss

- This strategy is very intuitive, but it is not easy to implement consistently in continuous time
- However, we can evaluate its performance in the discrete, simulating it with the Monte Carlo method
- The problem in this case is that we cannot buy or sell the underlying at the exact strike price
  - we buy at a price greater than K, as soon as we find out that the price is higher than that strike price
  - for the same reason, we sell at a price slightly lower than K
- Thus, even without considering transaction costs, there is a potential problem with the stop-loss strategy

Stop-loss in practice

- We can estimate the average cost of a stop-loss strategy as follows
- We use a *covered* status indicator to detect when the underlying is higher or lower than the strike price
- We record a cashflow every time we buy or sell the underlying
  - if the value of the underlying is higher than the strike price at maturity, the holder will exercise the option
  - therefore the strike price received must be added to the cashflows
- Cashflows are negative when we buy the underlying and positive when we sell
- As a result, the cost of the strategy is valued as the average of the total discounted cashflows with inverted sign
- MATLAB code (function): `AE_StopLoss.m` (`AE_SL`)

Table of contents

Multidimensional options

- The Monte Carlo method is easily adapted to multidimensional options
- For illustrative purposes, we consider a European exchange option written on two assets
- The risk-neutral asset value process follows a two-dimensional geometric Brownian motion

$$\begin{aligned} dU(t) &= rU(t)\,dt + \sigma_U U(t)\,dW_U(t) \\ dV(t) &= rV(t)\,dt + \sigma_V V(t)\,dW_V(t) \end{aligned}$$

- We assume the Wiener processes have instantaneous correlation ρ

Definition of exchange option

- The payoff of the option at maturity T is

$$\max\left(V\left(T\right) - U\left(T\right), 0\right)$$

- We can note that this option is a particular case of a spread option, whose payoff depends on the assets' price differential
- It is called exchange because it allows you to exchange one activity for another at maturity
- For example, if we hold a portfolio comprising U and an exchange option, the payoff at maturity is

$$U\left(T\right) + \max\left(V\left(T\right) - U\left(T\right), 0\right) = \max\left(U\left(T\right), V\left(T\right)\right)$$

Exact pricing of an exchange option

- For this option there exists an *analytical* evaluation formula based on a generalization of the Black-Scholes formula

$$P = V_0 N(d_1) - U_0 N(d_2)$$

  - the values of the Gaussian culumative distribution are

$$\begin{aligned} d_1 &= \frac{\ln(V_0/U_0) + \hat{\sigma}^2 T/2}{\hat{\sigma}\sqrt{T}} \\ d_2 &= d_1 - \hat{\sigma}\sqrt{T} \end{aligned}$$

  - the volatility of the option is given by

$$\hat{\sigma} = \sqrt{\sigma_V^2 + \sigma_U^2 - 2\rho\sigma_V\sigma_U}$$

- We write a MATLAB function (`exchange`) to implement the formula

Correlated Wiener processes

- We discuss another issue before applying the Monte Carlo method
- We impose that the joint sampling of the paths generated by the related Wiener processes has a normal multivariate distribution
- To this end, we carry out a **Cholesky decomposition** of the covariance matrix

en.wikipedia.org/wiki/Cholesky_decomposition

  - we write the correlation matrix, with correlation coefficient ρ

$$\mathbf{\Sigma} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

  - we impose $\mathbf{\Sigma} = \mathbf{L}\mathbf{L}'$, and obtain that the factors generating $\mathbf{\Sigma}$ are

$$\mathbf{L} = \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{bmatrix}$$

Correlated Wiener processes

- It is easy to verify that the definition of **L** obeys the identity

$$
\begin{aligned}
\mathbf{L}\mathbf{L}' &= \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} 1 & \rho \\ 0 & \sqrt{1-\rho^2} \end{bmatrix} \\
&= \begin{bmatrix} 1 \cdot 1 + 0 \cdot 0 & 1 \cdot \rho + 0 \cdot \sqrt{1-\rho^2} \\ \rho \cdot 1 + 0 \cdot \sqrt{1-\rho^2} & \rho \cdot \rho + \sqrt{1-\rho^2} \cdot \sqrt{1-\rho^2} \end{bmatrix} \\
&= \begin{bmatrix} 1 & \rho \\ \rho & \rho^2 + 1 - \rho^2 \end{bmatrix} = \Sigma
\end{aligned}
$$

- The two related Wiener processes are then generated by two *independent* normal random variables, Z_1 and Z_2

$$
\begin{aligned}
\varepsilon_1 &= Z_1 \\
\varepsilon_2 &= \rho Z_1 + \sqrt{1-\rho^2} Z_2
\end{aligned}
$$

Exchange option in practice

- In the case of the exchange option, we need to simply generate the joint sampling of the prices of the two assets at maturity
- After introducing the function, we calculate the values of the two related Wiener processes, a pair for each simulation
- The resulting vectors are used to compute the sample values at maturity of the geometric Brownian motions
- We then calculate the current values of the sample payoffs of the exchange option
- We also calculate the sample mean and its 95% confidence interval

- MATLAB code (function): `AE_Exchange.m` (`AE_EX`)

Definition of barrier option

- We now turn to consider an option whose valuation depends on the time path followed by the value of the underlying
- The **barrier option** is one type of **exotic option**

  en.wikipedia.org/wiki/Barrier_option

  en.wikipedia.org/wiki/Exotic_option

  - it activates ('*in*' type) or extinguishes ('*out*' type) if the price of the underlying reaches a predetermined level (the **barrier**)
  - the barrier can be placed above ('*up*' type) or below ('*down*' type) of the initial price of the underlying
  - there are therefore four main types of barrier options: *up-and-out*, *down-and-out*, *up-and-in*, and *down-and-in*
- An activated or extinguished option remains such even if the price of the underlying crosses the barrier again

Down-and-out put option

- Barier options are based on ordinary options, such as American, Bermudan, or European call or put option

  en.wikipedia.org/wiki/Option_style

- We examine here a European “down-and-out” put option
- The barrier is controlled at the end of each trading day
- There exists an analytical formula for continuous monitoring, which can be adjusted to reflect discrete monitoring
- As before, we use this formula to check the result of the simulation obtained with the Monte Carlo method

Adjustment of analytic formula to discrete-time pricing

- When computing the exact valuation of the option, the barrier value must be corrected by the expression

$$\hat{S}_b = S_b e^{-0.5826 \cdot \sigma \sqrt{\delta}}$$

- The analytic formula refers to continuous-time monitoring
- Under discrete-time monitoring, a down-and-out option will get a higher valuation than in continuous-time monitoring
- The reason is that crossing the barrier is less likely

Adjustment of analytic formula to discrete-time pricing

- In the (approximate) correction applied in the code
  - the term 0.5826 derives from the **Riemann zeta function**

    en.wikipedia.org/wiki/Riemann_zeta_function

  - δ is the time interval between two consecutive monitoring instants
- The sign of the correction depends on the type of option
- For a down-and-out option, the sign is *minus* so that the artificially lower barrier reflects the lower barrier-crossing probability

Barrier option in practice

- We implement the Monte Carlo simulation for the evaluation of a down-and-out put option as follows
- A parameter is used to determine how many times the price of the underlying should be compared with the barrier
- The payoff is zero if the price crosses the barrier at least once
  - the code simulates the complete path, though part of it may actually be useless if the barrier is crossed
  - nevertheless, this allows to streamline the implementation by refraining from using loops
- We also compute the number of paths in which the barrier is crossed
- MATLAB code (function): `AE_DownAndOut.m` (`AE_DO`)

Definition of Asian option

- An Asian option payoff is a function of the average price of the underlying asset in a certain period of time

en.wikipedia.org/wiki/Asian_option

- For this reason, it is also called an average value option
- There are several variations, the most basic being
  - *fixed* strike price: the average value of the underlying defines the amount of resources due *to* the option holder
  - *floating* strike price: the average value of the underlying defines the strike price due *by* the option holder
- The average can be calculated in many ways
  - conventionally, it refers to an *arithmetic* mean
  - but Asian options with a *geometric* mean are also relatively popular

en.wikipedia.org/wiki/Geometric_mean

Geometric Asian option

- We consider an Asian call option with geometric mean and fixed strike price calculated at discrete intervals of time
- The option payoff is

$$\max\left(\left[\prod_{t=1}^{N} S(t)\right]^{\frac{1}{N}} - K, 0\right)$$

- We assume that T is the maturity date of the option
- the index t is the progressive value of the intervals $\delta = T/N$ between the beginning of the contract and the t^{th} monitoring
- For simplicity, we assume that monitoring is carried out at equidistant dates (although this is not always the case)

Asian option in practice

- To apply the Monte Carlo method
  - we simply generate sample paths of the price of the underlying asset
  - we obtain, as usual, the sample mean of the discounted payoff
- A parameter represents the number of monitoring instant sampled to calculate the arithmetic mean
- It should not be confused with the number of simulations
- We compare the result with an exact evaluation formula

- MATLAB code (function): `AE_AsianGeomFixed.m` (`AE_AGF`)