Asset Pricing

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Preliminary requirements (Module A)

- Please email TODAY your CV with passport-like photo at

  merella@unica.it

- In the email text, rate your knowledge of the following topics

  [1-10, 1 = awful, 10 = excellent]

  - microeconomics
  - macroeconomics
  - mathematics (optimization and linear algebra)
  - statistics and econometrics
  - computational methods

Preliminary information (Module A)

- Course material

  unica.it/unica/page/it/economia_finanziaria_it_1

- Office hours

  Fridays, 10-11, meet.google.com/fjz-edgx-jrh

  - please email me in advance

Exam rules

- The exam consists of two sections, one per module, each yielding 50% of the final mark
- Both sections of the exam must be sit within the same session

Exception The students may sit each section of the exam *separately* in the **June**, **July** and **September** exam sessions

The exception applies **only once** per student

Should the student **fail** or **reject the final mark**, (s)he will *re-sit the exam as a whole in a single exam session*

Examination format (Module A)

- The exam features 12 on-screen written questions, answered orally
- The questions are divided in 4 groups, each covering a different topic
  - the 3 questions in each group are in an increasing order of difficulty
  - each question yield up to three points
    - you can still score 27/30 by failing to reply to one whole topic!
    - and still 24/30 by failing to reply to the most difficult questions!
- The duration of the exam is 12 minutes per candidate
  - this gives you one minute per question on average
  - practising on the course exercises is of paramount importance!
- A mock exam will be provided at the end of week 5

Objective

- We try to understand the values of claims to *uncertain* payments
  - a low price implies a high rate of return
  - so we also try to explain the distribution of asset returns
- An asset value accounts for the *delay* and the *risk* of its payments
- Risk is the most important determinant of many assets' values
- Over the last 50 years, the average real return on U.S. stocks is 9%
  - only about 1% is due to interest rates
  - the remaining 8% is a premium earned for holding risk
- Uncertainty makes asset pricing interesting and challenging

Paradigm

- Asset pricing theory all stems from one simple concept

price equals expected discounted payoff

- There are two polar approaches to this elaboration
- *Absolute* pricing relates asset prices to their exposure to fundamental sources of risk
  - the consumption-based model is the purest example of this approach
  - it is the paradigm for most of the topics covered in this course
- *Relative* pricing relates asset prices to the prices of some other assets
  - APT and Black-Scholes are the classic examples of this approach
  - used only towards the end of this course, it will be the paradigm for next year's course on Financial Derivatives and Risk Management

Approach

- We study financial assets
- Financial assets correspond to contractual claims involving a current price and a future, possibly contingent, payoff
- We focus on assets traded in financial (*high liquidity*) markets
- We narrow our examination to (secondary) markets in which individuals (as opposed to firms) trade assets
- We refer to individuals as **investors**
  - both the supply side and demand side of the market are populated by investors
  - any investor might act as a "supplier" or "consumer" of assets at different points in time
  - what matter is then the investor's **net demand** of assets

Preview of findings

- The central task of absolute asset pricing is to understand and measure the sources of aggregate risk that drive asset prices
- A lot of empirical work has documented tantalizing stylized facts and links between macroeconomics and finance, for example...
  - expected returns relate, across time or assets, to macroeconomic variables, or variables that also forecast macroeconomic events
  - a wide class of models suggests that a "recession" or "financial distress" factor lies behind many asset prices
- Yet theory lags behind
- We *do not* yet have a well-described model that explains these interesting correlations

Overview

- Week 1: Financial markets and investors
- Week 2: Consumption-based model
- Week 3: Asset pricing and the macreconomy
- Week 4: Alternative consumption-based models
- Week 5: Complete vs incomplete financial markets
- Week 6: Other approaches to finance

Preliminary definitions

Definition (Financial economics)

Financial economics is the branch of economics studying financial assets, as opposed to goods and services.

en.wikipedia.org/wiki/Financial_economics

Definition (Financial asset)

A financial asset is a non-physical asset whose value is derived from a contractual claim, such as bank deposits, bonds, and stocks.

en.wikipedia.org/wiki/Financial_asset

Preliminary definitions

Definition (Financial market)

A financial market is a market in which financial assets are traded.

en.wikipedia.org/wiki/Financial_market

Definition (Secondary financial market)

The secondary market is the financial market in which previously issued financial assets are traded.

en.wikipedia.org/wiki/Secondary_market

Table of contents

Financial market structure

- Two elements characterize financial assets
  - **price**, the *current* payment involved in any asset's contractual claim
  - **payoff**, the *future* payment involved in any asset's contractual claim
- Two notions determine the economics of financial assets
  - **time**, since assets involve a current payment and a future payoff
  - **uncertainty**, since payoff are possibly contingent
- Two concepts govern the functioning of financial markets
  - **free portfolio formation**, which sets investor's trading possibilities
  - **law of one price**, which regulates asset pricing behavior

Notation for time, price and payoff

- We indicate the payoff with x, which we also use to identify the asset under consideration
- We indicate the price with $p(x)$ or, when it does not generate confusion, simply by p
- We indicate time with $t = 1, 2, ..., T$
  - a generic t typically indicates the current period
  - a generic $t + \tau$, with $\tau = 1, 2, ..., T - t$, indicates a future period
  - whenever necessary, time stands as the subscript of any given variable (e.g. p_t, $x_{t+\tau}$)

Notation for uncertainty

- We model uncertainty by means of different *scenarios* possibly occurring at a given point in time
- We refer to a scenario as a **state of nature**, or simply **state**, indicated by $s = 1, 2, ..., S$
- A variable taking different values across states is a **random variable**
  - we indicate a random variable simply by the relevant letter (e.g. x)
  - we indicate a random variable **realization** by the relevant state in parenthesis [e.g. $x(s)$]
  - whenever useful, we collect a random variable's realizations in a **vector** {e.g. $\mathbf{x} \equiv [x(1), x(2), ..., x(S)]'$)}

en.wikipedia.org/wiki/Row_and_column_vectors

Complete notation structure

	time	random variable	state-s realization	vector of realizations
current date	t	p_t	p_t	p_t
future dates	$t+\tau$	$x_{t+\tau}$	$x_{t+\tau}(s)$	$\mathbf{x}_{t+\tau}$

- The current state is *always* known, hence current values do not carry the relevant state in parenthesis
- For the same reason, the vector of realizations degenerates to a **scalar** (a single value)

en.wikipedia.org/wiki/Scalar_(mathematics)

Uncertainty

- We refer to the possibility that a given state s occurs as a **probability**, indicated by $\pi(s)$
- The probability $\pi(s)$ is *always* known for every state $s = 1, 2, ..., S$
  - when $\pi(s) = 0$, the state cannot occur
  - when $\pi(s) = 1$, the state occurs with certainty
- The sum of every event's probability *must* amount to certainty, hence

$$\sum_{s=1}^{S} \pi(s) = 1$$

Free portfolio formation

Definition (Payoff space)

The payoff space, denoted by $\mathcal{X}$, is the set of *every* payoff that investors may trade.

Definition (Free portfolio formation)

If two payoffs belong to the payoff space, then so does every combination of those two payoffs; formally

$$x^i, x^j \in \mathcal{X} \;\Rightarrow\; ax^i + bx^j \in \mathcal{X}, \forall a, b \in \mathcal{R}$$

- Free portfolio formation rules out short sales constraints, bid/ask spreads and leverage limitations

Law of one price

Definition (Law of one price)

The price of a combination of assets equals the (same) combination of assets' prices; formally

$$p\left(ax^i + bx^j\right) = ap\left(x^i\right) + bp\left(x^j\right), \forall a, b \in \mathcal{R}$$

- The law of one price rules out instantaneous profits made by repackaging portfolios
- For the law to hold, it suffices that at least one investor values a package by its contents, regardless how it is marketed
- Hence, if a violation of the law exists, traders quickly eliminate it so it cannot survive in equilibrium

The investor

- Our objective is to formulate the investor's net demand of assets within the framework outlined so far
- The starting point is to evaluate the payoff **x** from the investor's perspective
- We consider investors as a typical economic agents
  - investors assign value to *resources* according to a **utility function**

    en.wikipedia.org/wiki/Utility

  - the resources available to investors are expressed by **budget constraints**

    en.wikipedia.org/wiki/Budget_constraint

Utility function

- In this context, we refer to resources as **consumption**, indicated by c
- The utility function, indicated by $u(\cdot)$, assigns the value $u(c)$ to consumption c
- The typical properties characterize the function $u(\cdot)$
  - higher c means higher $u(c)$

$$u'(c) \equiv \frac{\partial u(c)}{\partial c} > 0$$

  - the variation in $u(c)$ for a unit raise in c is no more than proportional as c increases

$$u''(c) \equiv \frac{\partial^2 u(\cdot)}{(\partial c)^2} \leq 0$$

Isoelastic utility function

- An explicit function is required for a quantitative analysis
- We consider the isoelastic specification

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma} \quad (1)$$

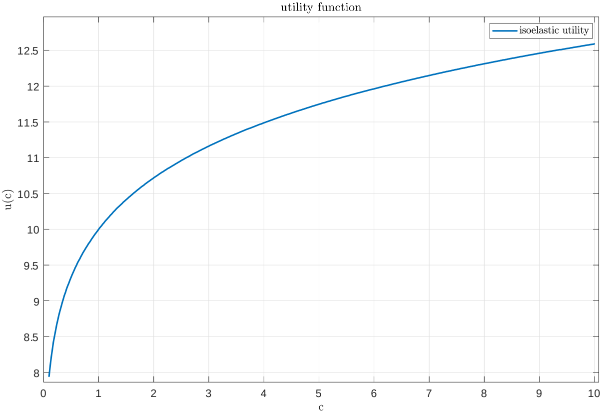
where the curvature parameter γ represents the *relative* **risk aversion** coefficient

en.wikipedia.org/wiki/Isoelastic_utility

en.wikipedia.org/wiki/Risk_aversion

Isoelastic utility function

- MATLAB code: `isoelastic_utility.m`



Exercise Replicate with other utility function specifications

Utility function: other specifications

en.wikipedia.org/wiki/Isoelastic_utility#Special_cases

en.wikipedia.org/wiki/Exponential_utility

en.wikipedia.org/wiki/Stone-Geary_utility_function

en.wikipedia.org/wiki/Hyperbolic_absolute_risk_aversion

en.wikipedia.org/wiki/Hyperbolic_absolute_risk_aversion#Special_cases

Lifetime utility

- The utility function $u\left(c_t\left(s\right)\right)$ measures investor's valuation of consumption separately for each date t and state s
- **Lifetime utility**, from time t onwards indicated by U_t, aggregates these values by the weighted average

$$U_t = \sum_{t=1}^{T} \beta^{t-1} \sum_{s=1}^{S} \pi\left(s\right) u\left(c_t\left(s\right)\right) \tag{2}$$

  - the **subjective discount factor** β assigns different weights to valuation of consumption at different dates
  - the probability π assigns different weights to valuation of consumption in different states, for any given date

Lifetime utility

- In most applications, we consider the one-period two-state ($s = 1, 2$) case of (2)

$$U_t = u(c_t) + \beta [\pi(1) u(c_{t+1}(1)) + \pi(2) u(c_{t+1}(2))] \quad (3)$$

- The single period runs from date t to date $t + 1$, thus involving two time-layers of valuations, $u(c_t(\cdot))$ and $u(c_{t+1}(\cdot))$
- The state at the current date t is realized, thus we have only one valuation in t, $u(c_t)$

Lifetime utility

- Using (1) in (3), we can identify three indifference curves, between
  1. current and state-1 future consumption levels, isolating $c_{t+1}(1)$

$$c_{t+1}(1) = \left(\frac{(1-\gamma)U - \beta\pi(2)\, c_{t+1}(2)^{1-\gamma}}{\beta\pi(1)} - \frac{1}{\beta\pi(1)} c_t^{1-\gamma} \right)^{\frac{1}{1-\gamma}}$$

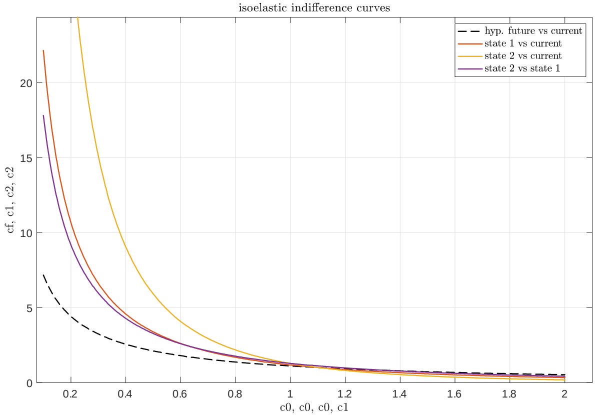
  2. current and state-2 future consumption levels, isolating $c_{t+1}(2)$
  3. state-1 and state-2 future consumption levels, isolating $c_{t+1}(2)$

$$c_{t+1}(2) = \left(\frac{(1-\gamma)U}{\beta\pi(2)} - \frac{1}{\beta\pi(2)} c_t^{1-\gamma} - \frac{\pi(1)}{\pi(2)} c_{t+1}(1)^{1-\gamma} \right)^{\frac{1}{1-\gamma}}$$

- In plotting the curves, beware of the choice of values!

Lifetime utility

- MATLAB code: `isoelastic_indifference_curves.m`



Exercise Replicate with other utility function specifications

Expected value

Definition (Expected value)

The expected value $E[z]$ of a random variable z is the probability-weighted average of the independent realizations of z

$$E[z] = \sum_{s=1}^{S} \pi(s)\, z(s)$$

en.wikipedia.org/wiki/Expected_value

Expected lifetime utility

- Using the definition of expected value for the future valuations of contingent consumption in (3), we have

$$E\left[u\left(c_{t+1}\right)\right]=\pi\left(1\right)u\left(c_{t+1}\left(1\right)\right)+\pi\left(2\right)u\left(c_{t+1}\left(2\right)\right)$$

- Lifetime utility (3) can thus be conveniently expressed as

$$U_t=u\left(c_t\right)+\beta E\left[u\left(c_{t+1}\right)\right] \tag{4}$$

- Under uncertainty, the investor evaluates future outcomes by means of the expected value $E\left[u\left(c_{t+1}\right)\right]$

en.wikipedia.org/wiki/Von_Neumann-Morgenstern_utility_theorem

Budget constraints

- The investor's net demand for assets, indicated by q, generates
  - at date t, the flow of resources

$$-p_t \cdot q$$

  - at date $t+1$, for each $s = 1, 2$, the (mutually exclusive) flows

$$x_{t+1}(1) \cdot q, \quad x_{t+1}(2) \cdot q$$

- The investor is exogenously assigned the endowments

$$e_t, \quad e_{t+1}(1), \quad e_{t+1}(2)$$

Budget constraints

- At date t, consumption is the sum of the endowment and the value of the asset's net demand

$$c_t = e_t - p_t q \tag{5}$$

- At date $t+1$, consumption is the sum of the endowment and the outcome of the asset's net demand
  - if the realized state is $s = 1$

$$c_{t+1}(1) = e_{t+1}(1) + x_{t+1}(1)\, q \tag{6}$$

  - if the realized state is $s = 2$

$$c_{t+1}(2) = e_{t+1}(2) + x_{t+1}(2)\, q \tag{7}$$

Table of contents

Investor's problem

- The investor maximizes lifetime utility (4) taking into account the constraints (5)-(7)

$$\begin{aligned} \max_{q} \quad & U = u\left(c_t\right) + \beta E\left[u\left(c_{t+1}\right)\right] \\ sub \quad & c_t = e_t - p_t q \\ & c_{t+1} = e_{t+1} + x_{t+1} q \\ & c_t, c_{t+1} > 0 \end{aligned} \tag{8}$$

- We may obtain an unconstrained optimization problem by replacing the budget constraints into lifetime utility

$$\max_{q} \; u\left(e_t - p_t q\right) + \beta E\left[u\left(e_{t+1} + x_{t+1} q\right)\right]$$

Solution to the investor's problem

- Setting the derivative with respect to q equal to zero, we obtain

$$\frac{\partial u\left(e_t - p_t q\right)}{\partial q} + \beta E\left[\frac{\partial u\left(e_{t+1} + x_{t+1} q\right)}{\partial q}\right] = 0 \tag{9}$$

- The derivatives can be computed using the *chain rule*

en.wikipedia.org/wiki/Chain_rule

Solution to the investor's problem

- Applying the chain rule, we have

$$\frac{\partial u\left(c_t\left(q\right)\right)}{\partial q} = \frac{\partial u\left(c_t\right)}{\partial c_t}\frac{\partial\left(e_t - p_t q\right)}{\partial q} = u'\left(c_t\right)\left(-p_t\right)$$

$$\frac{\partial u\left(c_{t+1}\left(q\right)\right)}{\partial q} = \frac{\partial u\left(c_{t+1}\right)}{\partial c_{t+1}}\frac{\partial\left(e_{t+1} + x_{t+1} q\right)}{\partial q} = u'\left(c_{t+1}\right)x_{t+1}$$

- Using these expressions into (9), we get

$$u'\left(c_t\right)\left(-p_t\right) + \beta E\left[u'\left(c_{t+1}\right)x_{t+1}\right] = 0$$

First-order condition for the investor's problem

- Rearranging, we obtain the *first-order condition* for an optimal consumption and portfolio choice

$$p_t u'(c_t) = \beta E\left[u'(c_{t+1})\, x_{t+1}\right] \tag{10}$$

- The investor buys more or less of the asset until this condition holds
  - the investor will seek to buy more of the asset if

$$p_t u'(c_t) < \beta E\left[u'(c_{t+1})\, x_{t+1}\right]$$

  - the investor will seek to buy more of the asset if

$$p_t u'(c_t) > \beta E\left[u'(c_{t+1})\, x_{t+1}\right]$$

Basic pricing formula

- Define the **stochastic discount factor** (**SDF**) as

$$m_{t+1} \equiv \frac{\beta u'(c_{t+1})}{u'(c_t)} \tag{11}$$

- Dividing both sides of (10) by $u'(c_t)$ and using (11), we obtain the **basic pricing formula**

$$p_t = E_t[m_{t+1} x_{t+1}] \tag{12}$$

- The formula states that an asset's price should be the *expected discounted payoff* of that asset

Consumption-based SDF

- The consumption-based model uses the **intertemporal marginal rate of substitution** to discount the payoff
- Note that the generic realization of the random variable m_{t+1} reads

$$m_{t+1}(s) \equiv \frac{\beta u'(c_{t+1}(s))}{u'(c_t)} = \frac{\beta u'(e_{t+1}(s) + x_{t+1}(s)\,q)}{u'(e_t - p_t q)} \qquad (13)$$

- The pricing formula (12)-(13) can thus be expressed as

$$p_t = \sum_{s=1}^{S} \pi(s) \frac{\beta u'(e_{t+1}(s) + x_{t+1}(s)\,q)}{u'(e_t - p_t q)} x_{t+1}(s) \qquad (14)$$

- The formula states that an asset's price should be the *expected discounted payoff* of that asset

Asset return

- We frequently divide the payoff by the price to obtain a **gross return**

$$R_{t+1} \equiv \frac{x_{t+1}}{p_t} \tag{15}$$

  - capital letters to denote gross returns R, taking values like 1.05
  - lowercase letters *net returns* $r = R - 1$, taking values like 0.05
- The gross return is a payoff with price one
  - R is how many euro you get tomorrow for one euro paid today
  - dividing both sides of (12) by p_t and using (15), returns thus obey

$$1 = E\left[m_{t+1} R_{t+1}\right] \tag{16}$$

  - eq. (16) is the most important special case of the basic formula (12)
- Although expected returns can vary across time and assets, expected *discounted* returns should always be the same, 1

Example 1: investor's net asset demand

Setting one period ($t = 0, 1$), two states ($s = 1, 2$)
probabilities $\boldsymbol{\pi} = [0.24, 0.76]'$

Preferences isoelastic utility (1) with $\gamma = 0.9$
lifetime utility (3) with $\beta = 1$

Endowment current $e_0 = 100$
future $\mathbf{e}_1 = [100, 100]'$
$\longrightarrow$ no growth nor risk in the investor's earnings

Asset price $p(x) = 10$
payoffs $\mathbf{x} = [8, 14]'$
$\longrightarrow$ expected return $E[R] = \boldsymbol{\pi}' \cdot \mathbf{x} / p(x) = 1.256$

en.wikipedia.org/wiki/Dot_product

Example 1: investor's net asset demand

- We seek the investor's net demand, given the values

$$\begin{array}{ccc} \pi(1) = 0.24 & \beta = 1 & \gamma = 0.9 \\ p(x) = 10 & x(1) = 8 & x(2) = 14 \\ e = 100 & e(1) = 100 & e(2) = 100 \end{array}$$

- MATLAB code: `cons_based.m`

Consumption-based model results

q	c	$c(1)$	$c(2)$
1.1119	88.8807	108.8955	115.5671

Example 2: investor's reservation price

- Investor's net asset demand is positive
- The immediate reason is that the expected return is sufficiently large
- One way to check this is to compute the investor's **reservation price**

en.wikipedia.org/wiki/Reservation_price

- We modify the previous exercise
  - we set the investor's net demand to zero
  - we let the asset's price be our unknown

Example 2: investor's reservation price

- We seek the investor's reservation price, given the values

$$\begin{array}{ccc} \pi(1) = 0.24 & \beta = 1 & \gamma = 0.9 \\ q = 0 & x(1) = 8 & x(2) = 14 \\ e = 100 & e(1) = 100 & e(2) = 100 \end{array}$$

- MATLAB code: `cons_based_find_p0.m`

Consumption-based model results

$p(x)$	c	$c(1)$	$c(2)$
12.5600	100.0000	100.0000	100.0000

Example 3: state-1 probability

- Expected net return is zero!

$$E\left[R\right] = \frac{\boldsymbol{\pi}' \cdot \mathbf{x}}{p\left(x\right)} = 1$$

- This result has a quite intuitive explanation
  - the investor has a *constant* endowment
  - net saving needs incentive, which $E\left(R\right) = 1$ does not provide
- One way to confirm this is to compute the expected return
  - imposing that net demand is null when $p\left(x\right) = 10$
  - seeking some other figure characterizing our problem
- We modify the previous exercise
  - we set the asset's price back to its initial value
  - we let the state-1 probability be our unknown

Example 3: state-1 probability

- We seek the state-1 probability, given the values

$$
\begin{array}{ccc}
\beta = 1 & \gamma = 0.9 & q = 0 \\
p(x) = 10 & x(1) = 8 & x(2) = 14 \\
e = 100 & e(1) = 100 & e(2) = 100
\end{array}
$$

- MATLAB code: `cons_based_find_pi.m`

Consumption-based model results

$\pi(1)$	c	$c(1)$	$c(2)$
0.6667	100.0000	100.0000	100.0000

- Expected net return is once again zero!

$$E[R] = \frac{\boldsymbol{\pi}' \cdot \mathbf{x}}{p(x)} = 1$$

Example 4: endowment

- Another important concept springs from considering the net asset demand of an investor with a different endowment set

$$\begin{array}{ccc} \pi(1) = 0.24 & \beta = 1 & \gamma = 0.9 \\ p(x) = 10 & x(1) = 8 & x(2) = 14 \\ e = 77.6268 & e(1) = 100 & e(2) = 100 \end{array}$$

- MATLAB code: `cons_based.m`

Consumption-based model results

q	c	$c(1)$	$c(2)$
0.0000	77.6268	100.0000	100.0000

- With a 22.4% fall in the current endowment, demand falls to zero

Example 4: endowment

- Let i (j) denote the investor with the first (second) endowment set
  - *before* trading on the market, the two investors have SDF

$$
\begin{aligned}
\mathbf{m}^{i}_{t+1} &= \left[\left(\frac{100}{100}\right)^{-0.9}, \left(\frac{100}{100}\right)^{-0.9} \right]' = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
\mathbf{m}^{j}_{t+1} &= \left[\left(\frac{100}{77.6268}\right)^{-0.9}, \left(\frac{100}{77.6268}\right)^{-0.9} \right]' = \begin{bmatrix} 0.7962 \\ 0.7962 \end{bmatrix}
\end{aligned}
$$

  - *after* trading on the market, the two investors' SDFs become

$$
\begin{aligned}
\mathbf{m}^{i}_{t+1} &= \left[\left(\frac{108.8955}{88.8807}\right)^{-0.9}, \left(\frac{115.5671}{88.8807}\right)^{-0.9} \right]' = \begin{bmatrix} 0.8188 \\ 0.7921 \end{bmatrix} \\
\mathbf{m}^{j}_{t+1} &= \left[\left(\frac{100}{77.6268}\right)^{-0.9}, \left(\frac{100}{77.6268}\right)^{-0.9} \right]' = \begin{bmatrix} 0.7962 \\ 0.7962 \end{bmatrix}
\end{aligned}
$$

- The two investors' SDFs *tend to become equivalent*

Further examples: exercises!

Exercise What if the investor were more or less impatient?

Exercise What if the investor were more or less risk averse?

Exercise What if the asset's payoffs were different?

Exercise What if the endowment were different some other way?

Stochastic discount factor

- Example 4 shows that investors' SDFs "tend to become equivalent"
- Why they do not become *exactly* equivalent?
  - intuitively, the reason is that investors have only *one tool* (the asset) to deal with *two goals* (the SDF values in the two states)
  - more rigorously, the reason is that the financial market is *incomplete*

Definition (Complete market)

A financial market is said to be **complete** if an independent asset exists for every possible state of nature

en.wikipedia.org/wiki/Complete_market

- Later on, we'll give market completeness a proper formal definition

Risk-free rate

- Let's consider another financial asset: the risk-free return
- The return of a risk-free asset is know ex-ante, hence the pricing formula yields $1 = E\left[mR^f\right] = E\left[m\right]R^f$ and thus

$$R^f = 1/E\left[m\right] \tag{17}$$

- If a risk-free asset is not traded, we can define (17) as the **shadow** or **"zero-beta" risk-free return**
- Equation (17) is an additional first-order condition of the investor problem extended to take a risk-free asset into account

Two-state two-asset model

- To obtain (17), we may rewrite the investor problem as

$$\max_{q,q^f} \quad U = u(c_t) + \beta E\left[u(c_{t+1})\right]$$

$$\begin{aligned} sub \quad & c_t = e_t - p_t q - p_t^f q^f \\ & c_{t+1} = e_{t+1} + x_{t+1} q + q^f \\ & c_t, c_{t+1} > 0 \end{aligned}$$

- The risk-free payoff is $x_{t+1}^f = 1$
- p_t^f denotes the price of the risk-free return, such that $R_{t+1}^f \equiv 1/p_t^f$
- q^f is the investor's net demand for the risk-free asset

Two-state two-asset model

- As before, we may obtain an unconstrained optimization problem by replacing the budget constraints into lifetime utility

$$\max_{q,q^f} \; u\left(e_t - p_t q - p_t^f q^f\right) + \beta E\left[u\left(e_{t+1} + x_{t+1} q + q^f\right)\right]$$

- Equating the derivatives with respect to q and q^f to zero, we obtain

$$\begin{aligned} -u'\left(c_t\right) p_t + \beta E\left[u'\left(c_{t+1}\right) x_{t+1}\right] &= 0 \\ -u'\left(c_t\right)\left(p_t^f\right) + \beta E\left[u'\left(c_{t+1}\right)\right] &= 0 \end{aligned}$$

- Finally, rearranging and using (13) yields the two pricing formulas

$$\begin{aligned} p_t &= E\left[m_{t+1} x_{t+1}\right] \\ 1 &= E\left[m_{t+1} R^f\right] \end{aligned}$$

Two-state two-asset model

- Consider the same framework as in Example 1 augmented with a risk-free asset

$$\textbf{Asset} \quad \text{price } p\left(x^f\right) = 0.8$$

$$\text{payoffs } \mathbf{x}^f = [1, 1]'$$

$$\rightarrow \text{expected return } E\left[R^f\right] = \boldsymbol{\pi}' \cdot \mathbf{x}^f / p\left(x^f\right) = 1.25 = R^f$$

- We denote the net demand for the risk-free asset by q^f

Two-state two-asset model

- The budget constraints feature the additional net asset demand q^f

$$\begin{aligned} c_t &= e_t - q^f - p_t q \\ c_{t+1}(1) &= e_{t+1}(1) + R^f q^f + x_{t+1}(1) q \\ c_{t+1}(2) &= e_{t+1}(2) + R^f q^f + x_{t+1}(2) q \end{aligned}$$

- Using (17), the first-order condition for asset f reads

$$R^f = \frac{1}{E[m]} = \frac{1}{\sum_{s=1}^{S} \pi(s) \frac{\beta u'(e_{t+1}(s) + x_{t+1}(s) q)}{u'(e_t - p_t q)}}$$

Two-state two-asset model

- MATLAB code: `cons_based_2asset.m`

Consumption-based model results

q^f	q	c	$c(1)$	$c(2)$
		Investor i		
-0.1146	1.1232	88.8828	108.8422	115.5812
		Investor j		
-10.0440	0.9836	77.8351	95.3136	101.2150

Two-state two-asset model

- Recall that the investors' SDFs *before* trading on the market are

$$\mathbf{m}^i_{t+1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \mathbf{m}^j_{t+1} = \begin{bmatrix} 0.7962 \\ 0.7962 \end{bmatrix}$$

- *After* trading on the market, the SDFs now become

$$\begin{aligned} \mathbf{m}^i_{t+1} &= \left[\left(\frac{108.8422}{88.8828}\right)^{-0.9}, \left(\frac{115.5812}{88.8828}\right)^{-0.9}\right]' = \begin{bmatrix} 0.8333 \\ 0.7895 \end{bmatrix} \\ \mathbf{m}^j_{t+1} &= \left[\left(\frac{95.3136}{77.8351}\right)^{-0.9}, \left(\frac{101.2150}{77.8351}\right)^{-0.9}\right]' = \begin{bmatrix} 0.8333 \\ 0.7895 \end{bmatrix} \end{aligned}$$

- The two investors' SDFs are *exactly* equivalent!

More on the SDF

- How do modifications of β, c_t and $c_{t+1}(s)$ influence the SDF?

$$m_{t+1}(s) \equiv \frac{\beta u'(c_{t+1}(s))}{u'(c_t)}$$

- $\Delta\beta < 0$, $\Delta c_t < 0$, $\Delta c_{t+1} > 0 \quad \rightarrow \quad \Delta m_{t+1} < 0$, $\forall s$
  - a lower β means that the investor is more impatient
  - a lower c_t implies a larger current marginal utility
  - higher c_{t+1} implies a lower future marginal utility for all s
  - the investor values more current resources relative to future payments
  - weaker demand for assets, causing downward pressure on their prices
- $\Delta c_{t+1}(s) > 0 \quad \rightarrow \quad \Delta m_{t+1}(s) < 0$
  - the investor values less resources in the state of nature s
  - influence on asset prices heterogeneous, based on payoff structure

More on the risk-free return

- To think about the economics behind real interest rates in a simple setup, we use (1) and start by turning off uncertainty, in which case

$$R^f_{t+1} = \frac{1}{\beta}\left(\frac{c_{t+1}}{c_t}\right)^{\gamma}$$

- $\Delta\beta < 0$, $\Delta c_t < 0$, $\Delta c_{t+1} > 0$, $\Delta\gamma > 0 \quad \rightarrow \quad \Delta R^f_{t+1} > 0$
  - when investors are impatient, they want to consume now
  - when consumption growth is high, current resources are valued more
  - if utility is highly curved, investors seek smooth consumption paths
  - in all these instances, it takes a high interest rate to increase saving

More on the risk-free return

- To understand how interest rates behave under uncertainty, suppose that consumption growth is lognormally distributed; we obtain

$$r^f = \delta + \gamma E\left[\Delta \ln c_{t+1}\right] - \left(\gamma^2/2\right)\sigma^2\left(\Delta \ln c_{t+1}\right) \tag{18}$$

  - $r_t^f \equiv \ln R_t^f$ and $\delta \equiv -\ln \beta$ denote *net* rates
  - $\Delta \ln c_{t+1} \equiv \ln c_{t+1} - \ln c_t$, so Δ denotes the first difference operator
- Under consumption growth lognormality, we thus have

$$\begin{aligned} \ln\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} &= -\gamma\left(\ln c_{t+1} - \ln c_t\right) = -\gamma \Delta \ln c_{t+1} \\ &\equiv z \sim N\left(-\gamma E\left[\Delta \ln c_{t+1}\right], \gamma^2 \sigma_t^2\left(\Delta \ln c_{t+1}\right)\right) \end{aligned}$$

More on the risk-free return

- To derive expression (18) for the risk-free rate, start with

$$R^f = 1/E\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right]$$

- A normal random variable z has **moment generating function**

en.wikipedia.org/wiki/Moment-generating_function

$$E\left[e^z\right] \equiv e^{E[z]+(1/2)\sigma^2(z)}$$

  - using this expression with $z = \ln\left(c_{t+1}/c_t\right)^{-\gamma}$, we get

$$R^f = \left(\beta e^{-\gamma E[\Delta \ln c_{t+1}]+\left(\gamma^2/2\right)\sigma^2(\Delta \ln c_{t+1})}\right)^{-1}$$

  - applying logs and rearranging, we obtain (18)

More on the risk-free return

- Equation (18) delivers the same results as the deterministic case

$$r^f = \delta + \gamma E\left[\Delta \ln c_{t+1}\right] - \left(\gamma^2/2\right)\sigma^2\left(\Delta \ln c_{t+1}\right)$$

  - real interest rate is high if impatience/consumption growth is high
  - higher γ makes interest rates more sensitive to consumption growth
- The new term captures $\left(\gamma^2/2\right)\sigma^2\left(\Delta \ln c_{t+1}\right)$ precautionary savings
  - when consumption is volatile, low-consumption states worry investors more than high-consumption states please them
  - therefore, investors want to save more, driving down interest rates

Risk corrections

- Consider the definition of covariance, for m and x given by

  en.wikipedia.org/wiki/Covariance

$$cov\left(m, x\right) = E\left[mx\right] - E\left[m\right] E\left[x\right]$$

- We can write the basic formula (12) as

$$p = E\left[mx\right] = E\left[m\right] E\left[x\right] + cov\left(m, x\right) \tag{19}$$

Risk corrections

- Using the risk-free rate equation (17), we obtain

$$p = \frac{E\left[x\right]}{R^{f}} + cov\left(m, x\right) \tag{20}$$

- The first term is the **present value formula**, which gives asset prices in a risk-neutral world (constant consumption or linear utility)

en.wikipedia.org/wiki/Present_value

- The second is a *risk adjustment*, which entails payoffs covarying positively with the discount factor command higher asset prices

Risk corrections

- To understand the **risk adjustment**, replace m into (20) using (13)

$$p_t = \frac{E[x_{t+1}]}{R^f} + \frac{cov(\beta u'(c_{t+1}), x_{t+1})}{u'(c_t)} \tag{21}$$

($u'(c_t)$ is taken out of the covariance operator as it is non-random)
  - marginal utility declines as consumption rises
  - the asset price is lowered if its payoff covaries positively with c_{t+1}
- The reason is that investors *dislike* consumption uncertainty
  - if $cov(\beta u'(c_{t+1}), x_{t+1}) < 0$, the asset pays off well when the investor is wealthy, and badly when poor
  - the asset make the consumption stream more volatile, hence the investor requires a low price to be induced to buy it
  - insurance is an extreme example of the contrary: it pays off exactly when wealth and consumption would otherwise be low

Risk corrections

- We use returns so often that it is worth restating the same intuition for the special case that the price is one and the payoff is a return
- Start with the basic pricing equation for returns (16)

$$1 = E\left[m_{t+1} R_{t+1}\right]$$

- Applying the covariance decomposition, we get

$$1 = E\left[m_{t+1}\right] E\left[R_{t+1}\right] + cov\left(m_{t+1} R_{t+1}\right)$$

- Using (17) and (13), we obtain

$$E\left[R_{t+1}\right] - R^f = -R^f \frac{cov\left(\beta u'\left(c_{t+1}\right), R_{t+1}\right)}{u'\left(c_t\right)}$$

Risk corrections

- All expected returns equal the risk-free rate plus a risk adjustment
  - assets making consumption more volatile (positive covariance) must offer higher expected returns to induce investors to hold them
  - assets that covary negatively with consumption(e.g. insurance) can offer expected returns lower than the risk-free rate (even negative!)
- Much of finance focuses on expected returns
  - we offer intuition that *riskier* assets must offer higher expected returns to get investors to hold them
  - but we could equivalently say that *riskier* assets trade for lower prices so that investors will hold them
  - in fact, a low initial price for a payoff corresponds to a high expected return, so this is just different language for the same phenomenon

Idiosyncratic risk

- It is common to think that an asset with a high payoff (or return) variance is *risky* and thus should have a large risk correction
- But if the payoff is uncorrelated with the discount factor m, the asset receives no risk-correction to its price
- Hence, it pays an expected return equal to the risk-free rate!
- Formally, if $cov(m, x) = 0$, then for any $\sigma^2(x)$ (21) reduces to

$$p = \frac{E[x]}{R^f} \rightarrow E[R] \equiv \frac{E[x]}{p} = R^f$$

- This prediction holds even if the payoff x is highly volatile and investors are highly risk averse

Idiosyncratic risk

- The reason is simple: if you buy a little bit more of such an asset, it has no first-order effect on the variance of your consumption stream
- More generally, one gets *no compensation* for holding idiosyncratic risk: only **systematic risk** generates a risk correction

en.wikipedia.org/wiki/Systematic_risk

- To give meaning to these words, we can decompose any payoff x into
  - a part correlated with the SDF
  - an idiosyncratic part uncorrelated with the SDF

Idiosyncratic risk

- Formally, we can run the regression

$$x = proj\,(x|m) + \varepsilon$$

- Then, the price of the residual or idiosyncratic risk ε is zero, and the price of x is the same as the price of its projection on m
- The projection of x on m is the part of x perfectly correlated with m
- The idiosyncratic component ε is uncorrelated (**orthogonal** to) m

en.wikipedia.org/wiki/Orthogonality

- Thus only the systematic part of a payoff accounts for its price

Idiosyncratic risk

- Projection means linear regression without a constant

$$proj\left(x|m\right) = \frac{E\left[mx\right]}{E\left[m^2\right]}m$$

- We can verify that ε is orthogonal to m using the law of one price

$$\begin{aligned} p\left(proj\left(x|m\right)\right) &= p\left(\frac{E\left[mx\right]}{E\left[m^2\right]}m\right) = E\left[m^2\frac{E\left[mx\right]}{E\left[m^2\right]}\right] \\ &= E\left[m^2\right]\frac{E\left[mx\right]}{E\left[m^2\right]} = E\left[mx\right] = p\left(x\right) \end{aligned}$$

- The price of ε is zero, which means that $E\left[m\varepsilon\right] = 0$ (orthogonality)

$$E\left[m\varepsilon\right] = p\left(\varepsilon\right) = p\left(x\right) - p\left(proj\left(x|m\right)\right) = p\left(x\right) - p\left(x\right) = 0$$

Table of contents

Representative investor

- An important implication stems from the two-asset model above
- As after-trade SDF becomes equivalent across investors, we may refer to *any* investor to pin down the **pricing kernel** *in equilibrium*

en.wikipedia.org/wiki/Pricing_kernel

  - while SDFs are exactly equivalent only in *complete* markets
  - the same is approximately true also under *incomplete* markets
- The most convenient choice is to resort to the figure of the **representative agent** (in our case, the **representative investor**)

en.wikipedia.org/wiki/Representative_agent

Representative investor

- We define the representative investor as the hypothetical decision maker who holds the *entire wealth of the economy*
- We model the economy's wealth as a *stock*, so at each date t
  - it has a certain value p_t (the stock price)
  - it pays dividends y_t, which we'll identify with consumption ($y_t = c_t$)
  - the stock payoff is therefore $x_t = p_{t+1} + y_{t+1}$
- Using this setting, we can answer a rather ambitious question

Question What is the value of the economy of the whole country?

A macro-finance model

- To answer this question, we must modify the model in several ways

1. The time-horizon must be longer than one period
2. Budget constraints must obey the representative investor paradigm
3. Figures fed to the model must reflect real economy data

- We deal with each of these aspects in turn
- Our starting point is the two-asset one-period model

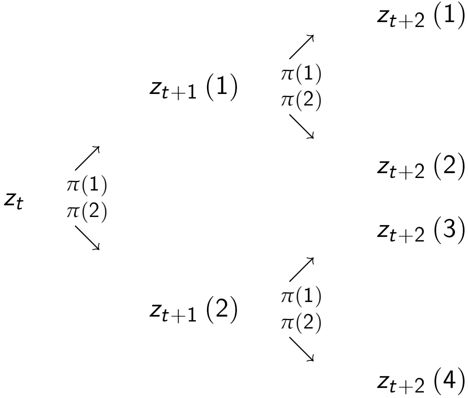
Multiperiod economy

- We extend the model to T periods
- We thus have $T+1$ dates ($t = 0, 1, 2..., T$)
- We still consider two states of nature ($s = 1, 2$) per period
- Notation remains unchanged
  - time stands as a variable's subscript (e.g. p_t)
  - the state is reported in parenthesis [e.g. $x_{t+1}(s)$]
- We represent the evolution of random variables with a **tree diagram**

en.wikipedia.org/wiki/Tree_diagram_(probability_theory)

Multiperiod economy

Tree diagram for a generic random variable



Multiperiod economy

- The tree diagram reveal a major computational issue
- At a generic date t, the investor faces
  - two scenarios at $t+1$ ($s_{t+1}=1$ or $s_{t+1}=2$ may occur)
  - two scenarios at $t+2$ $[(s_{t+1}, s_{t+2}) = (1,1), (1,2), (2,1), (2,2)]$
  - as time rolls on, the number of scenarios grows exponentially (specifically, we have 2^{τ} scenarios after τ periods)
- As we'll see, we settle this issue by considering stationary variables

en.wikipedia.org/wiki/Stationary_variable

Budget constraints

- If the stock stands for the economy's wealth, then it's its endowment
- Let q_{t-1} and q^f_{t-1} be the quantity of shares and bonds held by the representative investor during period t, that is
  - from date $t-1$, when the investor decides how many assets to hold
  - to date t, when the portfolio choice for period $t+1$ is made
- At time t, the investor
  - *inherits* q_{t-1} shares, each paying off $p_t + y_t$, and *chooses* q_t shares to hold for period $t+1$, each priced p_t
  - *inherits* q^f_{t-1} bond, each paying off 1, and *chooses* q^f_t bonds to hold for period $t+1$, each priced p^f_t
- The budget constraint will therefore be

$$c_t = (p_t + y_t)\, q_{t-1} - p_t q_t + q^f_{t-1} - p^f_t q^f_t \qquad (22)$$

Equilibrium conditions

- The representative investor solves

$$
\begin{array}{ll}
\max\limits_{\left\{q_t, q_t^f\right\}_{t\in[0,T]}} & U_0 = \sum_{t=0}^{T} \beta^t E\left[u\left(c_t\right)\right] \\
sub & c_t = \left(p_t + y_t\right) q_{t-1} - p_t q_t + q_{t-1}^f - p_t^f q_t^f, \\
 & c_t > 0, \text{ for all } t \in [0, T]
\end{array}
$$

- The solution yields same conditions found for the two-asset model

$$
\begin{aligned}
p_t &= E\left[m_{t+1}\left(p_{t+1} + y_{t+1}\right)\right] \\
1 &= E\left[m_{t+1} R_{t+1}^f\right]
\end{aligned}
$$

Exercise Solve the representative investor problem

Hints Given (22), q_t and q_t^f enter two successive addends of U_0
Use a single generic date t and again (13) to define m_{t+1}

Resolving the computational issue

- With isoelastic utility, the basic pricing formula reads

$$p_t = E\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}(p_{t+1}+y_{t+1})\right] \tag{23}$$

  - The dynamics of both p_t and y_t may be non-stationary, possibly leading to the computational issue discussed above
  - We avoid the issue by defining the **price-dividend ratio** $\omega_t \equiv p_t/y_t$

en.wikipedia.org/wiki/Dividend_yield

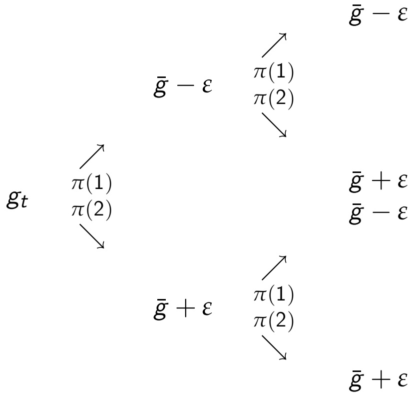
- Divide both sides of (23) by y_t, multiply and divide by y_{t+1} into $E[\cdot]$

$$\omega_t = E\left[\beta\left(g_{t+1}^c\right)^{-\gamma} g_{t+1}^y\left(1+\omega_{t+1}\right)\right] \tag{24}$$

- In the data, ω_t, $g_{t+1}^c \equiv c_{t+1}/c_t$ and $g_{t+1}^y \equiv y_{t+1}/y_t$ are stationary

Multiperiod economy

Tree diagram for a stationary variable



- We only need to keep track of $\bar{g}$ and ε

More on budget constraints

- By definition, the representative investor holds the economy's wealth
  - the investor's portfolio must thus constantly hold the whole stock
$$q_t = q, \text{ for all } t \in [0, T]$$
  - and, likewise, we portfolio must always contain all the bonds
$$q_t^f = q^f, \text{ for all } t \in [0, T]$$
- For simplicity, we impose:
  - $q = 1$ (y_t represents the totality of dividends distributed at each t)
  - $q^f = 0$ (all bonds held by one investor are issued by another)
- The budget constraint (22) at each date t then becomes
$$c_t = (p_t + y_t) - p_t = y_t \tag{25}$$
- Quite sensibly, the economy's yields c_t equal its wealth dividends y_t

Data

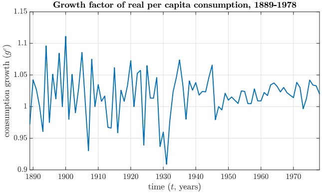
- One off-the-shelf source of data is Mehra-Prescott (1985) dataset

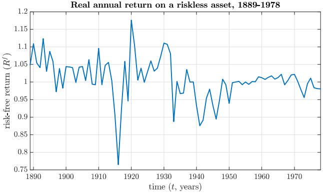
  academicwebpages.com/preview/mehra/pdf/equitydata.pdf

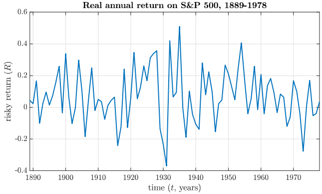
- Download the file and convert it to spreadsheet (**equitydata.xlsx**)
- We represent graphically and compute descriptive statistics for
  - the consumption growth factor, g^c
  - the risk-free return, R^f
  - the risky return, R
  - the price-dividend ratio, ω

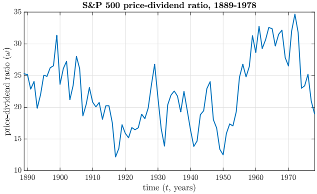
Data

- MATLAB code: `equitydata.m`









Data

- MATLAB code: `equitydata.m`

Mehra-Prescott (1985) descriptive statistics

	g^c	R^f	R	ω
mean	1.0183	1.0077	1.0698	22.4339
st.dev.	0.0359	0.0592	0.1654	5.4662
autocorr.	-0.1485	0.4507	0.1090	0.8267

- The data show a great deal of **autocorrelation**

en.wikipedia.org/wiki/Autocorrelation

Markov chain

- A convenient tool to handle persistent variables is the **Markov chain**

  en.wikipedia.org/wiki/Markov_chain

- This stochastic model parsimoniously reflects the time-dependence of the values that a variable can assume
- Formally, a Markov chain for a random variable g is

$$\Pr(g_{t+1} = g(j) \mid g_t = g(i), g_{t-1} = g(k), g_{t-2} = g(h), \ldots) = \Pr(g_{t+1} = g(j) \mid g_t = g(i))$$

  - the probability of a state at a given date $t+1$ depends only on the state attained in the previous date t
  - the information needed to make predictions thus depends only on the realized current state

Markov chain

- As suggested in the tree diagram above, we may suppose that our (stationary) random variable follows a two-state Markov process

$$g_t = \bar{g} + \varepsilon_t, \quad \text{con} \quad \varepsilon_t = \begin{cases} \varepsilon & \text{for} \quad s_t = 1, \\ -\varepsilon & \text{for} \quad s_t = 2, \end{cases} \quad \text{where} \quad \varepsilon > 0$$

  - $\bar{g}$ represents the mean of the stochastic process
  - ε the (symmetric) deviation of the process from its mean
  - (g may refer to g^c and g^y as well as ω)
  - [given (25), henceforth we let $g^c = g^y = g$]

- The **transition probability** from a state of nature to another is

$$\pi_{ij} = \Pr(\varepsilon_{\tau+1} = \varepsilon_j \mid \varepsilon_\tau = \varepsilon_i)$$

Conditional probability

- We may collect these figures in the **transition probability matrix**

  en.wikipedia.org/wiki/Transition_probability_matrix

$$\mathbf{\Pi} = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix}$$

- Each row of Π represents a **conditional probability** distribution, hence the elements of every row must add up to one

  en.wikipedia.org/wiki/Conditional_probability

$$\begin{aligned} \pi_{11} + \pi_{12} &= 1 \\ \pi_{21} + \pi_{22} &= 1 \end{aligned}$$

Unconditional probability

- We can also define the **stationary** (or **unconditional**) probability vector

en.wikipedia.org/wiki/Marginal_probability

$$\boldsymbol{\varpi} = \begin{bmatrix} \varpi_1 & \varpi_2 \end{bmatrix}$$

- This probability can be interpreted as the expectations that a particular state of nature will occur in a distant future
- The vector is *product-invariant* to the conditional probabilities, i.e.

$$\boldsymbol{\varpi} = \boldsymbol{\varpi}\boldsymbol{\Pi} \tag{26}$$

- For this reason, $\boldsymbol{\varpi}$ may be calculated as $\boldsymbol{\Pi}^n$, with n sufficiently large

Computation of unconditional probability

- We may expand (26) to get the two conditions

$$\begin{bmatrix} \varpi_1 & p_2 \end{bmatrix} = \begin{bmatrix} \varpi_1 & \varpi_2 \end{bmatrix} \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} = \begin{bmatrix} \pi_{11}\varpi_1 + \pi_{21}\varpi_2 & \pi_{12}\varpi_1 + \pi_{22}\varpi_2 \end{bmatrix}$$

- Imposing $\varpi_2 = 1 - \varpi_1$, the first condition yields

$$\varpi_1 = (1 - \pi_{12})\,\varpi_1 + \pi_{21}\,(1 - \varpi_1) \;\rightarrow\; \varpi_1 = \frac{\pi_{21}}{\pi_{12} + \pi_{21}}$$

- From the second, we obtain

$$\varpi_2 = \pi_{12}\,(1 - \varpi_2) + (1 - \pi_{21})\,\varpi_2 \;\rightarrow\; \varpi_2 = \frac{\pi_{12}}{\pi_{12} + \pi_{21}}$$

Probability and persistence

- The autocorrelation coefficient is a function of transition probabilities (hence persistence is reflected by the probability distribution)
- To see this, take the unconditional expectation of the coefficient

$$\rho\left(g_t, g_{t+1}\right) = E\left[\rho_t\left(g_t, g_{t+1}\right)\right] = E\left[\frac{E_t\left[\left(g_t - \bar{g}\right)\left(g_{t+1} - \bar{g}\right)\right]}{\sigma_t \sigma_{t+1}}\right]$$

  - expand the unconditional expectation, and impose $\sigma_t = \sigma_{t+1} = \varepsilon$

$$\rho\left(g_t, g_{t+1}\right) = \frac{\varpi_1\left[g(1) - \bar{g}\right] E_1\left[\left(g_{t+1} - \bar{g}\right)\right] + \varpi_2\left[g(2) - \bar{g}\right] E_2\left[\left(g_{t+1} - \bar{g}\right)\right]}{\varepsilon \cdot \varepsilon}$$

  - expand the unconditional expectation, and impose $g\left(s\right) - \bar{g} = \varepsilon$

$$\rho\left(g_t, g_{t+1}\right) = \frac{-\varpi_1 \varepsilon\left[\pi_{11}(-\varepsilon) + \pi_{12}\varepsilon\right]}{\varepsilon^2} + \frac{\varpi_2 \varepsilon\left[\pi_{21}(-\varepsilon) + \pi_{22}\varepsilon\right]}{\varepsilon^2}$$

- Then, simplify and rearrange to obtain

$$\rho\left(g_t, g_{t+1}\right) = 2\varpi_1 \pi_{11} + 2\varpi_2 \pi_{22} - 1$$

A symmetric setting

- Hereafter, the conditional probability matrix $\mathbf{\Pi}$ will be symmetric

$$\mathbf{\Pi} = \begin{bmatrix} \pi & 1-\pi \\ 1-\pi & \pi \end{bmatrix}$$

- Therefore, the unconditional probability distribution will be

$$\boldsymbol{\varpi} = \begin{bmatrix} \frac{1-\pi}{1-\pi+1-\pi} & \frac{1-\pi}{1-\pi+1-\pi} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

- As a result, the autocorrelation coefficient will read

$$\rho\left(g_t, g_{t+1}\right) = 2\frac{1}{2}\pi + 2\frac{1}{2}\pi - 1 = 2\pi - 1$$

Macro-finance model assessment

- We are now ready to produce and validate predictions regarding the behavior of observed financial assets
- The strategy is simple
  - we feed the model the observed moments of consumption growth
  - we deliver predictions regarding the risky and risk-free assets
  - we compare these predictions with the relevant figures in the data
- Specifically, using consumption growth moment we calibrate
  - the parameters $\bar{g}$ (using the mean) and ε (standard deviation)
  - the probability π (using the autocorrelation coefficient)

Calibration of consumption growth and probability

- Consumption growth: $\bar{g} = 1.0183$ and $\varepsilon = 0.0359$ yield

$$\begin{aligned} g(1) &= \bar{g} + \varepsilon = 1.0542 \\ g(2) &= \bar{g} - \varepsilon = 0.9824 \end{aligned}$$

- Probability: $\rho = -0.1485$ yields

$$\pi = \frac{1+\rho}{2} = 0.4258$$

- The transitional probability matrix is therefore

$$\mathbf{\Pi} = \begin{bmatrix} 0.4257 & 0.5743 \\ 0.5743 & 0.4257 \end{bmatrix}$$

Calibration of the SDF and risk-free return

- Given $g(1)$ and $g(2)$, the SDF can take the two values

$$m(s) = \beta [g(s)]^{-\gamma} = \begin{cases} \beta (1.0542)^{-\gamma} & \text{if } s = 1 \\ \beta (0.9824)^{-\gamma} & \text{if } s = 2 \end{cases}$$

- Given Π, its conditional expected values can also take two values

$$E_t[m_{t+1}] = \begin{cases} \beta \left[\frac{0.4257}{(1.0542)^{\gamma}} + \frac{0.5743}{(0.9824)^{\gamma}} \right] & \text{if } s_t = 1 \\ \beta \left[\frac{0.5743}{(1.0542)^{\gamma}} + \frac{0.4257}{(0.9824)^{\gamma}} \right] & \text{if } s_t = 2 \end{cases}$$

- By virtue of (17), the risk-free return also takes two values, given by the reciprocal of $E_t[m_{t+1}]$

Calibration of the price-dividend ratio

- Since $g^c = g^y = g$, we can rewrite (24) as

$$\omega_t = E_t\left[\beta\left(g_{t+1}\right)^{1-\gamma}\left(1+\omega_{t+1}\right)\right]$$

- As ω is a stationary variable, we have $\omega_t\left(s\right) = \omega_{t+1}\left(s\right) = \omega\left(s\right)$
- As a result, the price-dividend ratio also takes two values

$$\begin{aligned}
\omega\left(1\right) &= \beta\left[\frac{(0.4257)[1+\omega(1)]}{(1.0542)^{\gamma-1}} + \frac{(0.5743)[1+\omega(2)]}{(0.9824)^{\gamma-1}}\right] \quad \text{if } s = 1 \\
\omega\left(2\right) &= \beta\left[\frac{(0.5743)[1+\omega(1)]}{(1.0542)^{\gamma-1}} + \frac{(0.4257)[1+\omega(2)]}{(0.9824)^{\gamma-1}}\right] \quad \text{if } s = 2
\end{aligned}$$

  - in principle, we could works out the exact values for the two unknowns $\omega\left(1\right)$ and $\omega\left(2\right)$ from these two linear equations
  - however, the algebra is tedious and not particularly informative, so we'll solve the system with a MATLAB code instead

Calibration of the risky return

- Once we obtain the conditional values of the price-dividend ratio, we can work out those of the risky return
- The risky return relates to the price-dividend ratio by the expression

$$R_{t+1} = \frac{p_{t+1} + y_{t+1}}{p_t} = \frac{y_t}{p_t}\left(\frac{p_{t+1}}{y_{t+1}}\frac{y_{t+1}}{y_t} + \frac{y_{t+1}}{y_t}\right) = \frac{1+\omega_{t+1}}{\omega_t} g_{t+1}$$

- We thus have four conditional values for the risky return

$$R(i,j) = \frac{1+\omega(j)}{\omega(i)} g(j), \text{ for } i = s_t,\ j = s_{t+1},\ s_t, s_{t+1} = 1,2$$

- The two conditional expected risk return values are then obtained by the **dot product** of the s-th rows of $\boldsymbol{\Pi}$ and $\mathbf{R} \equiv [R(i,j)]$

en.wikipedia.org/wiki/Dot_product

Calibration of the preference parameters

- The numerical simulation of the model requires values for β and γ
- Several strategies exist to address this task (estimation, calibration)
- The easiest and most typically adopted strategy is to use *introspection* and/or reference to existing studies in the literature

en.wikipedia.org/wiki/Introspection

- Two sets of values find general consensus among economists
  - the subjective discount factor should be in the range $\beta \in (0.95, 1)$
  - the relative risk-aversion coefficient in the range $\gamma \in (1, 10)$

Exercise Modify the MATLAB code `cons_based_2asset.m` to simulate the Mehra-Prescott model
Use the moments calculated by the MATLAB code `equitydata.m` and pick a pair of values for the preference parameters from the intervals reported above

Table of contents

Simulation of the Mehra-Prescott model

- Preference parameters: $\beta = 0.97$ and $\gamma = 5$
- MATLAB code: `mehra_prescott_1985.m`

Mehra-Prescott (1985) model results
with $\beta = 0.97$ and $\gamma = 5$

	simul	actual		simul	actual
R^f	1.1086	1.0077	$std\left(R^f\right)$	0.0287	0.0592
$E[R] - R^f$	0.0099	0.0698	$std(R)$	0.0004	0.1654
ω	10.2814	22.4339	$std(\omega)$	0.1888	0.8267

- The simulated values are off by a factor of 2 to 14!
  [excluding the factor of 413 for $std(R)$...]

Simulation of the Mehra-Prescott model

- Are investors less impatient? Set $\beta = 0.99$
- MATLAB code: `mehra_prescott_1985.m`

Mehra-Prescott (1985) model results
with $\beta = 0.99$ and $\gamma = 5$

	simul	actual		simul	actual
R^f	1.0862	1.0077	$std\left(R^f\right)$	0.0281	0.0592
$E[R] - R^f$	0.0098	0.0698	$std(R)$	0.0004	0.1654
ω	13.3082	22.4339	$std(\omega)$	0.2437	0.8267

- Marginal improvement in simulating R^f, ω and $std(\omega)$

Simulation of the Mehra-Prescott model

- Are investors more risk-averse? Set $\gamma = 10$
- MATLAB code: `mehra_prescott_1985.m`

Mehra-Prescott (1985) model results
with $\beta = 0.99$ and $\gamma = 10$

	simul	actual		simul	actual
R^f	1.1349	1.0077	$std\left(R^f\right)$	0.0571	0.0592
$E\left[R\right] - R^f$	0.0274	0.0698	$std\left(R\right)$	0.0025	0.1654
ω	7.2665	22.4339	$std\left(\omega\right)$	0.2959	0.8267

- Some improvement in simulating $std\left(R^f\right)$, $E\left[R\right] - R^f$, $std\left(R\right)$ and $std\left(\omega\right)$, but further marked deterioration in simulating R^f and ω

Simulation of the Mehra-Prescott model

- No combination of values for β and γ reconciles the model with data
- Matching $E[R] - R^f$ requires huge levels of risk aversion ($\gamma = 19$), but the other figures are far off target
- Matching ω requires reasonable levels of risk aversion ($\gamma = 3$), but the other figures are again far off target
- R^f, $std(R)$, and $std(\omega)$ cannot possibly be matched
- Is there some way to improve these dreadful outcomes?

Alternative consumption-based models

Exercise Solve the consumption-based model with the following alternative *pseudo** SDF specifications

[* the SDF specifications have no scientific foundations and are proposed exclusively for illustrative purpose]

Stone-Geary	$m(g) = \beta (g-a)^{-\gamma}$, $a > 0$
Quadratic	$m(g) = \beta (a - b\,g)$, $b > 0$, $a > b\,g$
Negative exponential	$m(g) = \beta\, a\, e^{-a\,g}$, $a > 0$
HARA	$m(g) = \beta\, a \left(\frac{a\,g}{1-\gamma} + b\right)^{\gamma-1}$, $a > 0$, $b > -a\,g/(1-\gamma)$
Habit formation	$m(g_t g_{t-1}) = \beta (g_t g_{t-1})^{-\gamma}$

Alternative consumption-based models

- The two research lines are currently considered most promising at improving the consumption-based models
  - **rare disasters** (Rietz, 1988; Barro, 2006)
    [sciencedirect.com/science/article/pii/0304393288901729, academic.oup.com/qje/article/121/3/823/1917876]
  - **lung-run risks** (Bansal-Yaron, 2004; Bansal-Kiku-Yaron, 2012)
    [onlinelibrary.wiley.com/doi/full/10.1111/j.1540-6261.2004.00670.x, cfr.pub/published/cfr-005.pdf]
  - the two lines are also considered jointly (Barro-Jin, 2021)
    [sciencedirect.com/science/article/pii/S1094202520300740]
- In its basic representation, rare disasters are easily introduced in the basic model, so we'll explore the approach in detail
- Long-run risks models are more challenging
  - thus we'll only take some steps towards implementing them
  - specifically, we'll look into **Epstein-Zin** (**recursive**) **preferences**

en.wikipedia.org/wiki/Epstein-Zin_preferences

Rare disasters

- As the title suggests, the rare disasters approach considers an *additional* state of nature
  - features a *depression-like* consumption growth factor
  - occurs with extremely low probability
- We thus have three possible scenarios for consumption growth

$$g_t = g + \varepsilon_t, \text{ with } \varepsilon_t = \begin{cases} \varepsilon & \text{for } s = 1, \\ -\varepsilon & \text{for } s = 2, \\ k - g & \text{for } s = 3, \end{cases} \Bigg| \text{ where } \varepsilon > 0 \text{ and } k < 1$$

- The relevant transitional probability matrix is, with $0 < \delta < 1$

$$\Pi = \begin{bmatrix} \pi & 1-\pi-\delta & \delta \\ 1-\pi-\delta & \pi & \delta \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

Crash state

- The state of nature $s = 3$ is the *crash state*, which occurs with a very small probability δ
- In this state, the long-term growth rate g is reduced by the amount $k - g$ (becomes k)
- The crash state is not persistent: the economy transitions from the current $s = 3$ to the future $s = 1$ or $s = 2$ with equal probability
- The calibration of the crash state is based on global historical events of the XX century (e.g. the World Wars and the Great Depression)
- Specifically, the calibrated parameters are

$$k = 0.5 \quad \text{and} \quad \delta = 0.01$$

Simulation of the Rietz model

- MATLAB code: `rietz_1988.m`

Rietz (1988) model results
with $\beta = 0.97$, $\gamma = 3$, $k = 0.5$ and $\delta = 0.01$

	simul	actual		simul	actual
R^f	1.0070	1.0077	$std\left(R^f\right)$	0.0155	0.0592
$E[R] - R^f$	0.0415	0.0698	$std(R)$	0.0039	0.1654
ω	29.0990	22.4339	$std(\omega)$	0.2569	0.8267

- The model matches R^f and improves in simulating $E[R] - R^f$

Simulation of the Rietz model

- Are investors more risk-averse? Set $\gamma = 3.75$
- MATLAB code: `rietz_1988.m`

Rietz (1988) model results
with $\beta = 0.97$, $\gamma = 3.75$, $k = 0.5$ and $\delta = 0.01$

	simul	actual		simul	actual
R^f	0.9651	1.0077	$std\left(R^f\right)$	0.0199	0.0592
$E[R] - R^f$	0.0692	0.0698	$std(R)$	0.0068	0.1654
ω	49.9430	22.4339	$std(\omega)$	0.6184	0.8267

- We match $E[R] - R^f$ and record a marked improvement in explaining $std(\omega)$, but the simulated R^f and ω deteriorate

Simulation of the Rietz model

- Are investors more impatient? Set $\beta = 0.93$
- MATLAB code: `rietz_1988.m`

Rietz (1988) model results
with $\beta = 0.93$, $\gamma = 3.75$, $k = 0.5$ and $\delta = 0.01$

	simul	actual		simul	actual
R^f	1.0066	1.0077	$std\left(R^f\right)$	0.0208	0.0592
$E[R] - R^f$	0.0720	0.0698	$std(R)$	0.0071	0.1654
ω	15.6902	22.4339	$std(\omega)$	0.1951	0.8267

- With β out of range, we get close to R^f and $E[R] - R^f$, but record a marked deterioration in explaining the ω moments

Simulation of the Rietz model

- Revert back to the initial parameterization and set $k = 0.43$
- MATLAB code: `rietz_1988.m`

Rietz (1988) model results
with $\beta = 0.97$, $\gamma = 3$, $k = 0.43$ and $\delta = 0.01$

	simul	actual		simul	actual
R^f	0.9644	1.0077	$std\left(R^f\right)$	0.0170	0.0592
$E[R] - R^f$	0.0700	0.0698	$std(R)$	0.0069	0.1654
ω	47.5193	22.4339	$std(\omega)$	0.4379	0.8267

- Exacerbating the crash state severity yields remarkably similar results as having more risk-averse investors

Simulation of the Rietz model

- What if the crash state is less likely? Set $\delta = 0.006$
- MATLAB code: `rietz_1988.m`

Rietz (1988) model results
with $\beta = 0.97$, $\gamma = 3$, $k = 0.43$ and $\delta = 0.006$

	simul	actual		simul	actual
R^f	1.0074	1.0077	$std\left(R^f\right)$	0.0151	0.0592
$E[R] - R^f$	0.0463	0.0698	$std(R)$	0.0036	0.1654
ω	26.4025	22.4339	$std(\omega)$	0.2330	0.8267

- The result are similar to those obtained initially: likelihood and severeness of the crash state have similar consequences

Recursive utility

- With time-additive preferences, the **elasticity of intertemporal substitution** is the *reciprocal* of the relative risk aversion coefficient

  en.wikipedia.org/wiki/Elasticity_of_intertemporal_substitution

- This arbitrary restriction implies that investors' desires to smooth consumption across *dates* and across *states* are related
- The asset pricing literature identifies in this restriction one of the fundamental causes consumption-based models' poor performance
  - we need a high γ to explain the mean return on risky assets
  - but then this implies a very low elasticity of intertemporal substitution, which generates a very high risk-free return
- *Recursive utility* disentangles the two measures

Recursive utility

- Recursive preferences are represented by two nested functions

$$U_t = \Psi\left(c_t, \mu\left(U_{t+1}\right)\right)$$

- The *certainty equivalent operator* $\mu\left(U_{t+1}\right)$ translates random future utility into consumption units

$$\begin{aligned} \mu\left(U_{t+1}\right) &= \left(E_t\left[U_{t+1}^{1-\gamma}\right]\right)^{1/(1-\gamma)} \\ \gamma &= \text{relative risk aversion coefficient} \end{aligned}$$

- The *time aggregator* $\Psi\left(\cdot\right)$ combines current consumption and certainty equivalent into a measure of current utility

$$\begin{aligned} \Psi\left(c_t, \mu\left(U_{t+1}\right)\right) &= \left\{(1-\beta)\, c_t^{\rho} + \beta\left[\mu\left(U_{t+1}\right)\right]^{\rho}\right\}^{1/\rho} \\ 1/\left(1-\rho\right) &= \text{elasticity of intertemporal substitution} \end{aligned}$$

SDF under recursive utility

- With Epstein-Zin preferences we can write the Euler equation as

$$\frac{\partial U_t}{\partial c_t} = \frac{\partial U_t}{\partial U_{t+1}} E_t \left[\frac{\partial U_{t+1}}{\partial c_{t+1}} R_{t+1} \right]$$

  - giving up a unit of consumption today costs $\partial U_t / \partial c_t$ utility
  - this must equal the expected utility value of the future payoff, $E_t [\cdot]$
  - expressed in units of current utility, $\partial U_t / \partial U_{t+1}$

- After some tedious algebra, we can express the SDF as

$$m_{t+1} = \beta \left(\frac{c_{t+1}}{c_t} \right)^{\rho-1} \left[\frac{U_{t+1}}{\mu \left(U_{t+1} \right)} \right]^{1-\rho-\gamma}$$

- The last (additional) term is a function of continuation utility U_{t+1}, whose value must be computed iteratively (using a MATLAB code)

Recursive utility: standard vs long-run risks

- There are two approaches to relate $U_{t+1}/\mu(U_{t+1})$ to the future stream of consumption growth factors
  - Epstein-Zin: mean and variance are assumed time-invariant
  - Bansal-Yaron: consumption growth moments may vary over time
- To ease the illustration, here we adopt the first approach
- The consumption growth process is as in the Mehra-Prescott model
- We follow the computational method proposed by Weil (1989) [sciencedirect.com/science/article/abs/pii/0304393289900287]

Simulation of the Weil model

- MATLAB code: `ezw_1989.m`

Weil (1989) model results
with $\beta = 0.97$, $\gamma = 5$ and $1/\left(1-\rho\right) = 0.2$

	simul	actual		simul	actual
R^f	1.1086	1.0077	$std\left(R^f\right)$	0.0287	0.0592
$E\left[R\right] - R^f$	0.0099	0.0698	$std\left(R\right)$	0.0004	0.1654
ω	10.2814	22.4339	$std\left(\omega\right)$	0.1888	0.8267

- When $\gamma = 1 - \rho$, results are identical to Mehra-Prescott's
- However, here we can adjust γ independently of ρ

Simulation of the Weil model

- Are investors less impatient and more risk-averse? Set $\beta = 0.99$ and $\gamma = 10$
- MATLAB code: `ezw_1989.m`

Weil (1989) model results
with $\beta = 0.99$, $\gamma = 10$ and $1/(1-\rho) = 0.2$

	simul	actual		simul	actual
R^f	1.0681	1.0077	$std\left(R^f\right)$	0.0268	0.0592
$E[R] - R^f$	0.0177	0.0698	$std(R)$	0.0012	0.1654
ω	15.3255	22.4339	$std(\omega)$	0.2755	0.8267

- Relative to Mehra-Prescott's, the variations are far less marked
- We may now fine-tune the value of ρ separately

Simulation of the Weil model

- How well can the model perform? Set $\beta = 0.97$, $\gamma = 40$, $\rho = -9$
- MATLAB code: `ezw_1989.m`

Weil (1989) model results
with $\beta = 0.99$, $\gamma = 40$ and $1/(1-\rho) = 0.1$

	simul	actual		simul	actual
R^f	1.0064	1.0077	$std\left(R^f\right)$	0.0320	0.0592
$E[R] - R^f$	0.0581	0.0698	$std(R)$	0.0091	0.1654
ω	23.1267	22.4339	$std(\omega)$	0.6765	0.8267

- The model does reasonably well, yet still requires huge risk aversion
- Still, the approach is still considered the asset pricing workhorse model for the added flexibility it provides to researchers

Table of contents

Complete markets

- We now deepen our understanding of the basic formula

$$p = E[mx]$$

  - we use a simple market structure for **contingent claims**
  - the market structure's representation is the **state-space diagram**
  - this leads to an interpretation of the basic formula as a **dot product**
- We show that the stochastic discount factor *exists*, is *positive*, and that the function that determines m is linear
  - we simply consider prices and payoffs in a complete market
  - we have *no* need to introduce any utility function

Preliminary definitions

Definition (Payoff space)

The payoff space $\mathcal{X}$ is the set of *every* payoff that investors may trade

Definition (Free portfolio formation)

If two payoffs belong to $\mathcal{X}$, then so does every combination of the payoffs

$$x^i, x^j \in \mathcal{X} \;\Rightarrow\; ax^i + bx^j \in \mathcal{X}, \forall a, b \in \mathcal{R}$$

Definition (Law of one price)

The price of an assets' combination equals the asset prices' combination

$$p\left(ax^i + bx^j\right) = ap\left(x^i\right) + bp\left(x^j\right), \forall a, b \in \mathcal{R}$$

Description of the model

Definition (Contingent claim)

The state-$\bar{s}$ contingent claim is a security that pays one unit of resources in the future if and only if state $s = \bar{s}$ occurs

- A contingent claim is also referred to as a state-price security, an Arrow–Debreu security, a pure security, or a primitive security

en.wikipedia.org/wiki/State_prices

- Suppose that S states of nature may occur, indexed by $s = 1, 2, ..., S$
- We denote by $x^{\bar{s}}(s)$ the payoff of the state-$\bar{s}$ contingent claim in the generic state of nature s
  - $x^{\bar{s}}(\bar{s}) = 1$ in the only state, i.e. for $s = \bar{s}$
  - $x^{\bar{s}}(s) = 0$ in all other states, i.e. for $s \neq \bar{s}$
- We denote by $p^{\bar{s}} \equiv p\left(x^{\bar{s}}\right)$ the price of the state-$\bar{s}$ contingent claim

Contingent claims and stochastic discount factor

- In complete markets investors face contingent claims for each state s
  - they do not necessarily have to trade *explicit* contingent claims
  - it suffices that other payoff exist to synthesize x^s for all s
- In terms of SDF identification, we can prove the following claim

Claim If there exist a contingent claim for each state s, a stochastic discount factor *exists* and equals, for every s, the state-s contingent claim price over probability

$$m(s) = \frac{p^s}{\pi(s)} \tag{27}$$

Contingent claims and a generic asset

- Using **free portfolio formation**, we can think of a generic payoff x as a *portfolio* of contingent claims

$$\begin{aligned}\mathbf{x} &= \sum_{s=1}^{S} x(s) \cdot \mathbf{x}^s = x(1)\,\mathbf{x}^1 + x(2)\mathbf{x}^2 + ... + x(S)\,\mathbf{x}^S \\ &= x(1)\begin{bmatrix}1\\0\\...\\0\end{bmatrix} + x(2)\begin{bmatrix}0\\1\\...\\0\end{bmatrix} + \; ... \; + x(S)\begin{bmatrix}0\\0\\...\\1\end{bmatrix} = \begin{bmatrix}x(1)\\x(2)\\...\\x(S)\end{bmatrix}\end{aligned}$$

  - since $x^s(s) = 1$, holding $x(s)$ units of state-s contingent claim yields the asset's payoff in state of nature s

$$x(s)\,x^s(s) = x(s) \cdot 1 = x(s)$$

  - since $x^s(s') = 0$ for any $s' \neq s$, holding state-s contingent claims does not affect the asset's payoff in any state s'

$$x(s)\,x^s\left(s'\right) = x(s) \cdot 0 = 0$$

Contingent claims and a generic asset

- Using the **law of one price**, the asset's price must equal the value of the contingent claims held in the portfolio

$$\begin{aligned} p(x) &= x(1)\,p\left(x^1\right) + x(2)p\left(x^2\right) + ... + x(S)\,p\left(x^S\right) \\ &= \sum_{s=1}^{S} p^s x(s) \end{aligned} \tag{28}$$

- We may think of equation (28) as a *happy-meal theorem*

  A happy meal price (in a frictionless market) should equal the price of one hamburger, one small fries, one small drink and the toy

Contingent claims and basic formula

- It is easier to take expectations rather than sum over states
  - multiply and divide (28) by the relevant probabilities $\pi(s)$

$$p = \sum_{s=1}^{S} \pi(s) \left[\frac{p^s}{\pi(s)} \right] x(s)$$

  - define m as the ratio of contingent claim price to probability (27)

$$p = \sum_{s=1}^{S} \pi(s)\, m(s)\, x(s) = E[mx]$$

- Thus, in a complete market, the stochastic discount factor m exists
  - it is just the set of contingent claims prices scaled by probabilities
  - the combination SDF-probability is called **state-price density**

Contingent claims and investor's problem

- Our objective here is to price assets without utility functions
- Nevertheless, looking at the investor's choice in the contingent claim context shed lights on the results' economic interpretation
  - suppose that the investor is endowed with initial wealth y and a future state-contingent income $y(s)$
  - the investor may trade contingent claims x^s for each possible state
- The investor's problem is formalized as

$$\max_{\{c,c(s)\}} \quad u(c) + \sum_{s=1}^{S} \beta\pi(s)\, u(c(s))$$

$$s.t. \quad c + \sum_{s=1}^{S} p^s c(s) = y + \sum_{s=1}^{S} p^s y(s)$$

- The *intertemporal* budget constraint is given in present-value terms

Investor's problem solution

- Introducing a Lagrange multiplier λ on the budget constraint, we may write

$$\mathcal{L} = u(c) + \sum_{s=1}^{S} \beta\pi(s)\, u(c(s)) + \lambda \left[y + \sum_{s=1}^{S} p^s y(s) - c - \sum_{s=1}^{S} p^s c(s) \right]$$

- set the derivatives of $\mathcal{L}$ wrt c and $c(s)$ equal to zero

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c} &= u'(c) - \lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial c(s)} &= \beta\pi(s)\, u'(c(s)) - \lambda p^s = 0 \end{aligned}$$

- The first order conditions are

$$\begin{aligned} u'(c) &= \lambda \\ \beta\pi(s)\, u'(c(s)) &= \lambda p^s \end{aligned}$$

Investor's problem solution

- Eliminate the Lagrange multiplier dividing the second condition by the first

$$\beta\pi(s)\frac{u'(c(s))}{u'(c)} = \frac{\lambda}{\lambda}p^s$$

- Rearrange to obtain

$$p^s = \beta\pi(s)\frac{u'(c(s))}{u'(c)}$$

- Finally, use the definition (27)

$$m(s) \equiv \frac{p^s}{\pi(s)} = \beta\frac{u'(c(s))}{u'(c)}$$

- Thus, we obtain the consumption-based model again

A graphical representation

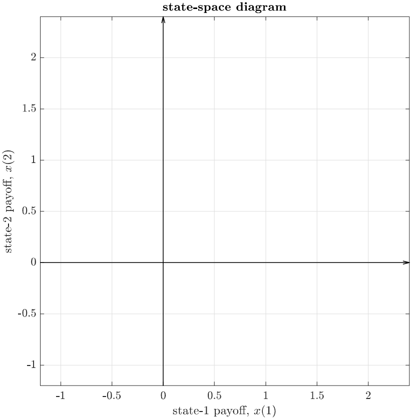
- We develop a graphical representation of the results achieved so far
- The **state-space diagram** is an intuitive tool to analyze asset prices
  - the diagram is a Cartesian representation
  - each axis measures asset payoffs in a given state of nature
  - prices and payoffs are represented as *vectors*

en.wikipedia.org/wiki/Euclidean_vector

- In a simple two-state diagram, we consider several elements in turn
  - state-1 contingent claim
  - state-2 contingent claim
  - set of contingent claim prices, denoted by pc
  - a generic asset, with payoff x^a

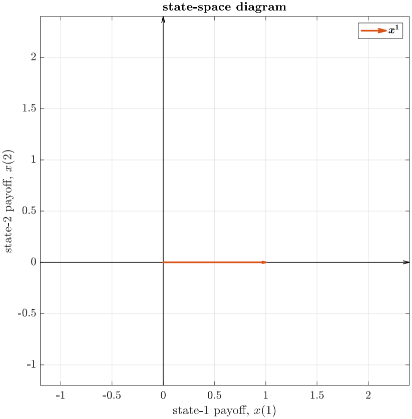
State-space

- MATLAB code: `state_space.m`



State-space and state-1 contingent claim

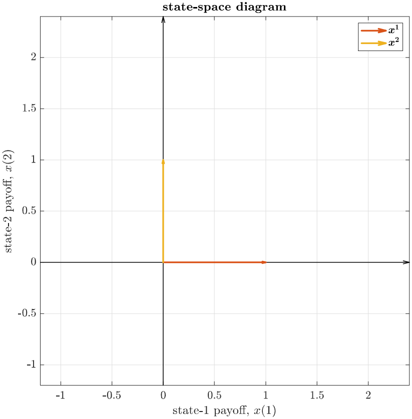
- MATLAB code: `state_space.m`



- The state-1 contingent claim is defined by the payoff $\mathbf{x}^1 = [1, 0]'$
- The vector in red represents the payoff in the state-space

State-space and state-2 contingent claim

- MATLAB code: `state_space.m`



- The state-2 contingent claim is defined by the payoff $\mathbf{x}^2 = [0, 1]'$
- The vector in yellow represents the payoff in the state-space

State-space and contingent claim prices

- Since there is a contingent claim for each state, we can also represent the set of contingent claim prices as a vector
- Let **pc** denote this vector (p stands for price, c for contingent claim)

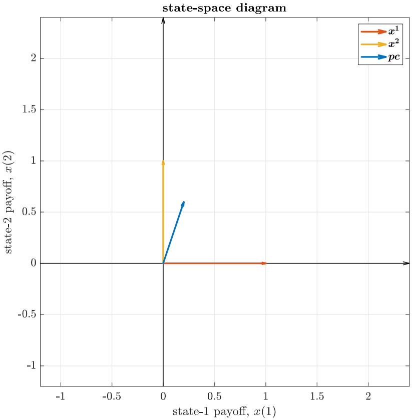
$$\mathbf{pc} = \left[p^1 \; p^2 \; ... \; p^S\right]'$$

- Each element of the vector represents a different coordinate valued

$$p^s = \pi(s)\, m(s)$$

State-space and contingent claim prices

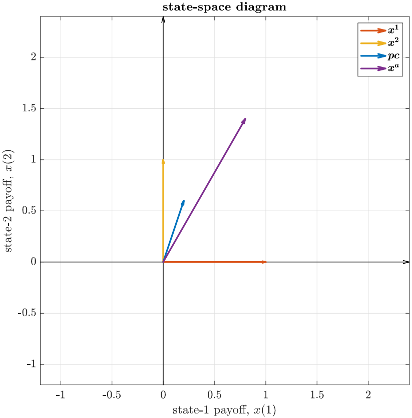
- MATLAB code: `state_space.m`



- Contingent claim prices set defined by the vector $\mathbf{pc} = [0.2, 0.6]'$
- The vector in blue represents the set in the state-space

State-space and a generic asset

- MATLAB code: `state_space.m`



- The generic asset a is defined by the payoff $\mathbf{x}^a = [0.8, 1.4]'$
- The vector in purple represents the payoff in the state-space

State-space diagram implications

- In our example, the vector **pc** lies in the first quadrant
- This is rather intuitive, since the contingent claim prices
  - represent prices of securities that have non-negative payoffs: hence, it is legitimate to expect their values to be positive
  - are proportional to $m(s)$, which in the consumption-based model is proportional to the state-s marginal utility of consumption
    - since marginal utility is always positive, the marginal rate of substitution and the stochastic discount factor are also always positive
    - that is $\mathbf{m} > 0$, and consequently $\mathbf{pc} > 0$

      [since **m** and **pc** are vectors, the inequalities mean that *all* their elements are positive, i.e. the variables take positive values in every state of nature]
- Note, however, we need *another* assumption for $\mathbf{pc} > 0$ to hold: the **no-arbitrage condition**

en.wikipedia.org/wiki/Arbitrage

Basic formula as a dot product

- Recall that an asset x can be seen as a contingent claim portfolio

$$\mathbf{x} = [x(1), x(2), ..., x(S)]'$$

- Each $x(s)$ represents the quantity of x^s held in portfolio
- Recall also that the asset price must equal the total current value of the contingent claims of which it represents the combination

$$p(x) = \sum_{s=1}^{S} p^s x(s)$$

Basic formula as a dot product

- Finally, recall that we interpret both pc and x as vectors
- We may then notice that the price of the asset corresponds to the **scalar product** between the two vectors

$$
\begin{aligned}
\mathbf{pc}'\mathbf{x} &\equiv \begin{bmatrix} p^1 \ p^2 \ \dots \ p^S \end{bmatrix} \begin{bmatrix} x(1) \\ x(2) \\ \dots \\ x(S) \end{bmatrix} \\
&= p^1 x(1) + p^2 x(2) + \dots + p^S x(S) \\
&= \textstyle\sum_{s=1}^{S} p^S x(s) = p(x)
\end{aligned} \tag{29}
$$

Geometric meaning of the dot product

- The dot product $\mathbf{pc}'\mathbf{x}$ is also equal to the product between
  - the length of the first vector ($\mathbf{pc}$), which we indicate with $|\mathbf{pc}|$
  - the length of the *projection* of the second vector ($\mathbf{x}$) on the first ($\mathbf{pc}$), which we denote by $|proj\,(\mathbf{x}|\mathbf{pc})|$

$$\mathbf{pc}'\mathbf{x} = |\mathbf{pc}| \times |proj\,(\mathbf{x}|\mathbf{pc})|$$

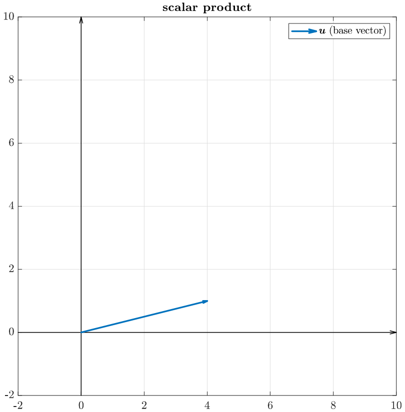
- As a result, the price of x can also be computed as

$$p\,(x) = |\mathbf{pc}| \times |proj\,(\mathbf{x}|\mathbf{pc})|$$

- We can give the dot product a graphical illustration
- For convenience, we label **pc** and **x** *base* and *projected* vector

The base vector **u**

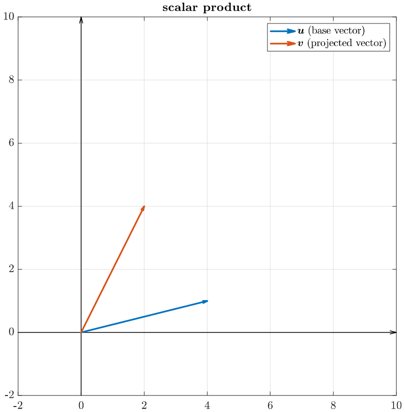
- MATLAB code: `scalar_product.m`



- The vector in blue represents the base vector $\mathbf{u} = [4, 1]'$ in the state-space

The projected vector $\mathbf{v}$

- MATLAB code: `scalar_product.m`



- The vector in red represents the projected vector $\mathbf{v} = [2, 4]'$ in the state-space

The projection of **v** onto **u**

- The *algebraic* scalar product between the two vectors is equal to

$$\mathbf{u}'\mathbf{v} = [4, 1] \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 4 \cdot 2 + 1 \cdot 4 = 12$$

  - MATLAB command: `u'*v`

- The *geometric* scalar product between the two vectors is equal to

$$\mathbf{u}'\mathbf{v} = |\mathbf{u}| \times |proj\,(\mathbf{v}|\mathbf{u})|$$

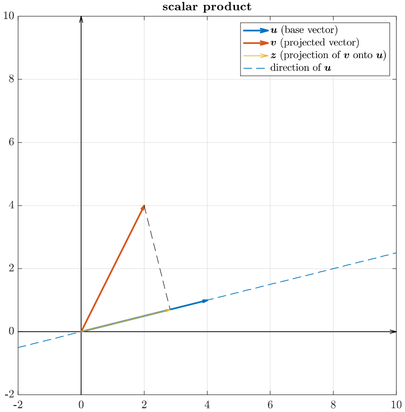
- Vector **u**'s length is given by

$$|\mathbf{u}| = \sqrt{4^2 + 1^2} = 4.1231$$

  - MATLAB command: `norm(u)`

The projection of **v** onto **u**

- MATLAB code: `scalar_product.m`



- The dashed line performs the orthogonal projection of **v** onto **u**
- The vector in yellow represents the projection **z** in the state-space

The projection of **v** onto **u**

- Vector **z**'s length is given by

$$|\mathbf{z}| \equiv |proj\left(\mathbf{v}|\mathbf{u}\right)| = \frac{\mathbf{u}'\mathbf{v}}{|\mathbf{u}|} = \frac{12}{4.1231} = \sqrt{\frac{12^2}{17}} = 2.9104$$

- To find **z**'s coordinates, proceed as follows
  - **z** lies in the same *direction* as **u**, hence the two vectors' coordinates must be proportional, i.e. $z\left(2\right)/z\left(1\right) = u\left(2\right)/u\left(1\right)$ and

$$z\left(2\right) = \left[u\left(2\right)/u\left(1\right)\right]z\left(1\right) = \left[1/4\right]z\left(1\right)$$

  - travel $|\mathbf{z}|$ along the direction of **u**

$$\sqrt{12^2/17} = |z| = \sqrt{z\left(1\right)^2 + z\left(2\right)^2} = \sqrt{4^2 z\left(2\right)^2 + 4^2 z\left(2\right)^2} = z\left(2\right)\sqrt{17}$$

$$z\left(2\right) = \sqrt{12^2/17^2} = 12/17 = 0.7059 \;\rightarrow\; z\left(1\right) = 4z\left(2\right) = 2.8235$$

  - MATLAB command: `z = projection(u,v)` (`scalar_product.m`)

The projection of 2$\mathbf{v}$ onto $\mathbf{u}$

- Any projection onto the direction of $\mathbf{u}$ is valid, even if it won't lie *within* the vector $\mathbf{u}$
- Consider the vector 2$\mathbf{v}$, whose coordinates are twice as those of $\mathbf{v}$

$$2\mathbf{v} = [4, 8]'$$

  - the dot product with $\mathbf{u}$ is

$$\mathbf{u}'(2\mathbf{v}) = 2\mathbf{u}'\mathbf{v} = 24$$

  - the projection onto $\mathbf{u}$ is

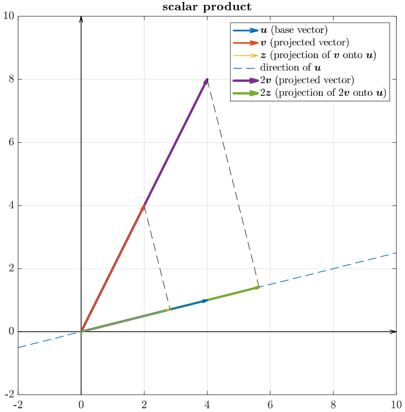
$$|proj\,(2\mathbf{v}|\mathbf{u})| = 2\,|proj\,(\mathbf{v}|\mathbf{u})| = 2\,|\mathbf{z}| = 2 \cdot 2.9104 = 5.8208$$

- The projection of 2$\mathbf{v}$ onto $\mathbf{u}$ lies in the same direction as $\mathbf{u}$ and $\mathbf{z}$, thus it *must* be represented by

$$2\mathbf{z} = [5.6470, 1.4118]'$$

The projection of 2**v** onto **u**

- MATLAB code: `scalar_product.m`



- The vector in purple is the projected vector 2**v** in the state-space
- The vector in green represents the projection 2**z** in the state-space

Special projections

- The projection of a vector onto itself is equal to the vector itself, hence the dot product $\mathbf{u}'\mathbf{u}$ equals the square of the vector length

$$\mathbf{u}'\mathbf{u} = |\mathbf{u}| \times |\mathit{proj}\,(\mathbf{u}|\mathbf{u})| = |\mathbf{u}| \times |\mathbf{u}| = |\mathbf{u}|^2$$

- The projection of a vector onto another orthogonal to it is *zero*
  - in our example, the vector $\mathbf{h} = [-1, 4]'$ is orthogonal to $\mathbf{u}$, so the scalar product between the two vectors is

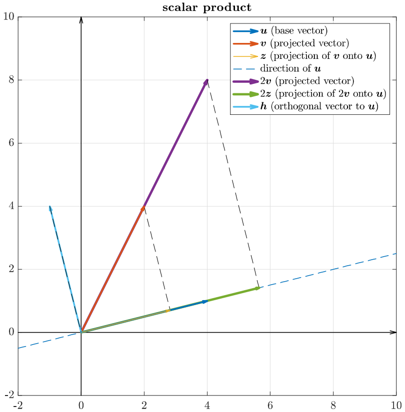
$$\mathbf{u}'\mathbf{h} \equiv [4, 1] \begin{bmatrix} -1 \\ 4 \end{bmatrix} = 4 \cdot (-1) + 1 \cdot 4 = 0$$

  - the projection of $\mathbf{h}$ onto $\mathbf{u}$ is therefore

$$|\mathit{proj}\,(\mathbf{h}|\mathbf{u})| = \frac{\mathbf{u}'\mathbf{h}}{|\mathbf{u}|} = \frac{0}{4.1231} = 0$$

Special projections

- MATLAB code: `scalar_product.m`



- The vector in cyan represents the orthogonal vector $\mathbf{h}$ in the state-space

Pricing asset a

- Turn back to the state diagram representing the vectors

$$\begin{aligned} \mathbf{pc} &= [0.2, 0.6]' \\ \mathbf{x}^a &= [0.8, 1.4]' \end{aligned}$$

- Using the basic formula (or equivalently the dot product) we obtain

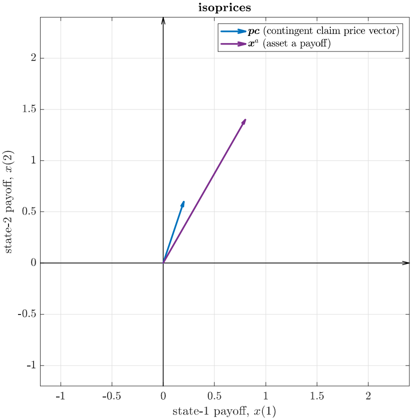
$$\begin{aligned} p^a &\equiv p(x^a) = \sum_{s=1}^{S} \pi(s)\, m(s)\, x^a(s) = \sum_{s=1}^{S} p^s x^a(s) = \mathbf{pc}'\mathbf{x}^a \\ &= \frac{1}{5}\frac{8}{10} + \frac{3}{5}\frac{14}{10} = \frac{8+42}{50} = 1 \end{aligned}$$

- We could also obtain this result by exploiting the geometric definition of scalar product

$$p^a = \mathbf{pc}'\mathbf{x}^a = |\mathbf{pc}| \times |proj(\mathbf{x}^a|\mathbf{pc})|$$

Asset a and vector **pc** in the state-space

- MATLAB code: `isoprice.m`



- The vectors in blue and purple represent **pc** and $\mathbf{x}^a$ in the state-space, respectively

Pricing assets with the same projection onto **pc** as *a*

- Define a vector $\mathbf{x}^b \equiv proj\left(\mathbf{x}^a | \mathbf{pc}\right)$
  - the length of **pc** is $|\mathbf{pc}| = \sqrt{0.2^2 + 0.6^2} = 0.6325$
  - the length of $\mathbf{x}^b$ is

$$\left|\mathbf{x}^b\right| = \left|proj\left(\mathbf{x}^a | \mathbf{pc}\right)\right| = p^a / |\mathbf{pc}| = 1/0.6325 = 1.581$$

- The coordinates of $\mathbf{x}^b$ are such that

$$\begin{aligned} 1.581 &= \left|\mathbf{x}^b\right| = \sqrt{x^b(1)^2 + \left[\frac{p^2}{p^1} x^b(1)\right]^2} \\ &= \sqrt{\left(1 + 3^2\right) x^b(1)^2} = x^b(1)\sqrt{10} \end{aligned}$$

- Hence, $x^b(1) = 1.581/\sqrt{10} = 0.5$ and $x^b(2) = 1.5$

Pricing assets with the same projection onto **pc** as *a*

- Note that asset *b* has the same price as *a*

$$
\begin{aligned}
p^b &= \mathbf{pc}'\mathbf{x}^b = [0.2, 0.6] \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix} \\
&= 0.2 \cdot 0.5 + 0.6 \cdot 1.5 = 1 = p^a
\end{aligned}
$$

- This is due to $\left|proj\left(\mathbf{x}^a|\mathbf{pc}\right)\right| = \left|\mathbf{x}^b\right| = \left|proj\left(\mathbf{x}^b|\mathbf{pc}\right)\right|$, which implies

$$
\begin{aligned}
p^b &= \mathbf{pc}'\mathbf{x}^b = |\mathbf{pc}| \times \left|proj\left(\mathbf{x}^b|\mathbf{pc}\right)\right| \\
&= |\mathbf{pc}| \times \left|proj\left(\mathbf{x}^a|\mathbf{pc}\right)\right| = \mathbf{pc}'\mathbf{x}^a = p^a
\end{aligned}
$$

Pricing assets with the same projection onto **pc** as *a*

- This result extends to any asset d such that $\left|proj\left(\mathbf{x}^d|\mathbf{pc}\right)\right| = \left|\mathbf{x}^b\right|$

$$\begin{aligned} p^d &= \mathbf{pc}'\mathbf{x}^d = |\mathbf{pc}| \times \left|proj\left(\mathbf{x}^d|\mathbf{pc}\right)\right| = |\mathbf{pc}| \times \left|\mathbf{x}^b\right| \\ &= |\mathbf{pc}| \times |proj\left(\mathbf{x}^a|\mathbf{pc}\right)| = \mathbf{pc}'\mathbf{x}^a = p^a \end{aligned}$$

- For example, suppose $x^d(1) = -0.5$ and $\left|proj\left(\mathbf{x}^d|\mathbf{pc}\right)\right| = \left|\mathbf{x}^b\right|$
- The above claim implies that $p^d = p^b = p^a = 1$, hence we have

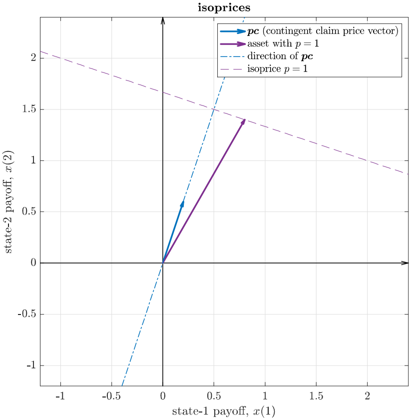
$$1 = p^d = \mathbf{pc}'\mathbf{x}^d = [0.2, 0.6]\begin{bmatrix}-0.5 \\ x^d(2)\end{bmatrix} = -0.1 + 0.6 \cdot x^d(2)$$

$$x^d(2) = 1.1/0.6 = 1.8333$$

- All payoffs lying on the perpendicular to **pc** passing through $\mathbf{x}^b$ share the same price, *because* $\mathbf{x}^b$ represents their projections onto **pc**

Isoprice $p = 1$ in the state-space

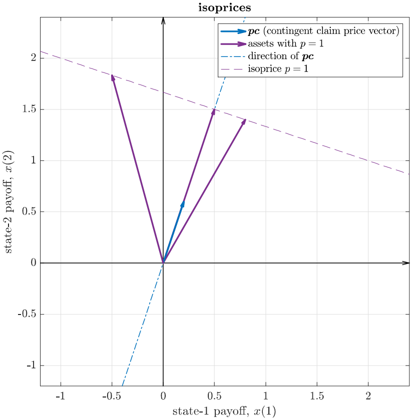
- MATLAB code: `isoprice.m`



- The dashed line in purple represents the isoprice $p = 1$ in the state-space

Assets with $p = 1$ in the state-space

- MATLAB code: `isoprice.m`



- The vectors in purple represent asset with price $p = 1$ in the state-space

Pricing assets with a zero projection onto **pc**

- For perpendicular vectors **z** and **u**, $|proj\,(\mathbf{z}|\mathbf{u})| = 0$
- We can thus identify payoffs with price (and the isoprice) $p = 0$
- Any payoff $\mathbf{x}^h$ orthogonal to **pc** has zero projection length on **pc**

$$p^h = \mathbf{pc}'\mathbf{x}^h = |\mathbf{pc}| \times \left| proj\left(\mathbf{x}^h|\mathbf{pc}\right)\right| = |\mathbf{pc}| \times 0 = 0$$

- For example, the vector $\mathbf{x}^h = \left[-p^2, p^1\right] = [-0.6, 0.2]$

$$p^h = \mathbf{pc}'\mathbf{x}^h = 0.2 \cdot (-0.6) + 0.6 \cdot 0.2 = 0$$

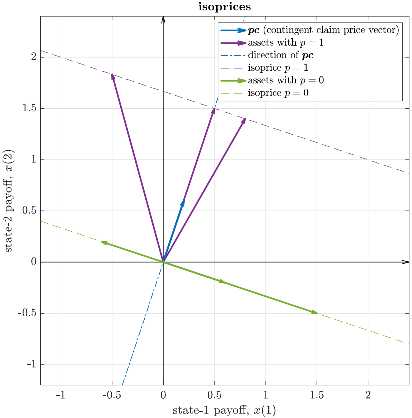
- *Every* payoff lying on the direction of the vector $\mathbf{x}^h$ has zero price

Exercise Compute the prices of the payoffs

$$[1.5, -0.5]' \quad \text{and} \quad [0.6, -0.2]'$$

Assets with $p = 0$ in the state-space

- MATLAB code: `isoprice.m`



- The dashed line in green is the isoprice $p = 0$ in the state-space
- The vectors in green are assets with zero price in the state-space

Assets with negative prices

- Assets may also have negative prices
- This happens when the payoffs have projections generating vectors with opposite orientation to that of **pc**
- Consider a short position on asset a in quantity 1/2

$$\mathbf{x}^l = -\mathbf{x}^a/2 = [-0.4, -0.7]'$$

- By the law of one price, we know that $p^l = -p^a/2 = -1/2$, the same result as given by

$$p^l = \mathbf{pc}'\mathbf{x}^l = 0.2 \cdot (-0.4) + 0.6 \cdot (-0.7) = -0.08 - 0.42 = -0.5$$

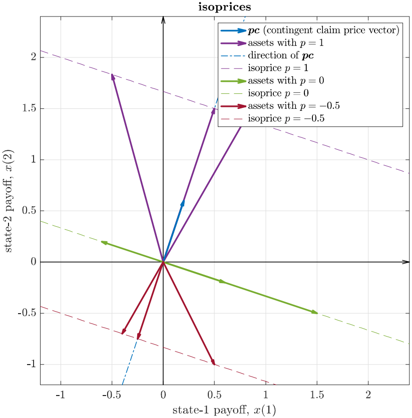
- Once again, all payoffs lying on the line perpendicular to **pc** through $\mathbf{x}^l$ have the same price as l
- The resulting line represents the isoprice $p = -1/2$

Exercise Compute the prices of the payoffs

$$[-0.25, -0.75]' \quad \text{and} \quad [0.5, -1]'$$

Assets with negative prices in the state-space

- MATLAB code: `isoprice.m`



- The dashed line in red is the isoprice $p = -0.5$ in the state-space
- The vectors in red are assets with $p = -0.5$ in the state-space

Some considerations about complete markets

- Three facts should be clear regarding complete markets

1. The price of any asset can be calculated through the dot product of its payoff the vector of contingent claim prices
2. The orthogonal directions to the contingent claim price vector represent set of assets with the same price
3. The vector **pc** represents the stochastic discount factor within the payoff space

- These are the reasons that motivate a space-diagram representation of the financial market

Pricing contingent claims

- Even if the financial market were complete, we wouldn't normally observe contingent claim prices
- We would instead observe the prices of more typical financial assets
- These prices, together with the respective payoffs, allow to triangulate the position of **pc**
- Suppose we observe two assets a and z such that

$$\begin{array}{ll} \mathbf{x}^a = [4/5, 7/5]' & p^a = 1 \\ \mathbf{x}^z = [-1/4, -3/4]' & p^z = -1/2 \end{array}$$

- For each asset, the price must obey the dot product between payoff and **pc**

$$\begin{aligned} p^a &= \mathbf{pc}'\mathbf{x}^a = p^1 \cdot 4/5 + p^2 \cdot 7/5 = 1 \\ p^z &= \mathbf{pc}'\mathbf{x}^z = p^1 \cdot (-1/4) + p^2 \cdot (-3/4) = -1/2 \end{aligned}$$

Algebraic identification of **pc**

- We obtain the coordinates of **pc** by solving the resulting system of two linear equations in two unknowns
  - we solve the first equation for p^1

$$p^1 = \frac{1}{4/5} - \frac{7/5}{4/5} p^2 = \frac{5}{4} - \frac{7}{4} p^2$$

  - we use the resulting expression to replace p^1 in the second equation

$$-\frac{1}{4}\left[\frac{5}{4} - \frac{7}{4} p^2\right] - \frac{3}{4} p^2 = -\frac{1}{2}$$

  - we solve for p^2

$$p^2 = \frac{(1/4)(5/4) - 1/2}{(1/4)(7/4) - 3/4} = \frac{5-8}{16} \frac{16}{7-12} = \frac{3}{5}$$

- We thus have

$$\mathbf{pc} = [0.2, 0.6]'$$

Geometric identification of **pc**

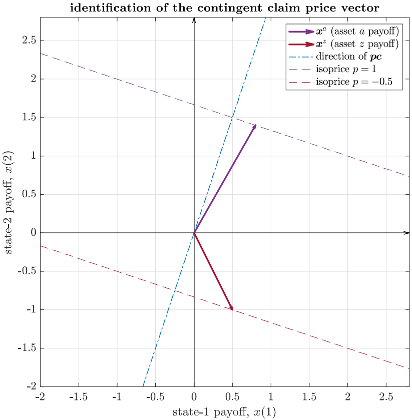
- To solve the problem geometrically, we draw two parallel lines
  - line A passes through $\mathbf{x}^a$
  - line Z passes through $\mathbf{x}^z$
- The common slope of A and Z must be such that
  - they are orthogonal to some line M passing through the origin
  - the lengths of the vectors that A and Z project onto M must be proportional to the prices of assets a and z

$$\frac{p^a}{p^z} = \frac{|\mathbf{pc}| \times |proj\,(\mathbf{x}^a|\mathbf{pc})|}{|\mathbf{pc}| \times |proj\,(\mathbf{x}^z|\mathbf{pc})|} = \frac{|proj\,(\mathbf{x}^a|\mathbf{pc})|}{|proj\,(\mathbf{x}^z|\mathbf{pc})|}$$

- Line M so identified represents the direction of **pc**

Direction of **pc** in the state-space

- MATLAB code: `find_pc.m`



- The dashed lines in purple and red are lines A and Z, respectively
- The line in blue represents the direction of **pc**

Coordinates of **pc**

- The length of the projection vector is found as follows
  - the segment connecting the projected and projection vectors is the difference between the vectors

$$\mathbf{x}^a - proj\left(\mathbf{x}^a | \mathbf{pc}\right)$$

  - exploiting the properties of right triangles, we have

$$\left|proj\left(\mathbf{x}^a | \mathbf{pc}\right)\right|^2 = \left|\mathbf{x}^a\right|^2 - \left|\mathbf{x}^a - proj\left(\mathbf{x}^a | \mathbf{pc}\right)\right|^2$$

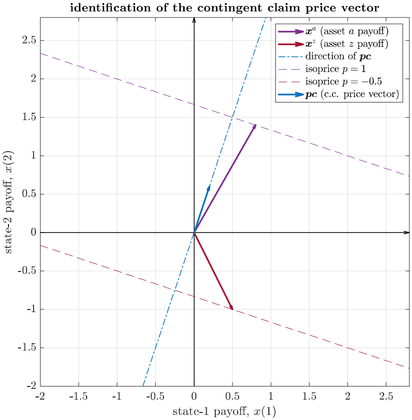
  - using this relation and the direction of **pc**, we find coordinates and length of $proj\left(\mathbf{x}^a | \mathbf{pc}\right)$
- We pinpoint the length of **pc** through one the pricing equation

$$|\mathbf{pc}| = \frac{p^a}{\left|proj\left(\mathbf{x}^a | \mathbf{pc}\right)\right|}$$

- The same result can be obtained using $\mathbf{x}^a$ instead

Identification of **pc** in the state-space

- MATLAB code: `find_pc.m`



- The vector in blue represents the contingent claim vector **pc**

Independent assets

- The reason why we can take full advantage of the initial assumptions is that the market is complete
- In turn, this is due to the number of *independent* assets being equal to the number of states of nature
  - independent means that assets are not combinations of one another
  - an asset cannot be obtained by holding a portfolio of others assets
- In our example, we have only two assets, yet neither can be obtained as combination of the other

$$\mathbf{x}^z \neq q \cdot \mathbf{x}^a, \ \forall q$$

Independent assets and portfolios

- In practical terms, applying the free formation of portfolios requires the solution of the system of equations

$$\begin{cases} \mathbf{x}^z(1) = q \cdot \mathbf{x}^a(1) \\ \mathbf{x}^z(2) = q \cdot \mathbf{x}^a(2) \end{cases}$$

- The solution should in turn satisfy the conditions

$$\frac{\mathbf{x}^z(1)}{\mathbf{x}^a(1)} = q = \frac{\mathbf{x}^z(2)}{\mathbf{x}^a(2)}$$

- The ensuing equation cannot possibly hold since

$$\frac{\mathbf{x}^z(1)}{\mathbf{x}^a(1)} = \frac{-1/4}{4/5} \neq \frac{-3/4}{7/5} = \frac{\mathbf{x}^z(2)}{\mathbf{x}^a(2)}$$

Non-independent assets and portfolios

- What would happen if the observed activities were not independent?
- Suppose that we observe the payoff $\mathbf{x}^y = [-0.4, -0.7]'$ instead of $\mathbf{x}^z$
- With assets a and y, it is not possible to identify **pc**
  - assume the opposite, and form the portfolio

$$\mathbf{pc} = q^a \cdot \mathbf{x}^a + q^y \cdot \mathbf{x}^y$$

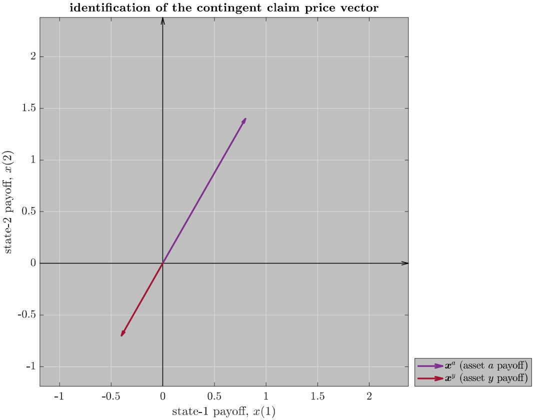
  - we should then solve the system of equations

$$\begin{cases} \frac{1}{5} = \frac{4}{5}q^a - \frac{2}{5}q^y & \rightarrow \ \frac{1}{2} = 2q^a - q^y \\ \frac{3}{5} = \frac{7}{5}q^a - \frac{7}{10}q^y & \rightarrow \ \frac{6}{7} = 2q^a - q^y \end{cases} \rightarrow \ \frac{1}{2} = \frac{6}{7}$$

  - it is straightforward that the last equation cannot possibly hold

Identification of **pc** in the state-space

- MATLAB code: `incomplete_markets.m`



- The vectors in purple and red represent asset a and y, respectively

Incomplete market

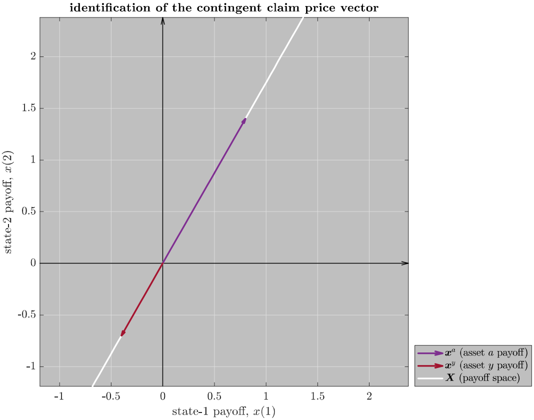
- The reason this happens is that asset y is merely a 0.5-unit short position in a
- That is, there is only one independent asset, for example asset a
- Asset y can be synthesized by resorting to free portfolio formation

$$\mathbf{x}^y = q \cdot \mathbf{x}^a = -0.5 \cdot \mathbf{x}^a$$

- In the presence of two states of nature and observing only one independent activity, the financial market is thus **incomplete**
- The concept can be generalized to an arbitrary number of states S
- The market is incomplete if the number of independent assets $n < S$
- The subset of *observable* asset is called **payoff space X**

The payoff space **X** in the state-space

- MATLAB code: `incomplete_markets.m`



- The line in white represents the payoff space **X**

Can we find **pc** outside **X**?

- The key consequence of the market incompleteness is that we cannot *correctly* determine the prices of unobserved financial assets
- If we suppose for a moment that this is not true, we could think that
  - although **pc** cannot be synthesized (it is therefore *non-observable*)
  - the vector could be identified (*outside* **X**) using the basic formula
- The system of equations resulting from the formula is

$$\begin{cases} p^a = p^1 x^a(1) + p^2 x^a(2) \\ p^y = p^1 x^y(1) + p^2 x^y(2) \end{cases}$$

$$\downarrow$$

$$\begin{cases} 1 = \frac{4}{5} p^1 + \frac{7}{5} p^2 \\ -\frac{1}{2} = -\frac{2}{5} p^1 - \frac{7}{10} p^2 \end{cases}$$

- As we've seen above, the system of equations delivers no solution

Representation of m in $\mathbf{X}$

- Yet, isolating p^2 in either equation of the system, we obtain

$$p^2 = \frac{p^a}{x^a(2)} - \frac{x^a(1)}{x^a(2)} p^1 = \frac{5}{7} - \frac{4}{7} p^1$$

- Let $\mathbf{pc}^*$ denote the line that obeys this relationship
- *Any* payoff with coordinates meeting this condition correctly prices the *observed* assets
- Among the infinite payoffs that satisfy this relation, one and only one, which we denote by $\mathbf{x}^*$, lies in $\mathbf{X}$
  - we identify $\mathbf{x}^*$ by imposing its coordinates to obey *also* the slope of $\mathbf{X}$
  - this means that the coordinates of $\mathbf{x}^*$ must satisfy

$$\begin{cases} x^*(2) = \frac{p^a}{x^a(2)} - \frac{x^a(1)}{x^a(2)} x^*(1) = \frac{5}{7} - \frac{4}{7} x^*(1) & : \mathbf{x}^* \in \mathbf{pc}^* \\ x^*(2) = \frac{x^a(2)}{x^a(1)} x^*(1) = \frac{7}{4} x^*(1) & : \mathbf{x}^* \in \mathbf{X} \end{cases}$$

Representation of m in $\mathbf{X}$

- Equalizing the right-hand sides of two equations, we obtain

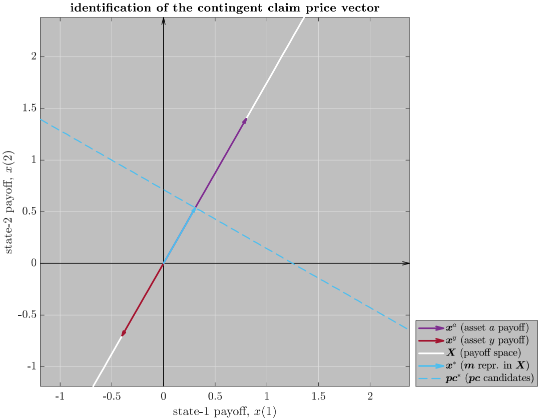
$$\begin{aligned}
\frac{p^a}{x^a(2)} &= \left(\frac{x^a(1)}{x^a(2)} + \frac{x^a(2)}{x^a(1)}\right) x^*(1) \\
&\rightarrow \quad x^*(1) = \frac{p^a x^a(1)}{x^a(1)^2 + x^a(2)^2} \\
&\rightarrow \quad x^*(2) = \frac{p^a x^a(2)}{x^a(1)^2 + x^a(2)^2}
\end{aligned}$$

- In our example, we thus have

$$\begin{aligned}
x^*(1) &= \frac{4/5}{16/25 + 49/25} = \frac{4}{13} \\
x^*(2) &= \frac{7/5}{16/25 + 49/25} = \frac{7}{13}
\end{aligned}$$

Representation of m in the state space

- MATLAB code: `incomplete_markets.m`



- The vector **pc** must lie on the dashed line in blue
- The vector in blue represents **m** within the payoff space **X**

Pricing assets in **X** using **x***

- The vector $\mathbf{x}^*$ correctly determines the asset prices in **X**

$$p^a = x^*(1)\,x^a(1) + x^*(2)\,x^a(2) = \tfrac{4}{13}\tfrac{4}{5} + \tfrac{7}{13}\tfrac{7}{5} = 1$$
$$p^y = x^*(1)\,x^y(1) + x^*(2)\,x^y(2) = -\tfrac{4}{13}\tfrac{4}{10} - \tfrac{7}{13}\tfrac{7}{10} = -\tfrac{1}{2}$$

- This is because (unobserved) **pc** lies on $\mathbf{pc}^*$, *orthogonal* to **X** in $\mathbf{x}^*$
  - to prove this claim, compute the expected value

$$\begin{aligned} E\left[(m+\varepsilon)\,x\right] &= E\left[mx + \varepsilon x\right] = E\left[mx\right] + E\left[\varepsilon x\right] \\ &= E\left[mx\right] = p(x) \;\Leftrightarrow\; E(\varepsilon x) = 0 \end{aligned}$$

  - the basic formula is valid with $m+\varepsilon$ as SDF as long as $E(\varepsilon x) = 0$
- In fact, $x^* \equiv \pi(m+\varepsilon)$ for some ε such that $E(\varepsilon x) = 0$, hence it correctly determines the price of any **x** in **X**

$$\begin{aligned} p(x) &= E\left[(m+\varepsilon)\,x\right] = \textstyle\sum_s \pi(s)\left[m(s) + \varepsilon(s)\right] x(s) \\ &= \textstyle\sum_s x^*(s)\,x(s) = \mathbf{x}^*\mathbf{x} \end{aligned}$$

The link between x^* and pc

- Since by definition $pc \equiv \pi \cdot m$, letting $\varepsilon^* \equiv \pi \cdot \varepsilon$ we have that x^* is the sum of pc and ε^*

$$x^* \equiv \pi (m + \varepsilon) = \pi m + \pi\varepsilon = pc + \varepsilon^*$$

- The vector $\boldsymbol{\varepsilon}^*$ is by construction *orthogonal* to $\mathbf{x}$, since
  - two vectors are orthogonal if their dot product equals zero

$$\boldsymbol{\varepsilon}^* \mathbf{x} = 0$$

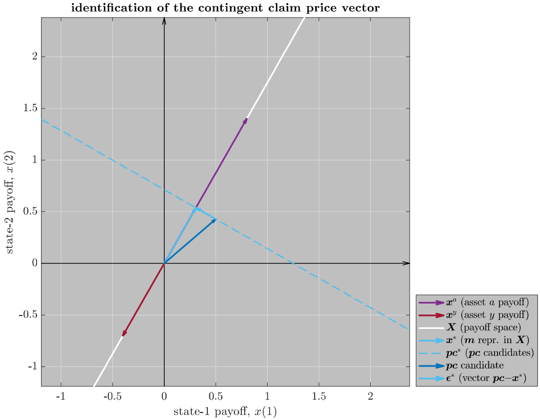
  - from the definition of ε, we indeed obtain

$$0 = E(\varepsilon x) = \sum_s \pi(s)\, \varepsilon(s)\, x(s) = \sum_s \varepsilon^*(s)\, x(s) = \boldsymbol{\varepsilon}^* \mathbf{x}$$

- Therefore we do know the *direction* of $\boldsymbol{\varepsilon}^*$ for any observed activity, but we ignore its *length* and *orientation*

One candidate **pc** in the state space

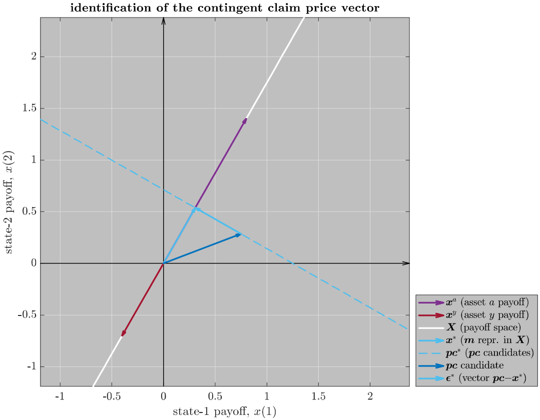
- MATLAB code: `incomplete_markets.m`



- The vector in blue represents **m** within the payoff space **X**
- The new vector in cyan equals the candidate **pc** minus $\mathbf{x}^*$

Another candidate **pc** in the state space

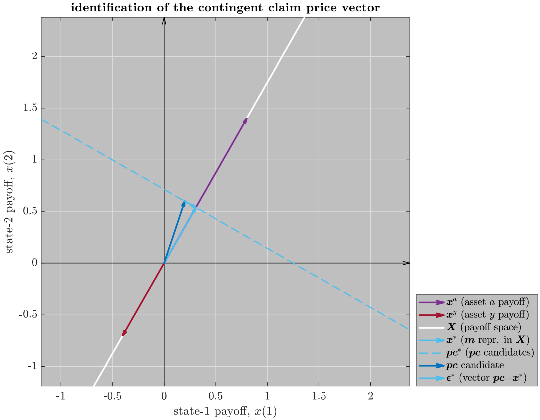
- MATLAB code: `incomplete_markets.m`



- The vector in blue represents **m** within the payoff space **X**
- The second vector in cyan is the new candidate **pc** minus $\mathbf{x}^*$

A third candidate **pc** in the state space

- MATLAB code: `incomplete_markets.m`



- The vector in blue represents **m** within the payoff space **X**
- The second vector in cyan is the newest candidate **pc** minus $\mathbf{x}^*$

Pricing assets outside **X** using **x***

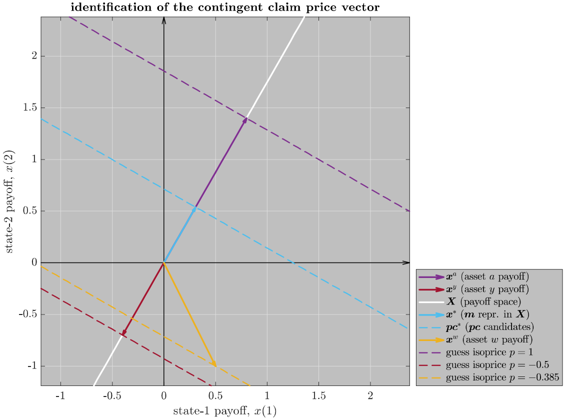
- It is tempting to use $\mathbf{x}^*$ to determine asset prices outside **X**
- Suppose that we are interested in pricing the payoff $\mathbf{x}^w = [0.5, -1]'$, and apply the basic formula using $\mathbf{x}^*$

$$p^w = x^*(1)\, x^w(1) + x^*(2)\, x^w(2) = \frac{4}{13}\frac{1}{2} + \frac{7}{13}(-1) = -\frac{5}{13}$$

- Is this price the correct one? We do not know!
  - if $\mathbf{x}^* \equiv \mathbf{pc}$, then it is the correct price; otherwise it isn't
  - if we *observed* this price, then it would hold $\mathbf{x}^* \equiv \mathbf{pc}$!
- All we *do know* is that the greater the distance between $\mathbf{x}^*$ and **pc** (the length of ε^*), the greater the error we will make in pricing $\mathbf{x}^w$

Pricing asset **w** using $\mathbf{x}^*$

- MATLAB code: `incomplete_markets.m`



- The vector in yellow represents $\mathbf{x}^w$ in the payoff space $\mathbf{X}$
- The dashed line in yellow is the (guess) isoprice $p = p^w$

Pricing assets outside X with a candidate pc

- Suppose that **pc** is such that $p^1 = 0.5$
- We may exploit the fact that **pc** must lie onto $\mathbf{pc}^*$ to obtain

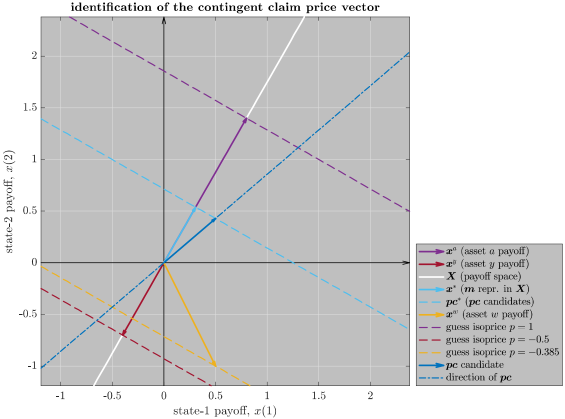
$$p^2 = \frac{p^a}{x^a(2)} - \frac{x^a(1)}{x^a(2)} p^1 = \frac{5}{7} - \frac{4}{7}\frac{1}{2} = \frac{3}{7}$$

- The price of asset w would then be

$$p^w = p^1 x^w(1) + p^2 x^w(2) = \frac{1}{2}\frac{1}{2} + \frac{3}{7}(-1) = -\frac{5}{28}$$

Pricing asset **w** using one candidate **pc**

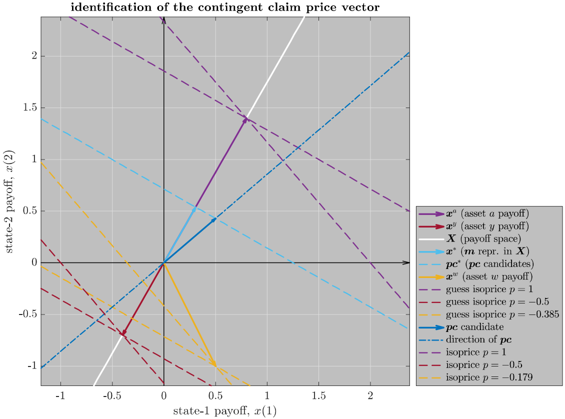
- MATLAB code: `incomplete_markets.m`



- The vector in blue represents one candidate **pc** in the state space
- The dashed line in blue is the direction of (candidate) **pc**

Pricing asset **w** using one candidate **pc**

- MATLAB code: `incomplete_markets.m`



- The *old* dashed line in yellow is the (guess) isoprice $p = p^w$
- The *new* dashed line in yellow is the (candidate) isoprice $p = p^w$

Pricing assets outside **X** with another candidate **pc**

- Suppose that **pc** is such that $p^1 = 0.75$
- We may exploit the fact that **pc** must lie onto $\mathbf{pc}^*$ to obtain

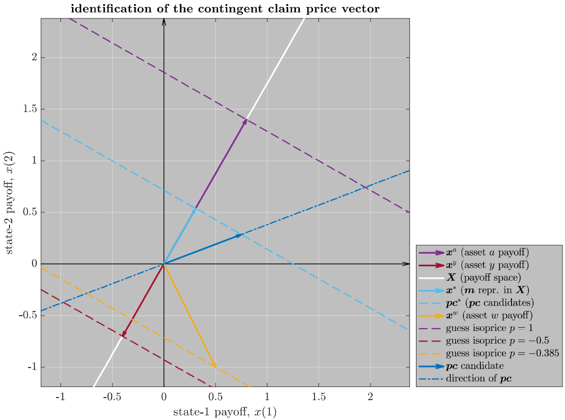
$$p^2 = \frac{p^a}{x^a(2)} - \frac{x^a(1)}{x^a(2)} p^1 = \frac{5}{7} - \frac{4}{7}\frac{3}{4} = \frac{2}{7}$$

- The price of asset w would then be positive!

$$p^w = p^1 x^w(1) + p^2 x^w(2) = \frac{3}{4}\frac{1}{2} + \frac{2}{7}(-1) = \frac{5}{56}$$

Pricing asset **w** using another candidate **pc**

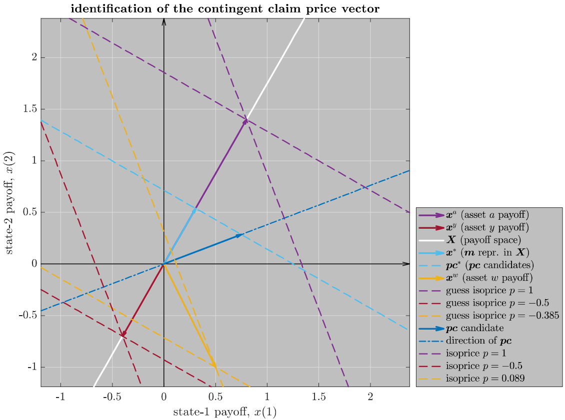
- MATLAB code: `incomplete_markets.m`



- The vector in blue represents another candidate **pc** in the state space
- The dashed line in blue is the direction of (the new candidate) **pc**

Pricing asset **w** using another candidate **pc**

- MATLAB code: `incomplete_markets.m`



- The *old* dashed line in yellow is the (guess) isoprice $p = p^w$
- The *new* dashed line in yellow is the (candidate) isoprice $p = p^w$

Pricing assets outside **X** with the real vector **pc**

- Suppose now that we *observe* asset w as in the case of complete markets: its price equals $p^w = -0.5$
- We may exploit the fact that **pc** must lie onto $\mathbf{pc}^*$ to obtain

$$\begin{aligned} -0.5 &= p^w = p^1 x^w(1) + \left[\frac{p^a}{x^a(2)} - \frac{x^a(1)}{x^a(2)} p^1\right] x^w(2) \\ &= p^1 \frac{1}{2} + \left[\frac{1}{7/5} - \frac{4/5}{7/5} p^1\right](-1) \end{aligned}$$

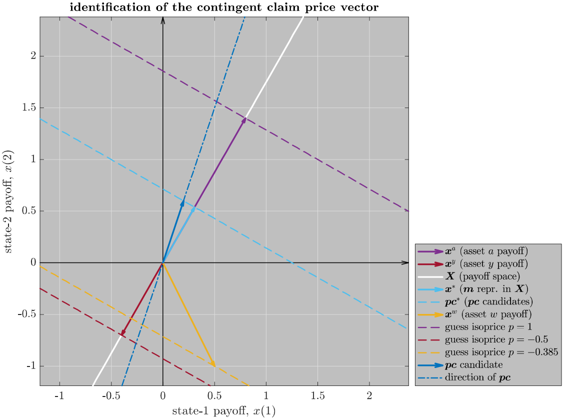
- We may isolate p^1 to get

$$p^1 = \frac{5/7 - 1/2}{1/2 + 4/7} = \frac{3/14}{15/14} = \frac{1}{5} \rightarrow p^2 = \frac{5}{7} - \frac{4}{7}\frac{1}{5} = \frac{3}{5}$$

- We obtain, once again, the vector **pc** that characterized our complete market above

Pricing asset **w** using the real vector **pc**

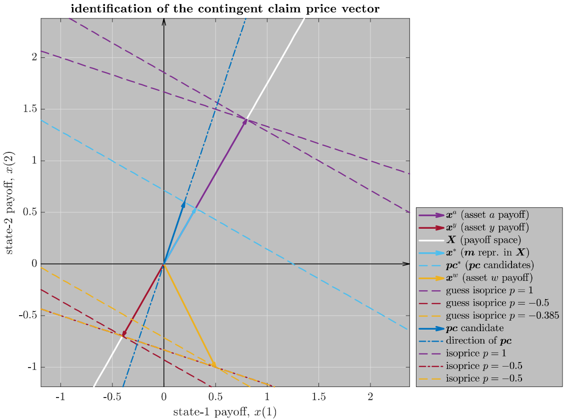
- MATLAB code: `incomplete_markets.m`



- The vector in blue represents the real vector **pc** in the state space
- The dashed line in blue is the direction of (the real vector) **pc**

Pricing asset **w** using the real vector **pc**

- MATLAB code: `incomplete_markets.m`



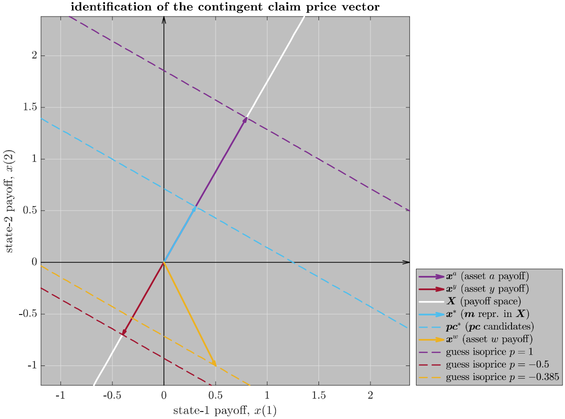
- The *old* dashed line in yellow is the (guess) isoprice $p = p^w$
- The *new* dashed line in yellow is the (real) isoprice $p = p^w$

Pricing asset in the real world

- In fact, the number of observed assets does not cover the universe of existing states of nature
- As a result, we have no way of assessing whether and to what extent the prices we determine are incorrect
- The statistical model tries to overcome this situation by trying to synthesize all possible financial assets using the observed ones
- In other words, it tries to *expand* the payoff space $\mathbf{X}$ in an attempt to make it coincide with the state space $\mathbf{R}$
  - whether this is or will be possible *cannot* be scientifically determined
  - this operation would require the observation of $\mathbf{R}$ to be able to compare it with $\mathbf{X}$
- Our knowledge is confined to what we prove *given* what we observe

Pricing asset in the real world

- MATLAB code: `incomplete_markets.m`



- What we observe is the vector $\mathbf{x}^*$ in the payoff space $\mathbf{X}$
- We can only *guess* the position of the isoprices in the state space $\mathbf{R}$

Table of contents

Relative pricing approach

- Our quantitative analysis has so far focused on *absolute* pricing
- That is, we have studied models that price assets by the exposure of their payoffs to fundamental sources of macroeconomic risk
- The complementary approach is *relative* pricing
  - Under this paradigm, we study what we can learn about an asset's value given the prices of some other assets
  - We explicitly refer the Arbitrage Pricing Theory (APT) model to exemplify this approach
- Asset pricing problems are often solved by mixing the two approaches
- The CAPM is a primary example of mixed approach
  - by construction, it theoretically reflects the absolute approach
  - in applications, the model prices assets *relative* to the market or other risk factors
  - it abstracts from what determines the market or factor risk premia, which are treated as free parameters

Factor models

- What most applied models (including APT and CAPM) have in common is their *factor-based* structure
- Factor models are characterized by a linear specification of the stochastic discount factor:

$$m_{t+1} = a + \mathbf{b}'\mathbf{f}_{t+1} \tag{30}$$

- The constants a and $\{b_j\}$ (elements of the vector $\mathbf{b}$) are free parameters
- Historically, the simplicity of these models' structure meant that they were the first to be empirically tested
- Today, their popularity is mostly due to the modest performance of models purely based on the absolute approach

Factor models and investor's marginal utility

- Via the basic pricing formula, it is possible to relate (30) to the consumption-based model's (11)
- While this parallel corresponds to the foundation of the CAPM, it is merely interpretative as far as the APT model is concerned
- By equating the right-hand sides of the two expression, it follows that the factors are linear approximations of marginal utility

$$\beta \frac{u'\left(c_{t+1}\right)}{u'\left(c_t\right)} \approx a + b' f_{t+1} \tag{31}$$

Identification of factors

- Depending on the nature of the information provided, factors are
  - *state variables*: indicate the relative need for resources across states of nature, based on current information
  - *forecast variables*: identify distinguishing feature across states based on future information
- In the APT, the factors are pinpointed by extracting relevant information from current and/or historical prices
- In the CAPM, the factors are determined by a two-step process
  - replacing consumption data with some other variable's
  - proving the linearity of the ensuing relationship

CAPM and the role of market portfolio

- Specifically, the CAPM relate the SDF to the **market portfolio**

$$m_{t+1} = a + bR^W_{t+1} \tag{32}$$

en.wikipedia.org/wiki/Market_portfolio

- Theoretically, a, b obtains by applying (32) to two return factors

$$\begin{aligned} 1 &= E\left[mR^W\right] = aE\left[R^W\right] + bE\left[\left(R^W\right)^2\right] \\ 1 &= E\left[m\right]R^f = aR^f + bE\left[R^W\right]R^f \end{aligned}$$

- Empirically, a, b are obtained as regression coefficients

Replacing consumption with wealth

- We can see *any* multiperiod model as a one-period model by resorting to two modifications
  - replace the end-of period utility function with a **value function**

$$V = V(W)$$

en.wikipedia.org/wiki/Value_function

  - impose market returns are **independent and identically distributed**

en.wikipedia.org/wiki/Independent_and_identically_distributed_random_variables

- The resulting lifetime utility is defined as a function of current consumption and future wealth W_{t+1}

$$U = u(c_t) + \beta E_t [V(W_{t+1})]$$

Value function and stochastic discount factor

- Solving the investor's problem with the new lifetime utility specification yields the first-order condition

$$p_t u'(c_t) = \beta E_t \left[V'(W_{t+1}) x_{t+1} \right]$$

- Accordingly, the investor equates at the margin
  - the decline in current utility due to the additional asset purchase
  - the increase in the value of available resources in the future
- We solution delivers a basic pricing formula with SDF

$$m_{t+1} = \beta \frac{V'(W_{t+1})}{u'(c_t)}$$

Quadratic value function

- We need an explicit functional form for the value function
- Since we intend to obtain a linear relationship between m and the factor, a natural choice is the quadratic function

$$V\left(W_{t+1}\right) = -\frac{\eta}{2}\left(W_{t+1} - W^{*}\right)^{2}$$

- We calculate the marginal increase in the value of available resources

$$V'\left(W_{t+1}\right) = -\eta\left(W_{t+1} - W^{*}\right)$$

- The resulting stochastic discount factor is

$$m_{t+1} = -\beta\eta\frac{W_{t+1} - W^{*}}{u'\left(c_{t}\right)}$$

Linear stochastic discount factor

- Let wealth be a function of current investment

$$W_{t+1} = R_{t+1}^{W} \left(W_t - c_t \right)$$

- We isolate the return R^W to obtain (32)

$$\begin{aligned} m_{t+1} &= \frac{\beta \eta W^*}{u'\left(c_t\right)} + \left[-\beta \eta \frac{\left(W_t - c_t\right)}{u'\left(c_t\right)} \right] R_{t+1}^{W} \\ &= a_t + b_t R_{t+1}^{W} \end{aligned}$$

Value function properties

- The CAPM builds on a value function subject to two key and rather restrictive assumptions
  - the function V depends on the marginal value of future resources
  - the marginal value of resources is a linear function
- However, the value function can be conveniently Interpreted as the *maximized* value of lifetime utility

$$V(W_t) \equiv \max_{\{c_t, c_{t+1}, \ldots, w_t, w_{t+1}\}} E_t\left[\sum_{s=t}^{\infty} \beta^{s-t} u(c_s)\right]$$

$$s.t. \quad W_{s+1} = R^W_{s+1}(W_s - c_s); \quad R^W_s = w_s' R_s; \quad w_s' 1 = 1$$

Linearization of the stochastic discount factor

- What are the conditions for a *utility* function to generate a linear marginal value of future resources?
- A *sufficient* condition is that the function is quadratic

$$u(c_t) = -\frac{1}{2}(c_t - c^*)^2$$

- However, this is not a *necessary* condition
- The quadratic form is empirically weak (the isoelastic form is more appropriate)
- Every model $m_{t+1} = g(f_{t+1})$ can be linearized when assuming normal distributions or using a Taylor expansion

Linearization of the stochastic discount factor

- For example, the Taylor expansion impose the following structure

$$m_{t+1} \approx g\left(E_t\left[f_{t+1}\right]\right) + g'\left(E_t\left[f_{t+1}\right]\right)\left(f_{t+1} - E_t\left[f_{t+1}\right]\right)$$

- In the case of a isoelastic function, we obtain

$$\begin{aligned} m_{t+1} &\approx \beta\left(\frac{E_t\left[c_{t+1}\right]}{c_t}\right)^{-\gamma} - \beta\gamma\frac{E_t\left[c_{t+1}\right]^{-\gamma-1}}{c_t^{-\gamma}}\left(c_{t+1} - E_t\left[c_{t+1}\right]\right) \\ &= \frac{(1+\gamma)\beta E_t\left[R^W(W_t-c_t)\right]^{-\gamma}}{c_t^{-\gamma}} - \frac{\beta\gamma E_t\left[R^W(W_t-c_t)\right]^{-\gamma-1}(W_t-c_t)}{c_t^{-\gamma}}R^W \\ &= a_t + b_t R^W \end{aligned}$$

CAPM extension to state variables

- The **intertemporal CAPM** (ICAPM) generalizes the CAPM by including additional state variables

$$m_{t+1} = a + \mathbf{b}'\mathbf{f}_{t+1}$$

en.wikipedia.org/wiki/Intertemporal_CAPM

- The state variables *predict* the future level of available resources
  - the current level of resources is an obvious example of state variable
  - other state variables may identify changes in the **production possibility frontier**

en.wikipedia.org/wiki/Production-possibility_frontier

  - goods' relative prices can also represent state variables
- The ICAPM can be implemented by defining the value function

$$V\left(W_{t+1}, z_{t+1}\right)$$

Stochastic discount factor in ICAPM models

- Under these conditions, the stochastic discount factor is

$$m_{t+1} = \beta \frac{V_W\left(W_{t+1}, z_{t+1}\right)}{V_W\left(W_t, z_t\right)}$$

- Once again, this equation can be obtained using a Taylor expansion

$$m_{t+1} \approx \beta \tfrac{V_W\left(W_{t+1}, z_{t+1}\right)}{V_W\left(W_t, z_t\right)} + \beta \tfrac{V_{WW}\left(W_{t+1}, z_{t+1}\right)}{V_W\left(W_t, z_t\right)} \left(W_{t+1} - E_t\left[W_{t+1}\right]\right)$$
$$+ \beta \tfrac{V_{Wz}\left(W_{t+1}, z_{t+1}\right)}{V_W\left(W_t, z_t\right)} \left(z_{t+1} - E_t\left[z_{t+1}\right]\right)$$

- What differentiate the ICAPM from other factor models is that factors must include the market return

Statistical characterization of asset prices

- An example of a purely relative pricing is the arbitrage pricing theory (APT) model

en.wikipedia.org/wiki/Arbitrage_pricing_theory

- The APT builds on the observed regularity that there is a significant common component to stock returns
- The market and most individual stocks comove
- Beyond the market, groups of stocks also move together
- In addition, each stock's return has some idiosyncratic movement

Strategy behind the APT

- The starting point is a characterization of *realized* returns or payoffs
- The objective of the APT is to derive expected values for returns or prices out of this statistical characterization
  - *idiosyncratic* movements in returns should imply *no* risk premium
  - risk premium should only relate to asset's covariance with the factors
- A mathematical model is constructed to describe the tendency for stocks to move together
  - the predictions of the model define the factors and residual idiosyncratic components
  - the idiosyncratic components *should* have zero (or small) risk prices
- The first step is to suppose that there are *no residuals*
  - an asset can be priced from the factors by the law of one price
  - the logic *hopefully* extends to small residuals implying no or negligible risk premia

Foundations of the APT

- The APT models the tendency of asset payoffs (returns) to move together via a statistical factor decomposition

$$x^i = a_i + \sum_{j=1}^{M} \beta_{ij} f_j + \varepsilon^i = a_i + \boldsymbol{\beta}_i' \mathbf{f} + \varepsilon^i \tag{33}$$

- In (33), f_j are the factors, β_{ij} are the betas or factor loadings, and ε_i are residuals
- It is customary to fold the factor means into the constant, and express the model in terms of the zero-mean factors $\tilde{f} \equiv f - E(f)$

$$x^i = E\left(x^i\right) + \sum_{j=1}^{M} \beta_{ij} \tilde{f}_j + \varepsilon^i \tag{34}$$

Exact factor specification

- The qualifying assumptions of the APT are the three equalities

$$E\left(\varepsilon^{i}\right) = E\left(\varepsilon^{i}\tilde{f}_{j}\right) = E\left(\varepsilon^{i}\varepsilon^{j}\right) = 0$$

- If there are *no idiosyncratic terms* ($\varepsilon^{i} = 0$ for all i), then (34) is called an *exact* factor model, expressed by

$$x^{i} = E\left(x^{i}\right)\mathbf{1} + \boldsymbol{\beta}_{i}'\tilde{\mathbf{f}} \qquad (35)$$

- Equation (35) states that the payoff x_i can be synthesized as a portfolio of the factors and a constant (the risk-free payoff)

Exact factor pricing

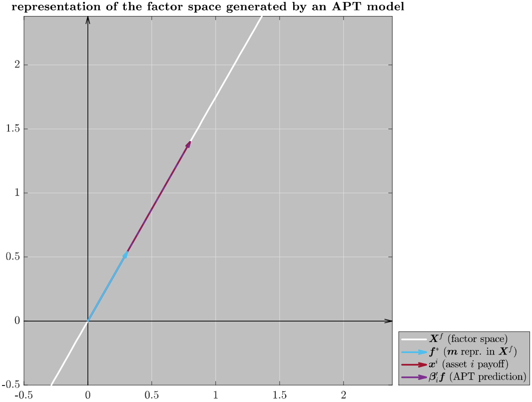
- By the law of one price, the price of x_i can only depend on the prices of the factors

$$p\left(x^i\right) = E\left(x^i\right) p\left(1\right) + \beta_i' p\left(\tilde{f}\right)$$

- Hence, the *exact* APT specification
- In practice, actual returns do not display an exact factor structure, but there is some idiosyncratic or residual risk
- As a result, a portfolio of a few large factors cannot exactly replicate the return of a given stock
- The idiosyncratic risks are, however, often small: this is the reason to hope that the APT holds *approximately*

Representation of exact factor pricing

- MATLAB code: `apt.m`



- The observed assets span the payoff space $\mathbf{X}^f$ (**factor space**)
- A portfolio of factors $\beta_i'\mathbf{f}$ *exactly* identifies (and prices) payoff $\mathbf{x}^i$

Approximate APT

- Consider again (34), but this time with a residual

$$x^i = E\left(x^i\right) 1 + \beta_i' \tilde{f} + \varepsilon^i$$

- By the law of one price, take prices of both sides

$$p\left(x^i\right) = E\left(x^i\right) p\left(1\right) + \beta_i' p\left(\tilde{f}\right) + E\left(m\varepsilon^i\right)$$

- The question is to investigate the role of the price of the residual

$$p\left(\varepsilon^i\right) = E\left(m\varepsilon^i\right)$$

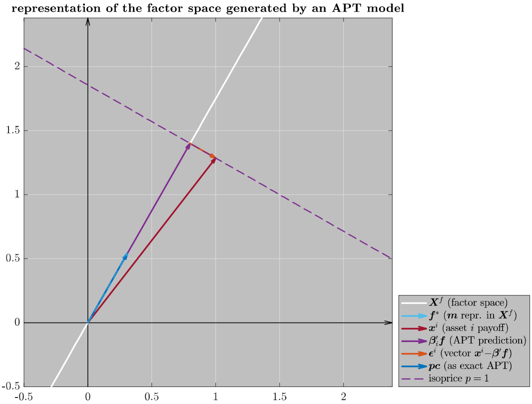
Exact pricing with approximate APT

- Payoffs whose residual ε^i is not zero are not in the factor space
- These payoffs are price correctly only if the vector representing m *happens* to coincide with f^*
- In this case, in fact, ε^i is by construction orthogonal to f^*
- Since by assumption f^* represents m in the state space

$$0 = E\left(m\varepsilon^i\right) = p\left(\varepsilon^i\right)$$

Representation of exact pricing with approximate APT

- MATLAB code: `apt.m`



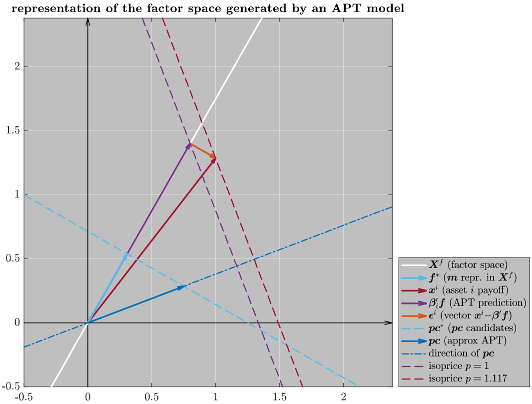
- If $\mathbf{pc} \equiv \mathbf{f}^*$, then the portfolio $\beta_i' \mathbf{f}$ *exactly* prices asset i even if the payoff $\mathbf{x}^i$ lies outside $\mathbf{X}^f$

Pricing error with approximate APT

- Unfortunately, the likelihood that $\mathbf{pc} \equiv \mathbf{f}^*$ is virtually null
- Even if that was the case, we would never be certain that the identity holds until we observe $\mathbf{x}^i$
- The discount factor f^* only prices correctly with certainty *observed* payoffs, i.e. those lying into $\mathbf{X}^f$
- The set of all possible discount factors that correctly price assets within $\mathbf{X}^f$ is the line $\mathbf{pc}^*$ perpendicular to $\mathbf{X}^f$ in $\mathbf{f}^*$
- The other candidate discount factors on line $\mathbf{pc}^*$ are not orthogonal to ε^i, so generate non-zero prices for it

Representation of pricing error with approximate APT

- MATLAB code: `apt.m`



- If $\mathbf{pc} \neq \mathbf{f}^*$, then the APT model generate the pricing error $p\left(\beta_i' \mathbf{f}\right) - E\left(mx^i\right)$ for a payoff $\mathbf{x}^i$ lying outside $\mathbf{X}^f$

Potential magnitude of the pricing error

- Moving along the line of discount factors m, any price from $-\infty$ to ∞ can in fact be generated for the residual
- As a result, the law of one price alone cannot pin down the price of the residual ε^i, nor the price of asset i
- It could be conjectured that the price of x^i may have to be *close to* the price of $\beta_i' f$
- This means that some appropriate *limit* may force the price of x^i to converge to the price of $\beta_i' f$
- In turn, this means that arbitrarily good accuracy can be achieved by going far enough in the direction of the limit

Theorems in support of the *limit* argument

Theorem

Fix a discount factor m that prices the factors. Then, as $\text{var}\left(\varepsilon^i\right) \longrightarrow 0$, $p\left(x^i\right) \longrightarrow p\left(\beta_i' f\right)$

- As the size of the ε^i vector gets smaller, x^i gets closer to $\beta_i' f$
- For any fixed m, the induced pricing function makes the price of x^i get closer and closer to $\beta_i' f$

Theorem

As the number of primitive assets increases, the variance of that portfolio falls towards zero

Interpretation of the theorems

- The two theorems can be interpreted in favour of the APT's pricing ability
- The APT holds approximately for well-diversified portfolios in large enough markets, and only using the law of one price
- These results are however obtained by fixing m and letting other things take limits
- Applying limits to m and fixing the elements of the observed assets turns the table downright
- For any nonzero residual ε^i, a discount factor m pricing the factors can be picked such that it assigns any price to x^i

Theorems in contradiction with the *limit* argument

Theorem

For any nonzero residual ε^i, there is a discount factor that prices the factors (consistent with the law of one price) and that assigns any desired price in $(-\infty, \infty)$ to the payoff x^i

- If you fix *m* and take limits of ε, the APT gets arbitrarily good
- If you fix ε, as one does in any application, the APT can get arbitrarily bad as you search over possible *m*

Additional Restrictions

- APT needs some restrictions beyond the law of one price
- Reasonable value may be imposed for the SDF, for instance by arbitrage ($m > 0$)
- This approach gives rise to finite upper and lower arbitrage bounds on the price of ε^i and hence x^i
- Arbitrage bounds are however too wide
  - assuming a positive m is equivalent to saying that a portfolio gives a higher payoff than another in every state of nature
  - state of nature are a great many, hence there are typically are no strictly dominating portfolios

Limiting SDF Variance

- A more effective strategy is limiting the variance of m
- A restricted range of discount factors produces a restricted range of prices for x^i

$$\min_{\{m\}} \left(\text{and} \quad \max_{\{m\}} \right) \quad p\left(x^i\right) = E\left(mx^i\right)$$
$$s.t. \quad E(mf) = p(f), \quad m \geq 0, \quad \sigma^2(m) \leq A$$

- This is the same as limiting the maximum Sharpe ratio available from portfolios of the factors and x^i

$$\frac{E(R^e)}{\sigma(R^e)} \leq \frac{\sigma(m)}{E(m)}$$

Ruling Out Good-Deal Opportunities

- The resulting bound can be interpreted as ruling out "good deals" as well as arbitrage opportunities

Theorem

As $\varepsilon^i \to 0$ and $\sigma^2\left(\varepsilon^i\right) \to 1$, the price $p\left(x^i\right)$ assigned by any discount factor m that satisfies $E\left(mf\right) = p\left(f\right)$, $m \geq 0$, $\sigma^2\left(m\right) \leq A$ approaches the factor portfolio price $p\left(\beta_i' f\right)$

Beta Representation for a Factor Pricing Model

- Compute the *coefficients* β in a multiple *time-series* regression of returns on factors

$$R_t^i = a_i + \beta_{i,a} f_t^a + \beta_{i,b} f_t^b + \cdots + \varepsilon_t^i, \qquad t = 1, 2, \ldots, T$$

- Use them into the *cross-sectional* model

$$E\left(R^i\right) = \gamma + \beta_{i,a}\lambda_a + \beta_{i,b}\lambda_b + \cdots, \qquad i = 1, 2, \ldots, N$$

  where $\beta_{i,a}$ measure the amount of exposure of asset i to factor a risks, and λ_a is the price of such risk exposure

- The risk-free rate is characterized by its beta being all zero (thereby called expected *zero-beta rate*), hence

$$R^f = \gamma$$