Esercizio 1

a) $\forall A, B \in W_1 \quad tr(A+B) = \underbrace{tr(A)}_{\substack{\| \\ 0}}^{A \in W_1} + \underbrace{tr(B)}_{\substack{\| \\ 0}}^{B \in W_1} = 0 + 0 = 0 \Rightarrow$

$A+B \in W_1$

$\forall A \in W_1 \quad \forall \lambda \in \mathbb{R} \quad tr(\lambda A) = \lambda\, tr(A) \underset{A \in W_1}{=} \lambda \cdot 0 = 0 \Rightarrow \lambda A \in W_1$

b) $W_1 = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a, b, c \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$

$= \left\{ a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$

$= L\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right)$

$\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$

$W_2 = L\left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right) =$

$= L\left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right)$

La somma NON è diretta perché $\mathbf{0} \neq \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in W_1 \cap W_2$

Si noti che

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \underset{\in W_1}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}} + \underset{\in W_2}{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}$$

$$= \underset{\in W_1}{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}} + \underset{\in W_2}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}$$

ESERCIZIO 2

$$\underset{B_c}{\mathbb{R}^2} \xrightarrow{f} \underset{B}{\mathbb{R}_2[x]} \xrightarrow{g} \underset{B'}{\mathbb{R}^3}$$

$B_c = \{ \overset{e_1}{(1,0)}, \overset{e_2}{(0,1)} \}$

$B = \{ \underset{v_1}{1-x}, \underset{v_2}{1+x^2}, \underset{v_3}{x} \}$

$B' = \{ \underset{v'_1}{(0,0,1)}, \underset{v'_2}{(1,0,1)}, \underset{v'_3}{(0,-1,0)} \}$

Cerco $M_{B_c B}(f)$.

$f(e_1) = f(1,0) = 1 + x^2 = v_2$

$f(e_2) = f(0,1) = 1 + x^2 = v_2$

$$M_{B_c B}(f) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$M_{B_c B'}(g \circ f) = M_{B B'}(g) \cdot M_{B_c B}(f) =$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 2 & 2 \end{pmatrix}$$

Allora $(g \circ f)(1,2) = x'_1 v'_1 + x'_2 v'_2 + x'_3 v'_3$ dove

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix} \Rightarrow (g \circ f)(1,2) = 3 \cdot (1,0,1) +$$

$$+ 6 \cdot (0,-1,0)$$

$$= (3,-6,3)$$

Esercizio 3

$$A = \begin{pmatrix} k & 2 & 0 \\ 1 & 1 & 0 \\ k & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \qquad A|B = \begin{pmatrix} k & 2 & 0 & k \\ 1 & 1 & 0 & 0 \\ k & 2 & 1 & 0 \\ 0 & 0 & 2 & k \end{pmatrix}$$

$$\det(A|B) \underset{\substack{\uparrow \\ 1^a \text{ RIGA} \to 1^a - 4^a}}{=} \det \begin{pmatrix} k & 2 & -2 & 0 \\ 1 & 1 & 0 & 0 \\ k & 2 & 1 & 0 \\ 0 & 0 & 2 & k \end{pmatrix} = k \cdot \det \begin{pmatrix} k & 2 & -2 \\ 1 & 1 & 0 \\ k & 2 & 1 \end{pmatrix}$$

$$\underset{\substack{\uparrow \\ 1^a \text{ RIGA} \to 1^a - 3^a}}{=} k \det \begin{pmatrix} 0 & 0 & -3 \\ 1 & 1 & 0 \\ k & 2 & 1 \end{pmatrix} = -3k \det \begin{pmatrix} 1 & 1 \\ k & 2 \end{pmatrix} = -3k(2-k)$$

$$= -3k \cdot (2-k)$$

Quindi per $k \neq 0$ e $k \neq 2$ $\rho(A|B) = 4 \neq \rho(A) \Rightarrow$ sistema incompatibile.

- Per $k = 0$

$$A = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \qquad A|B = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}$$

$$\underset{\substack{\parallel \\ \rho(A|B)}}{\rho(A) = 3} \quad \text{perché} \quad \det \begin{pmatrix} 0 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} = 1 \cdot \det \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} \neq 0$$

$\Rightarrow$ il sistema è compatibile e ammette un'unica soluzione, ed essendo il sistema omogeneo essa è la soluzione nulla $(0,0,0)$.

- Per $k=2$ $\qquad A = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \qquad A|B = \begin{pmatrix} 2 & 2 & 0 & 2 \\ 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 0 & 0 & 2 & 2 \end{pmatrix}$

$\rho(A) = 2$ (non può essere 3 perché ci sono due colonne uguali).

Passando a $A|B$ si ha che

$$\det \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 2 \end{pmatrix} = 2 \cdot \det \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = 2 \neq 0 \Rightarrow \rho(A|B) = 3$$

$\Rightarrow$ il sistema è incompatibile

Esercizio 4

$$V = \left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} : a,b,c \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} : a,b,c \in \mathbb{R} \right\}$$

$$= \left\{ a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} : a,b,c \in \mathbb{R} \right\}$$

$$= L\Big(\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{e_1}, \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{e_2}, \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{e_3} \Big)$$

e_1, e_2, e_3 sono lin. indipendenti, perché $\forall \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ t.c. che

$$\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 = \mathbf{0}$$

si ha

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_2 & \lambda_3 \end{pmatrix} \Rightarrow \begin{matrix} \lambda_1 = 0 \\ \lambda_2 = 0 \\ \lambda_3 = 0 \end{matrix}$$

Quindi $B = \Big\{ \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{e_1}, \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{e_2}, \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{e_3} \Big\}$ è base di V.

Trovo $M_B(f)$.

$$f(e_1) = f\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = e_2$$

$$f(e_2) = f\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = e_1 + e_3$$

$$f(e_3) = f\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = e_3$$

$$\Rightarrow M_B(f) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \Rightarrow$$

$$P(\lambda) = \det \begin{pmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 1 & 1-\lambda \end{pmatrix} = (1-\lambda) \cdot \det \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}$$

$$= (1-\lambda)(\lambda^2 - 1) = -(\lambda - 1)^2 (\lambda + 1)$$

Vi sono dunque due autovalori: 1 con $m_a(1) = 2$

-1 con $m_a(-1) = 1$

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \in V(1) \Leftrightarrow f\begin{pmatrix} a & b \\ b & c \end{pmatrix} = 1 \cdot \begin{pmatrix} a & b \\ b & c \end{pmatrix} \Leftrightarrow \begin{pmatrix} b & a \\ a & b+c \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} -a + b = 0 \\ b = 0 \end{cases} \quad c = s \qquad \Leftrightarrow \begin{cases} a = 0 \\ b = 0 \\ c = s \end{cases} \quad s \in \mathbb{R}$$

Quindi

$$V(1) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & s \end{pmatrix} : s \in \mathbb{R} \right\} = \mathcal{L}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow$$

$$m_g(1) = \dim V(1) = 1 \neq m_a(1)$$

$\Rightarrow f$ NON è diagonalizzabile