Università degli Studi di Cagliari

Corso di Laurea Magistrale in Ingegneria delle

Tecnologie per Internet

DIGITAL MODULATIONS

DETECTION

Detection

Binary Systems

→ **Coherent binary systems**

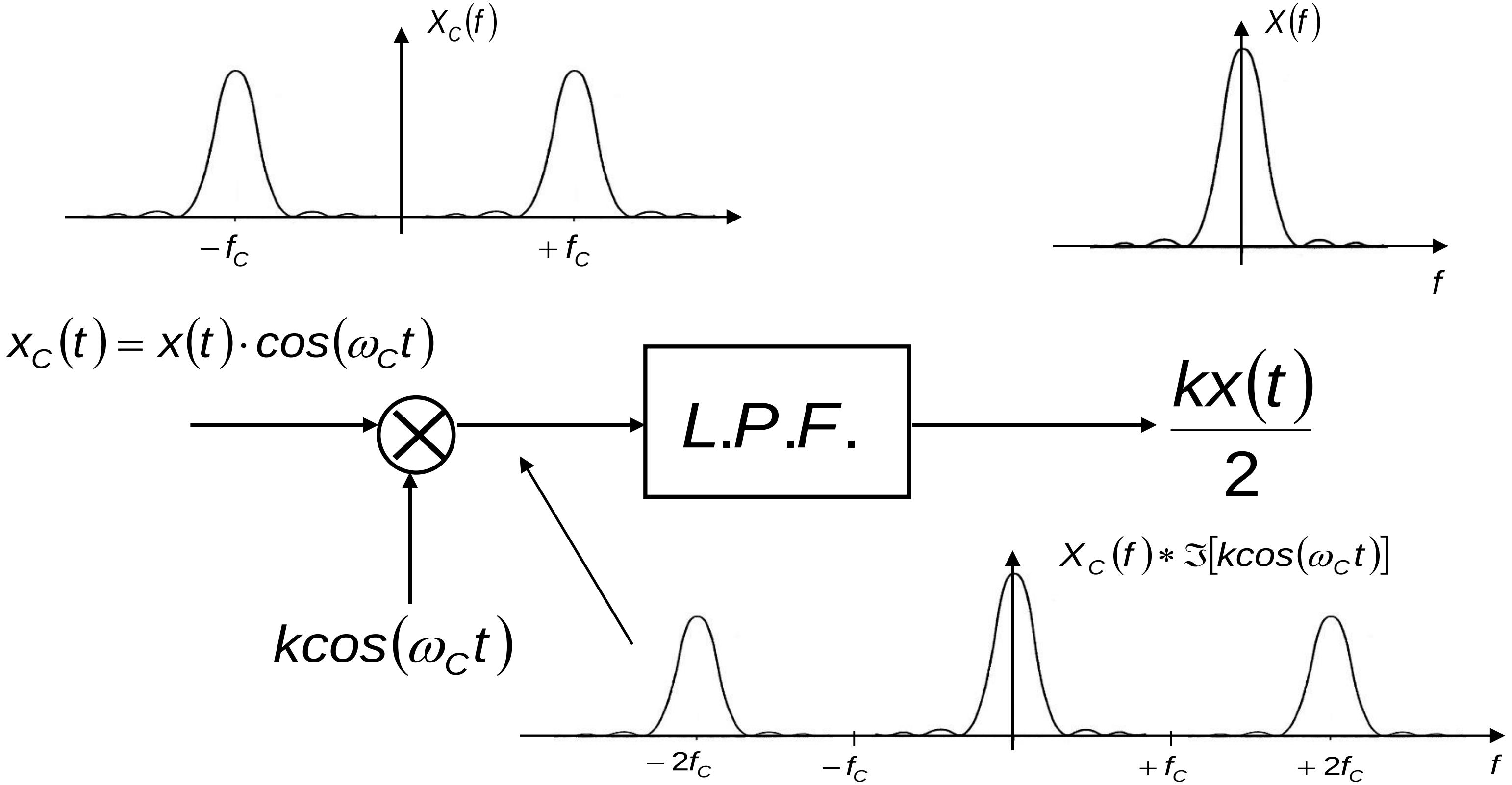
The receiver requires the carrier frequency and phase.

→ **Noncoherent binary systems**

The receiver does not require the carrier frequency and phase (envelope detection).

Coherent Binary Systems

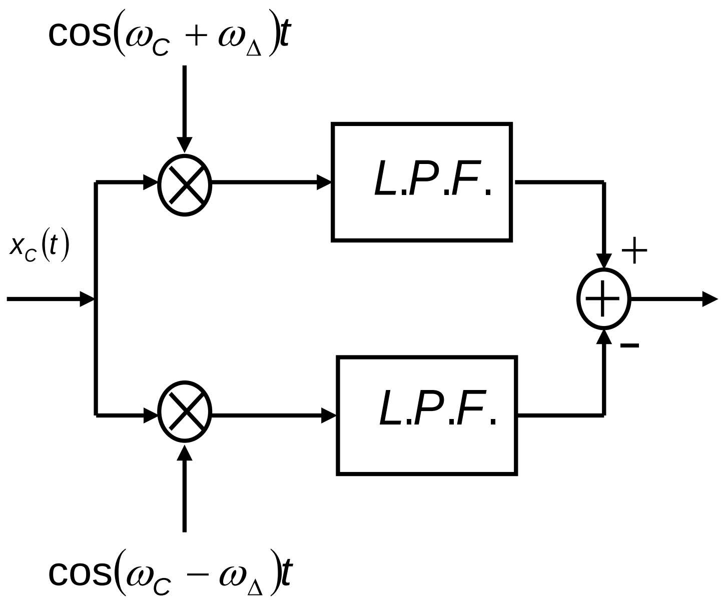
- ✓ **Ex**: PSK (the ASK spectrum has an impulse):



Coherent Binary Systems

✓ Binary FSK detection

Binary FSK consists of two interleaved OOK signals.



The difference between two outputs yields optimum the response

Coherent Binary Systems

✓ **Problems**

- Synchronization { Phase / Frequency }

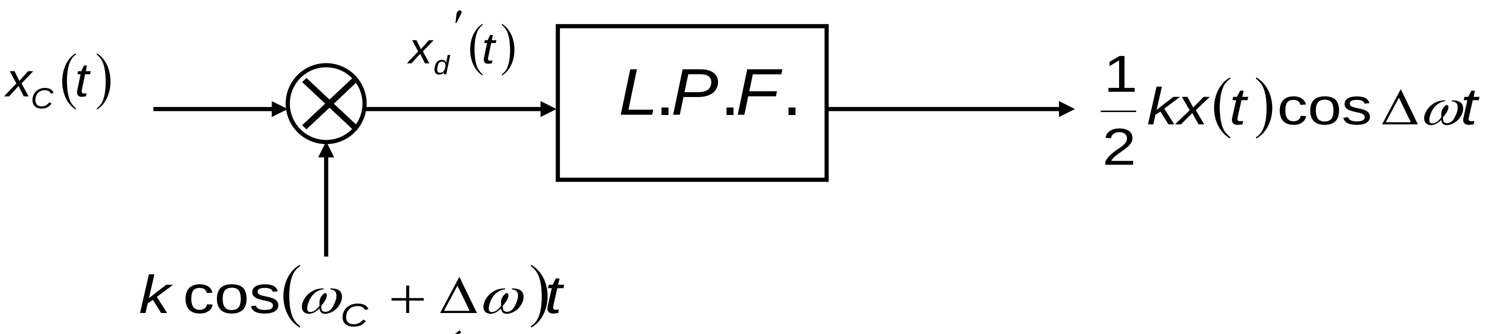
- Stability (due to thermal phenomena).

Coherent Binary Systems

- ✓ "Bad" **frequency** synchronization

Ex: ASK

$$x_C(t) = x(t)\cos\omega_C t$$



$$x_d'(t) = x(t)\cos\omega_C t \cdot k\cos(\omega_C + \Delta\omega)t = \frac{1}{2}kx(t)\{\cos(2\omega_C + \Delta\omega)t + \cos(\Delta\omega)t\}$$

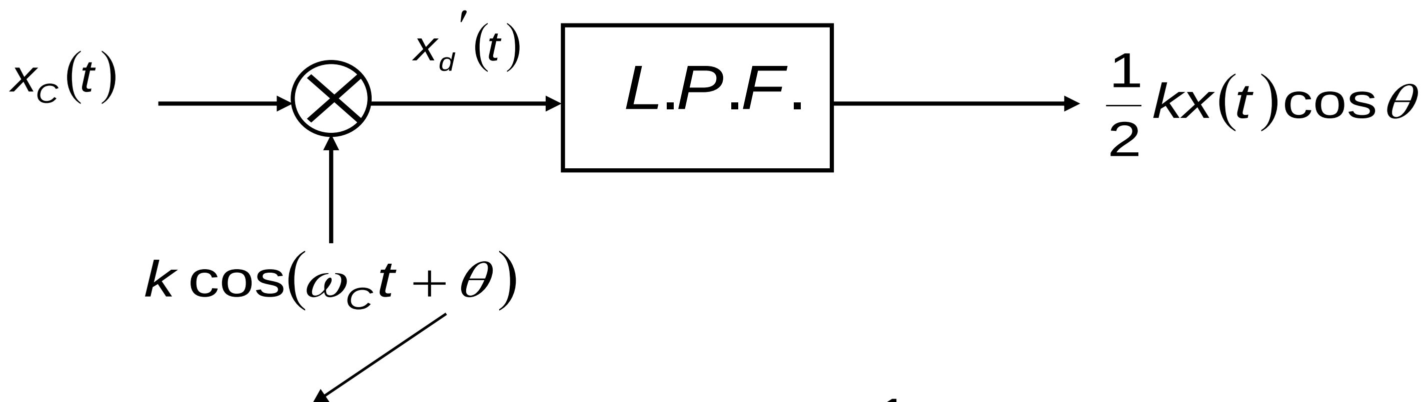
- ✓ Due to imperfect **frequency** synchronization

Coherent Binary Systems

- ✓ “Bad” **phase** synchronization

Ex: ASK

$$x_C(t) = x(t)\cos\omega_C t$$

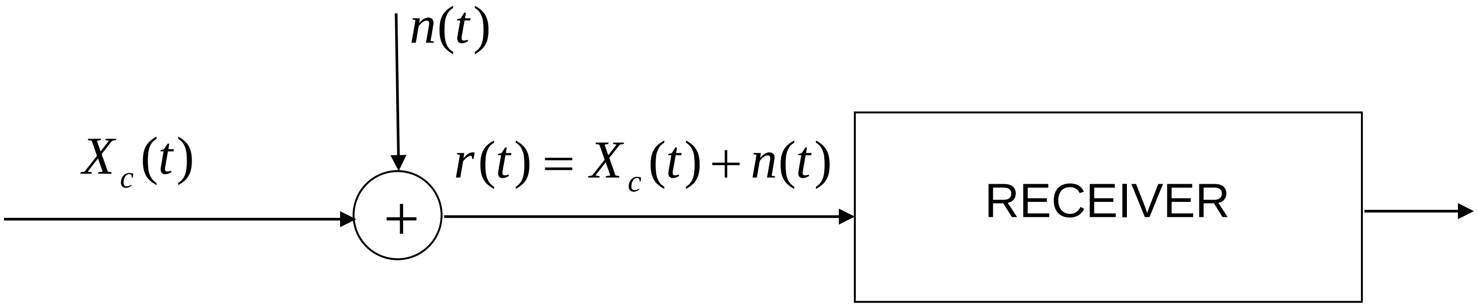


$$x_d{}'(t) = x(t)\cos\omega_C t \cdot k\cos(\omega_C t + \theta) = \frac{1}{2}kx(t)\{\cos(2\omega_C t + \theta) + \cos\theta\}$$

- ✓ Due to imperfect **phase** synchronization.

Decision Theory

- ✓ In general, the channel introduces a white Gaussian noise which corrupts the received signal



- ✓ We have to evaluate the influence of the noise on the $\Pr\{err\}$ for any modulation and receiver system.

Decision Theory

- ✓ The receiver will use the MAP criterion
  - ➢ We will handle a posteriori probabilities: given the received signal $r(t) = R(t)$ what is the probability to have transmitted the S_i symbol?

$$MAX_{S_i(t)} \quad \Pr\left\{\frac{S_i(t)}{r(t) = R(t)}\right\} = MAX_{S_i(t)} \quad \Pr\left\{S_i(t)\right\} \cdot \Pr\left\{\frac{r(t) = R(t)}{S_i(t)}\right\}$$

(see Bayes theorem)

- ✓ For any modulation system we have to find the pdf associated to each symbol, determine the decision region and then calculate the $\Pr\{err\}$

Optimum Binary Detection

- Any bandpass binary signal with keyed modulation can be expressed in the general quadrature-carrier form

$$X_c(t) = A_c\left\{\left[\sum_k I_k p_i(t-kT_b)\right]\cos(\omega_c t+\theta) - \left[\sum_k Q_k p_q(t-kT_b)\right]\sin(\omega_c t+\theta)\right\}$$

- For practical coherent systems, the carrier wave should be synchronized with the digital modulation.

$$f_c = \frac{N_c}{T_b} = N_c r_b \qquad (\mathrm{N_c} \text{ is usually a very large integer})$$

Optimum Binary Detection

- Accordingly, we will take $\theta = 0$ and impose the previous condition

$$X_c(t) = A_c \left\{ \left[\sum_k I_k p_i(t - kT_b) \right] \cos(t - kT_b) - \left[\sum_k Q_k p_q(t - kT_b) \right] \sin(t - kT_b) \right\}$$

- We can concentrate on a single bit interval by writing

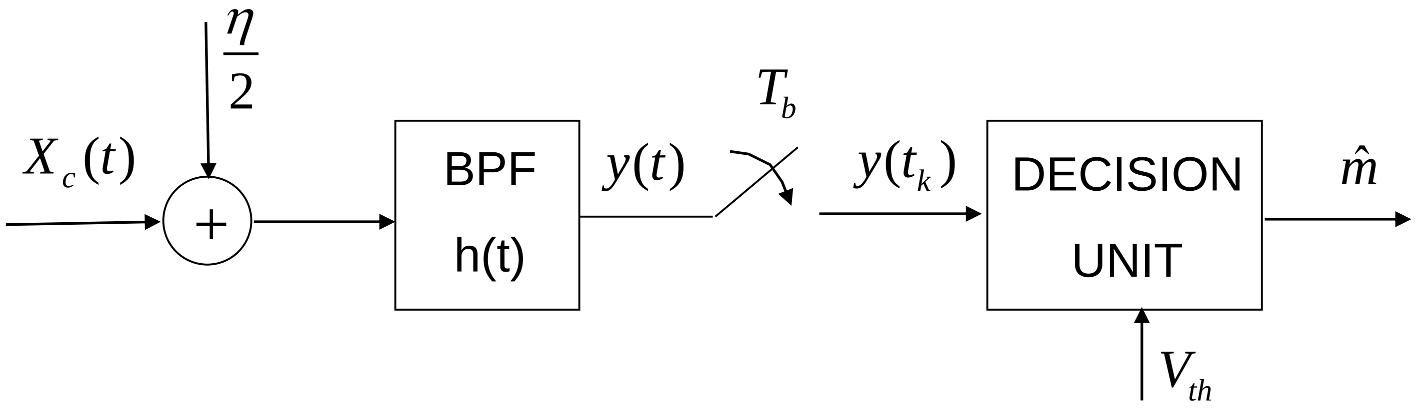
$$X_C(t) = S_m(t - kT_b) \qquad kT_b < t < (k+1)T_b$$

$$S_m \triangleq A_c\left[I_k p_i(t) cos\omega_c t - Q_k p_q(t) sin\omega_c t\right]$$

Optimum Binary Detection

- ✓ $S_m(t)$ stands for either of two signaling waveforms, $S_0(t)$ and $S_1(t)$ representing the message bits $m=0$ and $m=1$.

- ✓ Now consider the received signal $X_c(t)$ corrupted by AWGN.



Optimum Binary Detection

- The receiver decides between $m=0$ and $m=1$ according to the observed value of the random variable

$$Y = y(t_k) = z_m + n$$

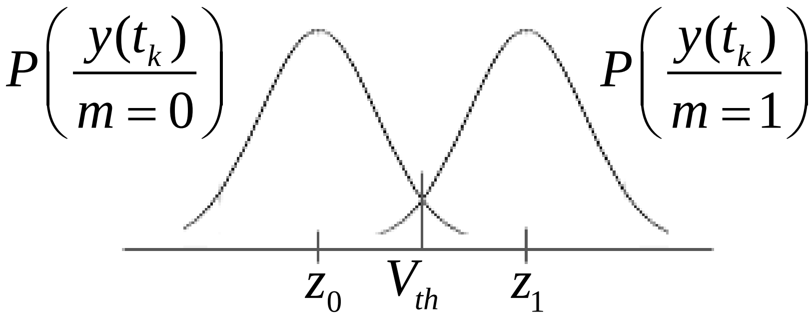
where

$$z_m \triangleq z_m(t_k) = \left[s_m(t-kT_b) * h(t)\right]\Big|_{t=t_k} = \int_{kT_b}^{(k+1)T_b} s_m(\xi - kT_b) * h(t_k - \xi)\,d\xi =$$

$$= \int_0^{T_b} s_m(\xi) h(T_b - \xi)\,d\xi$$

Optimum Binary Detection

- ✓ The noise sample $n = n(t_k)$ is a Gaussian RV with zero mean and variance σ^2



$$V_{th} = \frac{z_1 + z_0}{2} \Rightarrow P\left(\frac{err}{z_1}\right) = P\left(\frac{err}{z_0}\right) = Q\left(\frac{|z_1 - z_0|}{2\sigma}\right)$$

In which $m = 0$ and $m = 1$ are equally likely and the absolute-value notation includes the case of $z_1 < z_0$

Optimum Binary Detection

- What BPF impulse response h(t) maximizes the ratio $\frac{|z_1 - z_0|}{2\sigma}$ or, equivalently $\frac{|z_1 - z_0|^2}{4\sigma^2}$? To answer this question we note that:

$$|z_1 - z_0|^2 = \left| \int_{-\infty}^{+\infty} [s_1(\xi) - s_0(\xi)] h(T_b - \xi) d\xi \right|^2$$

where the infinite limits are allowed since $S_m(t) = 0$ outside of $0 < t < T_b$

$$\sigma^2 = \frac{\eta}{2} \int_{-\infty}^{+\infty} |h(t)|^2 dt = \frac{\eta}{2} \int_{-\infty}^{+\infty} |h(T_b - \xi)|^2 d\xi$$

Optimum Binary Detection

$$\frac{|z_1 - z_0|^2}{4\sigma^2} = \frac{\left| \int_{-\infty}^{+\infty} [s_1(\xi) - s_0(\xi)]\, h(T_b - \xi) d\xi \right|^2}{4 \cdot \frac{\eta}{2} \cdot \int_{-\infty}^{+\infty} \left| h(T_b - \xi) \right|^2 d\xi}$$

- ✓ Application of Schwarz's inequality yields:

$$\frac{|z_1 - z_0|^2}{4\sigma^2} \leq \frac{\int_{-\infty}^{+\infty} |s_1(\xi) - s_0(\xi)|^2\, d\xi \int_{-\infty}^{+\infty} \left| h(T_b - \xi) \right|^2 d\xi}{2\eta \int_{-\infty}^{+\infty} \left| h(T_b - \xi) \right|^2 d\xi} =$$

$$= \frac{1}{2\eta} \int_{-\infty}^{+\infty} |s_1(\xi) - s_0(\xi)|^2\, d\xi$$

Optimum Binary Detection

- Ratio is maximum if:

$$h_{opt}(t) = k[s_1(T_b - t) - s_0(T_b - t)]$$

with k an arbitrary constant.

- The filter for optimum detection should be matched to the dirrefence between the two signaling waveforms.

Optimum Binary Detection

- ✓ The error probability with optimum binary detection depends on:

$$\int_0^{T_b} [s_1(t) - s_0(t)]^2 dt = E_1 + E_0 + 2E_{10}$$

where $E_1 \triangleq \int_0^{T_b} s_1^2(t)dt$ and $E_0 \triangleq \int_0^{T_b} s_0^2(t)dt$

are the respective energies of the signals, while $E_{10} \triangleq \int_0^{T_b} s_1(t)s_0(t)dt$ is proportionals to their correlation coefficient.

Optimum Binary Detection

- ✓ Since zeros and ones are equally likely, the average signal energy per bit is:

$$E_b = \frac{E_1 + E_0}{2}$$

so:

$$\left(\frac{z_0 - z_1}{2\sigma}\right)^2_{MAX} = \frac{E_1 + E_0 - 2E_{10}}{2\eta} = \frac{E_b - E_{10}}{\eta}$$

and:

$$P(err) = Q[\sqrt{(E_b - E_{12})/\eta}]$$

Optimum Binary Detection

$$\left.\begin{aligned} z_m &= \int_0^{T_b} s_m(\xi)\, h(T_b - \xi) d\xi \\ h_{opt}(t) &= k[s_1(T_b - t) - s_0(T_b - t)] \end{aligned}\right\} \Rightarrow \begin{aligned} z_1 &= k(E_1 - E_{10}) \\ z_0 &= k(E_{10} - E_0) \end{aligned}$$

so:

$$V_{th} = \frac{z_1 + z_0}{2} = \frac{k}{2}(E_1 - E_0)$$

Note that the optimum threshold does not involve E_{10}

Optimum Binary Detection In Ook

$$OOK \rightarrow \begin{cases} s_0(t) = 0 \\ s_1(t) = A_c g_i(t) \cos \omega_c t \end{cases}$$

✓ The carrier-frequency condition $f_c = N_c / T_b$ means that, for every bit interval:

$$s_0(t - kT_b) = 0$$
$$s_1(t - kT_b) = A_c \cos \omega_c t$$

so:

$$E_0 = E_{10} = 0 \qquad E_1 = A_c^{\,2} \int_0^{T_b} \cos^2 \omega_c t dt = \frac{A_c^{\,2} T_b}{2}$$

$$E_b = \frac{E_1}{2} = \frac{A_c^{\,2} T_b}{4} \quad \Rightarrow \quad V_{th} = \frac{k(E_1 - E_0)}{2} = kE_b$$

Optimum Binary Detection In Ook

- Defyning:

$$\gamma_b = \frac{E_b}{\eta}$$

as the ratio between average bit energy and noise power spectral power,we obtain:

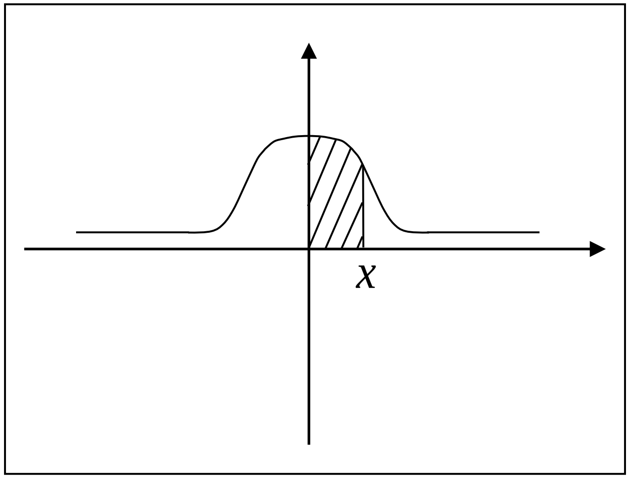
$$P(err) = Q(\sqrt{E_b / \eta}) = Q(\sqrt{\gamma_b})$$

with:

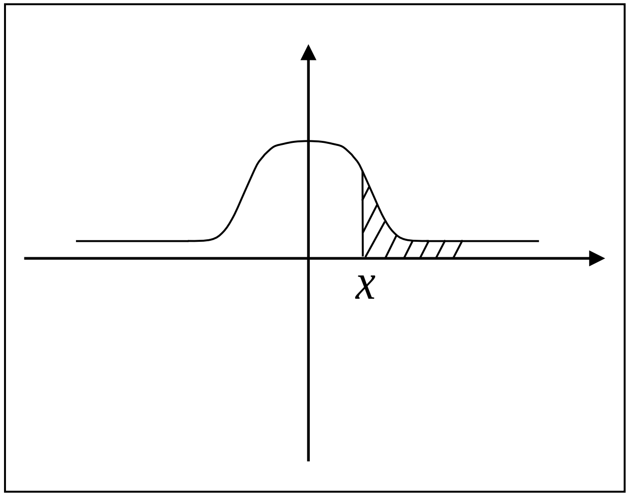
$$E_b = \frac{A_c^2 T_b}{4}$$

Signal Positioning

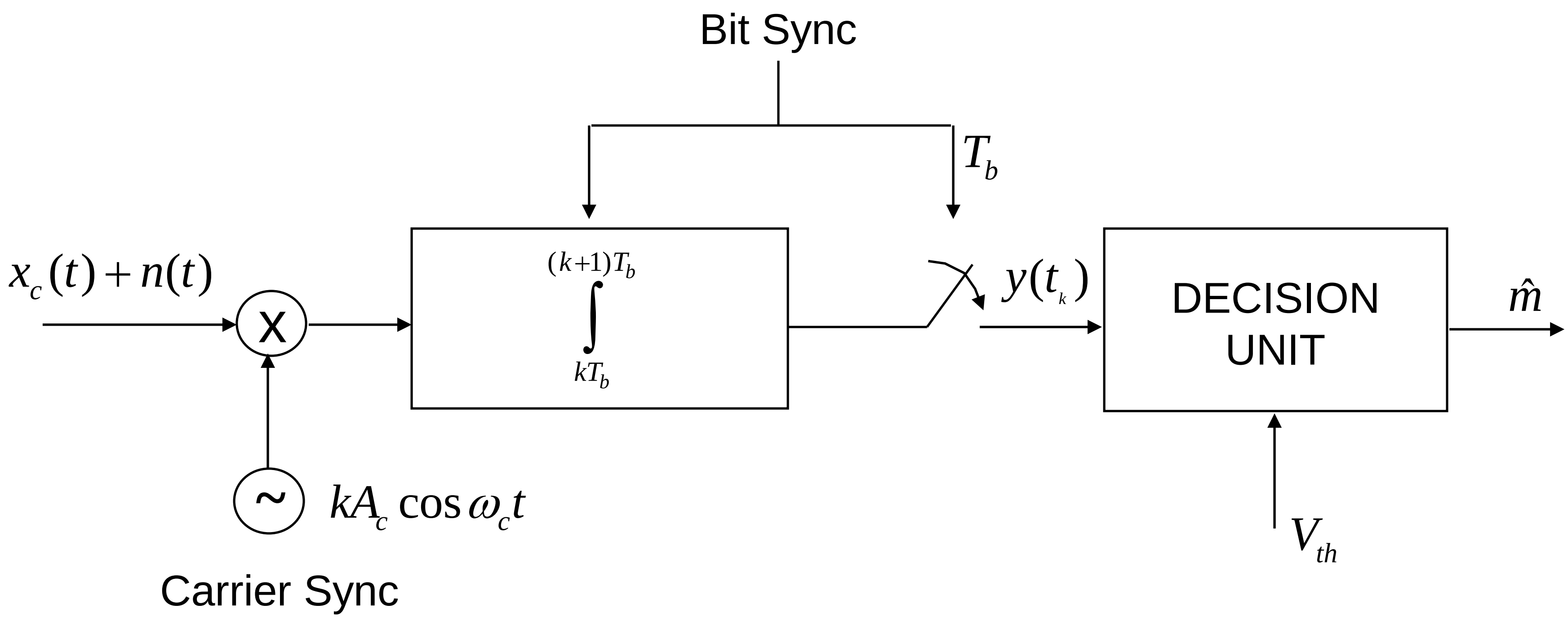
$$erf(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{\xi^2}{2}} d\xi \Rightarrow$$



$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{\xi^2}{2}} d\xi = \frac{1}{2} - erf(x) \Rightarrow$$



Optimum Binary Detection In Ook



Optimum Binary Detection

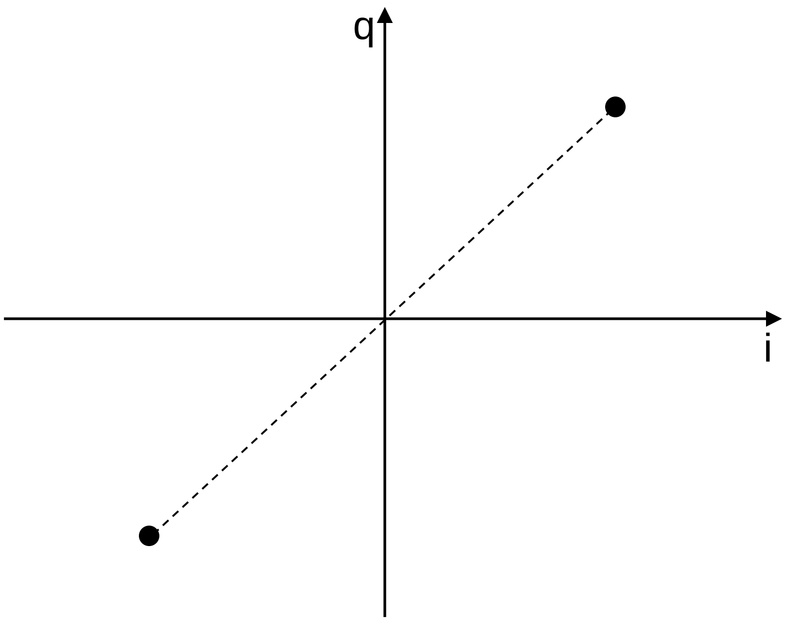
- ✓ Without whole calcutations, let’s see how to quantify $\Pr\{err\}$ in the main ditigal modulations we have studied.

- ✓ The Error Probability is expressed in terms of:

$$\gamma_b \triangleq \frac{E_b}{\eta} \qquad\qquad E_b \triangleq \frac{S_R}{r_b}$$

PRK

- It is a particular case of Binary PSK in which phase can have a shift of $\pm\pi$ radiants (Phase-Reversal Keying).



Optimum Binary Detection

✓ Coherent detection:

OOK $\Rightarrow$ $P_{be} = Q(\sqrt{\gamma_b})$

PRK $\Rightarrow$ $P_{be} = Q(\sqrt{2\gamma_b})$

Sunde's FSK $\Rightarrow$ $P_{be} \leq Q(\sqrt{1.22\,\gamma_b})$

QAM, QPSK $\Rightarrow$ $P_{be} = Q(\sqrt{2\gamma_b})$

Optimum Binary Detection

- ✓ Non Coherent detection:

OOK $\Rightarrow$ $$P_{be} = \frac{1}{2} e^{-\frac{\gamma_b}{2}}$$

FSK $\left(f_\Delta = \frac{r_b}{2} \right)$ $\Rightarrow$ $$P_{be} = \frac{1}{2} e^{-\frac{\gamma_b}{2}}$$

Non coherent PSK and QAM can not be used.

Comparison of Digital Modulations

- A performance comparison of digital modulation systems should consider several factors including:

  - Transmission bandwith B_T
  - Spectral efficiency r_b / B_T
  - Error Probability
  - Hardware Complexity

Comparison of Digital Modulations

✓ Binary digital modulation systems

Modulation	*Detector*	r_b / B_T	P_{be}
OOK and FSK $\left(f_\Delta = \frac{r_b}{2}\right)$	Envelope	1	$\frac{1}{2} e^{-\frac{\gamma_b}{2}}$
PRK	Coherent	1	$Q(\sqrt{2\gamma_b})$
QAM,QPSK	In Quadrature Coeherent	2	$Q(\sqrt{2\gamma_b})$

Comparison of Digital Modulations

✓ Example: comparison of Digital Modulation System with $P_{be} = 10^{-4}$

Modulation	*Detector*	r_b / B_T	$\gamma_b [dB]$
OOK and FSK $\left(f_\Delta = \frac{r_b}{2} \right)$	Envelope	1	12.3
PRK	Coherent	1	8.4
QAM,QPSK	In Quadrature Coherent	2	8.4
PSK (M=8)	In Quad. Co.	3	11.8
PSK (M=16)	In Quad. Co.	4	16.2

M-QAM error probability