Università degli Studi di Cagliari

Corso di Laurea Magistrale in Ingegneria delle Tecnologie per Internet

DIGITAL MODULATIONS

INTRODUCTION

Overview

Modulation is a nonlinear operation that allows to adapt the signal to be transmitted to the channel features.

Example:

$$x(t)\cdot p(t) = x_c(t)$$

- $x(t)$ = modulating signal
- $p(t) = A_c \cos\omega_c t$ (carrier)
- $x_c(t)$ = modulated signal

It does not exist the $H(\omega)$ of a modulator.

Overview

Linear Modulations $\Rightarrow$ the modulating signal spectrum is frequency translated without being changed.

Exponetial Modulations $\Rightarrow$ the signal *x(t)* modulates the angle of the carrier *ϑ(t):*

$$x_c(t) = A_c \cos\theta(t) \quad \theta(t) = \omega_c t + \varphi(t)$$

In general, a modulated signal (linear or angular) may be written as:

Band-pass signal $\Rightarrow$ $$x_c(t) = A(t)\cos[\omega_c t + \varphi(t)]$$

Outlines of Fourier transform

$$X(\omega)=\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt \overset{\Im^{-1}}{\longleftrightarrow} x(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)e^{j\omega t}d\omega$$

$$X(f)=\int_{-\infty}^{+\infty} x(t)e^{-j2\pi ft}dt \overset{\Im^{-1}}{\longleftrightarrow} x(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(f)e^{j2\pi\omega f}df \qquad \omega=2\pi f$$

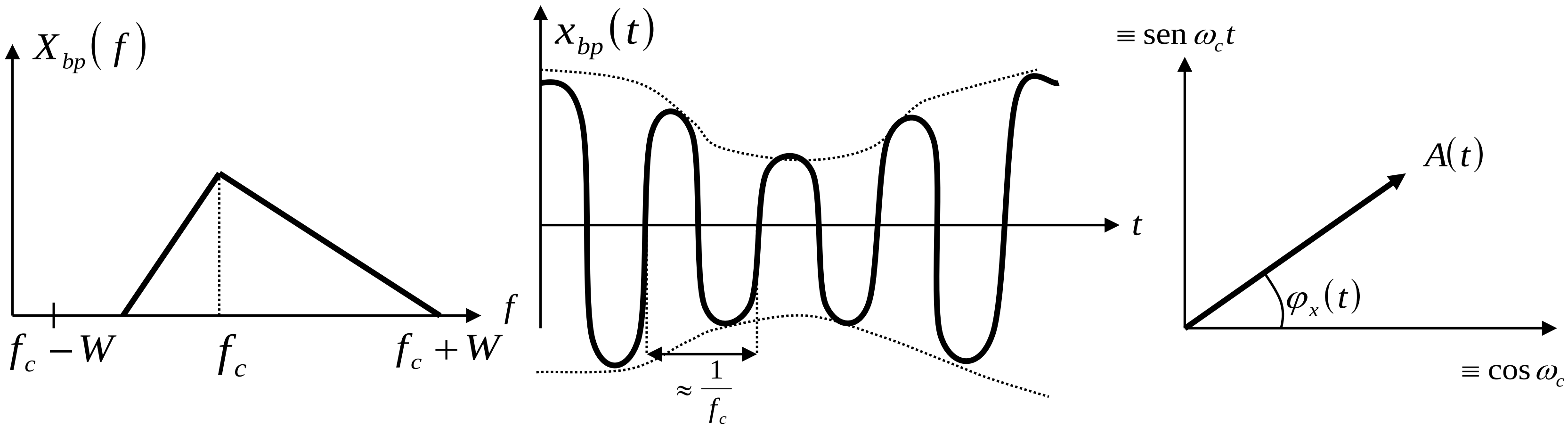
Examples:

$$x_1(t)\cdot x_2(t) \leftrightarrow \begin{cases} \frac{1}{2\pi} X_1(\omega)* X_2(\omega) \\ X_1(f)* X_2(f) \end{cases}$$

$$\cos\omega_0 t \leftrightarrow \begin{cases} \pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right] \\ \frac{1}{2}\left[\delta(f-f_0)+\delta(f+f_0)\right] \end{cases}$$

$$\sin\omega_0 t \leftrightarrow \begin{cases} -j\pi\left[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\right] \\ -\frac{j}{2}\left[\delta(f-f_0)-\delta(f+f_0)\right] \end{cases}$$

Band-pass signals



$$x_{bp}(t) = A(t)\cos[\omega_c t + \varphi_x(t)]$$

$$x_{bp}(t) = \overbrace{A(t)\cos\varphi_x(t)}^{x_i(t)}\cos\omega_c t - \overbrace{A(t)\sin\varphi_x(t)}^{x_q(t)}\sin\omega_c t =$$

$$= x_i(t)\cos\omega_c t - x_q(t)\sin\omega_c t$$

Band-pass signals

$$X_{bp}(f) = \frac{1}{2}\left[X_i(f - f_c) + X_i(f + f_c)\right] + \frac{1}{2}j\left[X_q(f - f_c) - X_q(f + f_c)\right] =$$

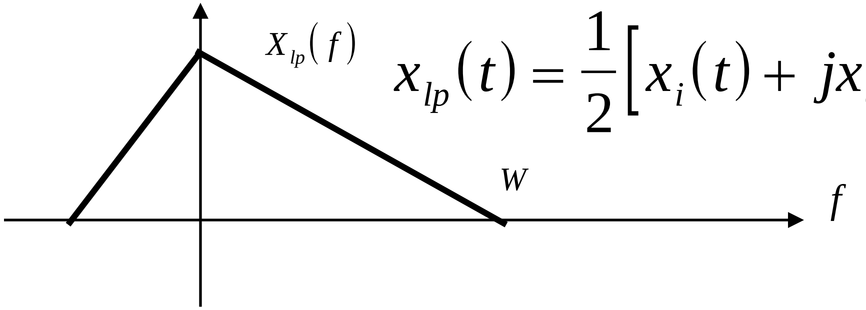
$$= \frac{1}{2}\left\{X_i(f - f_c) + jX_q(f - f_c)\right\} + \frac{1}{2}\left\{X_i(f + f_c) - jX_q(f + f_c)\right\}$$

The $X_{bp}(f)$ is composed by a part which is around $-f_c$ and by a part which is around $+f_c$. It is defined as the equivalent low-pass signal the part of the band-pass signal around $+f_c$ shifted down the origin:

$$X_{lp}(f) = \frac{1}{2}\left\{X_i(f) + jX_q(f)\right\} \quad |f| < f_c$$

Band-pass signals

In the time domain we have:



$$x_{lp}(t) = \frac{1}{2}\left[x_i(t) + jx_q(t)\right] = \frac{1}{2}A(t)e^{+j\varphi(t)}$$ ⇐ COMPLEX SIGNAL

Given the initial polar representation and the $x_{lp}(t)$ it may be written:

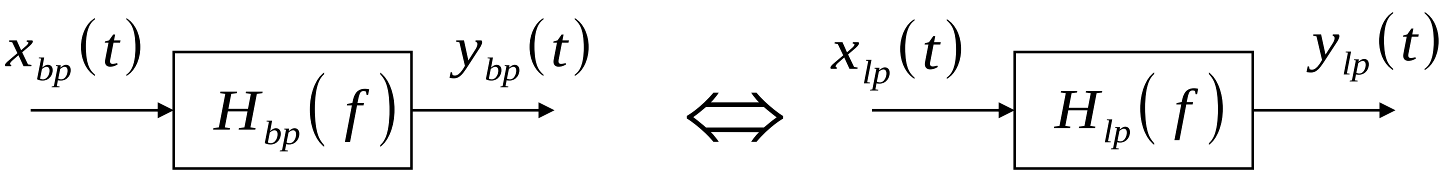
$$x_{bp}(t) = \mathrm{Re}\left\{A(t)e^{j[\omega_c t+\varphi(t)]}\right\} = \mathrm{Re}\left[2x_{lp}(t)e^{j\omega_c t}\right]$$

We may build up a low-pass representation from a band-pass representation and vice versa.

Band-pass signals

The low-pass representation is a "virtual" representation (i.e., it does not practically exist).

- ***Equivalent band-pass ⇔ low-pass:***



- We can work with the equivalent low-pass:

$$x_{bp}(t) = 2\left[\mathrm{Re}\{x_{lp}(t)\}\cos\omega_c t - \mathrm{Im}\{x_{lp}(t)\}\sin\omega_c t\right]$$

Band-pass signals

$$A(t) = 2\left|x_{lp}(t)\right| \qquad \varphi(t) = \arg\left[x_{lp}(t)\right]$$

$$Y_{lp}(\omega) = X_{lp}(\omega) \cdot H_{lp}(\omega) \Leftrightarrow y_{lp}(t)$$

From $y_{lp}(t)$ we can get, as mentioned above, $y_{bp}(t)$.

NOTE: we can get the same low-pass signal representation from many different band-pass representations (if only f_c changes).

Digital Modulation

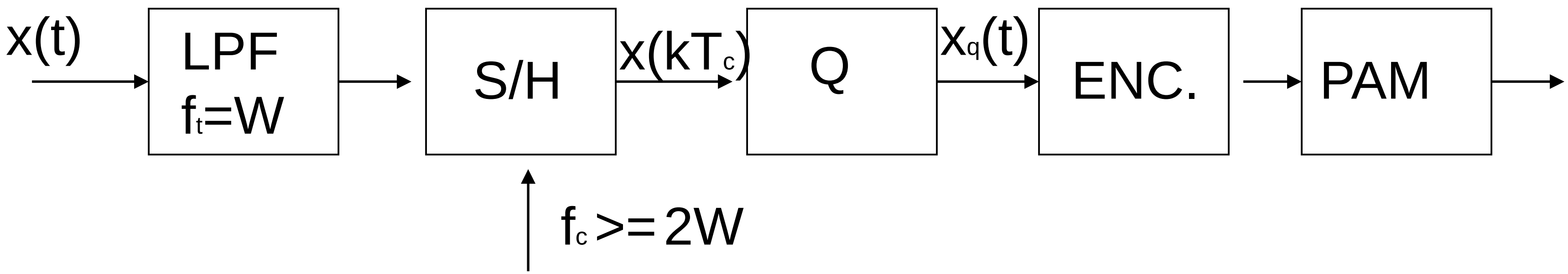
- ✓ An analogical signal can be transmitted in translated band using CW modulations (AM, FM).
- ✓ Modulation allows to fit the signal peculiarities to the channel characteristic.
- ✓ Digital modulations allow to transmit a numerical signal on translated band.

Numerical Transmission

- ✓ Digital message ⟶ Sequence of symbols generated by a discrete source.

- ✓ The discrete source can be an analogical signal changed into digital form (i.e. : Pulse Code Modulation).

Numerical Transmission



X(t) → Analogical signal

S/H → Sample & hold

LPF → Low Pass Filter with cut-off frequency f_t=W

Q → Quantizer

Numerical Transmission

- ✓ Encoder ⟶ associates to each quantization level a "codeword".
- ✓ PAM ⟶ Pulse Amplitude Modulation.

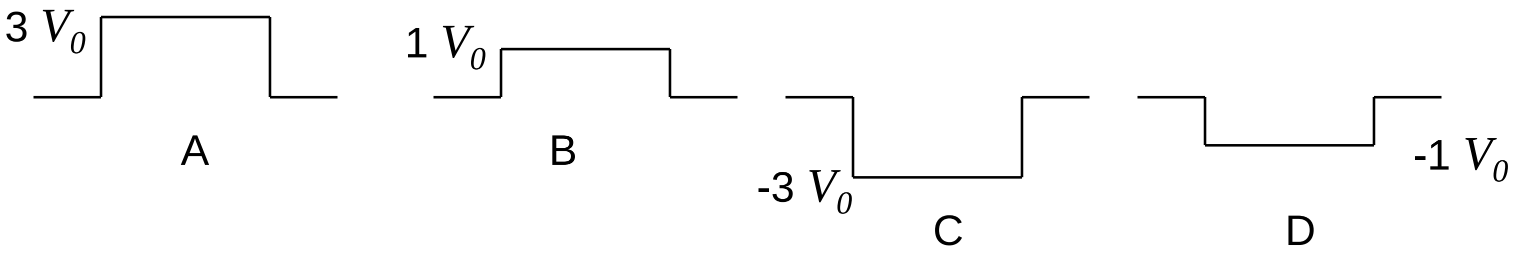
  Associates a particular waveform to a set of M bits.
- ✓ Such signal can be transmitted in base-band.

PAM

✓ Basic concept: the PAM coding technique uses a single wave form to encode the PCM (or another generic discrete source) output levels.

✓ The different levels are coded using the wave form amplitude.

✓ **Ex:** ***4 LEVELS, RECTANGULAR WAVE FORM***



PAM Spectrum

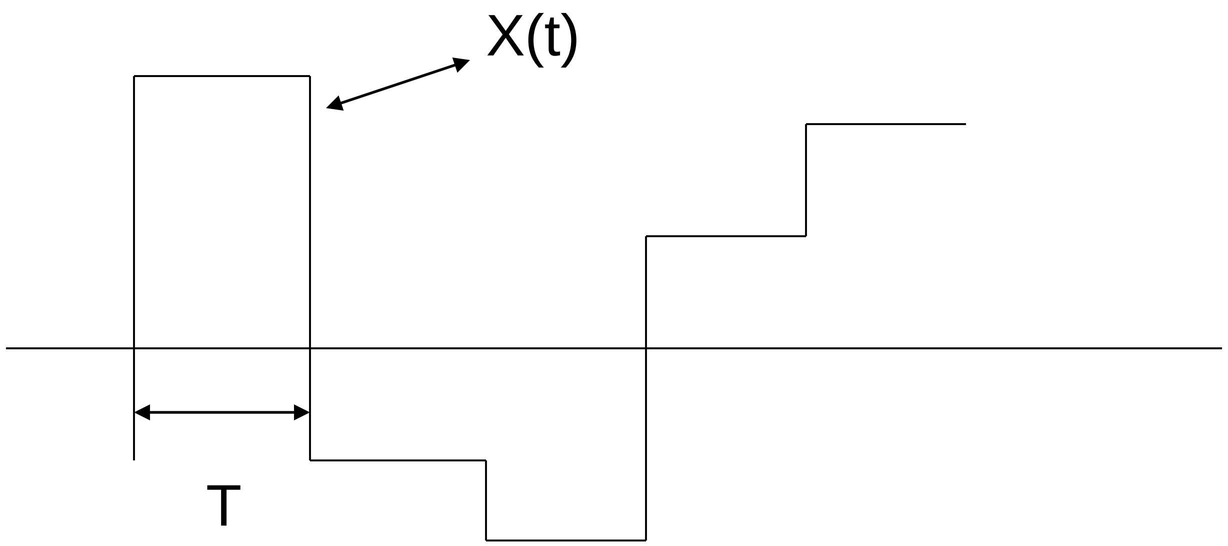
- If $g(t)$ is a PAM waveform (i.e., rectangular shape) and a_k are uncorrelated with $\overline{a_k} = 0$:

$$E\{a_k a_i\} = \begin{cases} \sigma_a^2 & k = i; \\ 0 & k \neq i; \end{cases}$$

- We want to obtain the power spectrum of a digital PAM signal *x(t).*

PAM Spectrum

✓ We consider $g(t)$ as a rectangular shape, then:



$E\{x(t)\} = E\{a_k\} = 0;$ $E\{x^2(t)\} = E\{(a_k)^2\} = \sigma_a^2$

PAM Spectrum

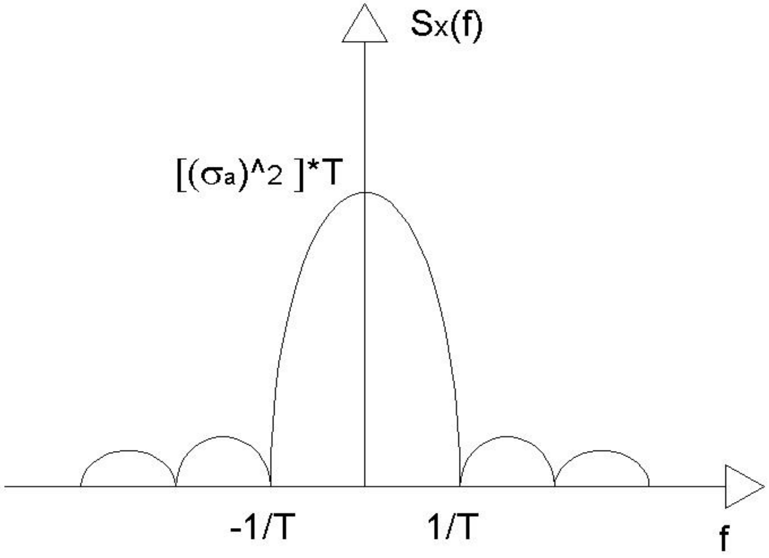
- The spectrum of a random process can be calculated by the Fourier transform of the autocorrelation function:

$$S_x(f) = \Im\left[R_x(\tau)\right] = \frac{\sigma_a^2}{T}\left|G(f)\right|;$$

PAM Spectrum

- ✓ In this case:

$$S_x(f) = \Im[R_x(\tau)] \Rightarrow \int_{-\infty}^{+\infty} R_x(\tau) e^{-j\omega\tau} d\tau = \sigma_a^2 T \mathrm{sinc}^2(\pi f T) = S_x(f);$$



PAM Spectrum

- To evaluate the general case with $E\{a_k\} \neq 0$ we consider again the autocorrelation function which becomes:

$$R_a(n) = E\{a_k a_{k-n}\};$$

- Equal to $R_x = E\{x(t)x(t+\tau)\} = E\{x(t)x(t-\tau)\}$ for stationary process.

PAM Spectrum

- The power spectrum of a PAM signal $x(t)$ can be expressed as:

$$S_x(f) = \frac{1}{T} |G(f)|^2 \sum_{n=-\infty}^{+\infty} R_a(n) e^{-j2\pi nfT};$$

- Hp: uncorrelated symbols, but $E\{a_k\} = m_a \neq 0$;

PAM Spectrum

- Therefore:

$$\sum_{n=-\infty}^{+\infty} R_a(n) \cdot e^{-j2\pi nfT} = \sigma_a^2 + m_a^2 \sum_{n=-\infty}^{+\infty} e^{-j2\pi nfT}$$

- By the Poisson's formula:

$$\sum_{n=-\infty}^{+\infty} e^{-j2\pi nfT} = \frac{1}{T} \sum_{n=-\infty}^{+\infty} \delta(f - \frac{n}{T})$$

PAM Spectrum

- ✓ We obtain:

$$S_x(f) = \sigma_a^2 r |G(f)|^2 + (m_a r)^2 \sum_{n=-\infty}^{+\infty} |G(nr)|^2 \delta(f - nr)$$

where we have inserted r=1/T and used the sampling property of impulse multiplication.

Bandpass Signals

- A band-pass modulated signal can be written as:

$$X_c(t) = A_c[X_i(t)\cos(\omega_c t + \theta)]$$

where, for every realization, θ is constant and uniformly distributed.

Bandpass Signals

$$X_c(t) = A_c[x_i(t)\cos(\omega_c t + \phi) - x_q(t)sen(\omega_c t + \phi)]$$

- ✓ The message is contained in $x_i(t)$ and $x_q(t)$
- ✓ To find the power spectral density, we must calculate the autocorrelation function
- ✓ **Example**

$$X_c(t) = X(t)\cos(\omega_c t + \phi)$$

(...)

$$R_{x_c}(t_1,t_2) = E\{X_c(t_1)X_c(t_2)\}$$

Bandpass Signals

(…example)

$$= \frac{1}{2} E\{X(t_1)X(t_2)\}\{\cos[\omega_c(t_1 - t_2)] + E\{\cos[\omega_c(t_1 + t_2) + 2\phi]\}\}$$

$$= \frac{1}{2} R_x(t_1, t_2)\{\cos[\omega_c(t_1 - t_2)] + 0\}$$

Bandpass Signals

- If $x(t)$ is a SSL process then:

$$R_{x_c}(\tau) = \frac{1}{2} R_x(\tau) \cos 2\pi f_c \tau$$

$$S_{x_c}(f) = \Im[R_{x_c}(\tau)] = \frac{1}{4}[S_x(f - f_c) + S_x(f + f_c)]$$

- The spectrum of a modulated process is the same as $x(t)$ base-band spectrum shifted by $\pm f_c$ and multiplied to 1/4.

Bandpass Signals

- In general, assuming that the x_i and x_q components are independent (and at least one of them which has zero mean) because of the superposition effects we have:

$$X_c(t) = A_c[X_i(t)\cos(\omega_c t + \phi) - X_q(t)\sin(\omega_c t + \phi)]$$

$$S_{x_c}(f) = \frac{A_c^2}{4}[S_{x_i}(f - f_c) + S_{x_i}(f + f_c) + S_{x_q}(f - f_c) + S_{x_q}(f + f_c)]$$

Bandpass Signals

- ✓ We define the low-pass equivalent spectrum as:

$$S_{lp} = S_{x_i}(f) + S_{x_q}(f)$$

- ✓ Then:

$$S_{x_c}(f) = \frac{A_c^2}{4}[S_{lp}(f - f_c) + S_{lp}(f + f_c)]$$

- ✓ We can obtain the pass-band spectrum from the low-pass equivalent one by shifting it at $\pm f_c$.