# Operazioni tra Vettori

- Prodotto scalare e vettoriale

$$\underline{A}\cdot\underline{B} = A_x B_x + A_y B_y + A_z B_z$$

$$\underline{A}\times\underline{B} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ \underline{i}_x & \underline{i}_y & \underline{i}_z \end{vmatrix} = \begin{vmatrix} A_r & A_\theta & A_\phi \\ B_r & B_\theta & B_\phi \\ \underline{i}_r & \underline{i}_\theta & \underline{i}_\phi \end{vmatrix} = \begin{vmatrix} A_R & A_\phi & A_z \\ B_R & B_\phi & B_z \\ \underline{i}_R & \underline{i}_\phi & \underline{i}_z \end{vmatrix}$$

$$\underline{A}\times\underline{B} = A_x \begin{vmatrix} B_y & B_z \\ \underline{i}_y & \underline{i}_z \end{vmatrix} - A_y \begin{vmatrix} B_x & B_z \\ \underline{i}_x & \underline{i}_z \end{vmatrix} + A_z \begin{vmatrix} B_x & B_y \\ \underline{i}_x & \underline{i}_y \end{vmatrix}$$

- Proprietà

$$\underline{A}\cdot\underline{B} = \underline{B}\cdot\underline{A}$$

$$\underline{A}\times\underline{B} = -\underline{B}\times\underline{A}$$

$$\underline{A}\cdot\left(\underline{B}+\underline{C}\right) = \underline{A}\cdot\underline{B} + \underline{A}\cdot\underline{C}$$

$$\underline{A}\times\left(\underline{B}+\underline{C}\right) = \underline{A}\times\underline{B} + \underline{A}\times\underline{C}$$

# Operazioni tra Vettori

- Prodotto misto

$$\underline{A} \cdot \underline{B} \times \underline{C} = \underline{C} \cdot \underline{B} \times \underline{A} = \underline{B} \cdot \underline{C} \times \underline{A}$$

- Doppio prodotto vettoriale

$$\underline{A} \times \left(\underline{B} \times \underline{C}\right) = \underline{B}\left(\underline{A} \cdot \underline{C}\right) - \underline{C}\left(\underline{A} \cdot \underline{B}\right)$$

- Parallelismo, perpendicolarità e complanarità tra vettori

$$\underline{A} \cdot \underline{B} = 0$$

$$\underline{A} \times \underline{B} = 0$$

$$\underline{A} \cdot \underline{B} \times \underline{C} = 0$$

$$\underline{\hat{n}} \cdot \underline{A} = 0$$

$$\underline{\hat{n}} \times \underline{A} = 0$$

# Identità Differenziali

- Definizioni:

  - Gradiente

$$\nabla f = \frac{\partial f}{\partial x}\underline{i}_x + \frac{\partial f}{\partial y}\underline{i}_y + \frac{\partial f}{\partial z}\underline{i}_z$$

  - Divergenza

$$\nabla \cdot \underline{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

  - Rotore

$$\nabla \times \underline{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\underline{i}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\underline{i}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\underline{i}_z$$

  - Laplaciano

$$\nabla^2 f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla^2 \underline{A} = \nabla^2 A_x \underline{i}_x + \nabla^2 A_y \underline{i}_y + \nabla^2 A_z \underline{i}_z$$

# Identità Differenziali

- Identità Differenziali di Ordine 1

$$\nabla \cdot \left( f\, \underline{A} \right) = f \nabla \cdot \underline{A} + \nabla f \cdot \underline{A}$$

$$\nabla \times \left( f\, \underline{A} \right) = f \nabla \times \underline{A} + \nabla f \times \underline{A} = f \nabla \times \underline{A} - \underline{A} \times \nabla f$$

$$\nabla \cdot \left( \underline{A} \times \underline{B} \right) = \underline{B} \cdot \nabla \times \underline{A} - \underline{A} \cdot \nabla \times \underline{B}$$

$$\nabla f \cdot \underline{\hat{u}} = \frac{\partial f}{\partial u}$$

- Identità Differenziali di Ordine 2

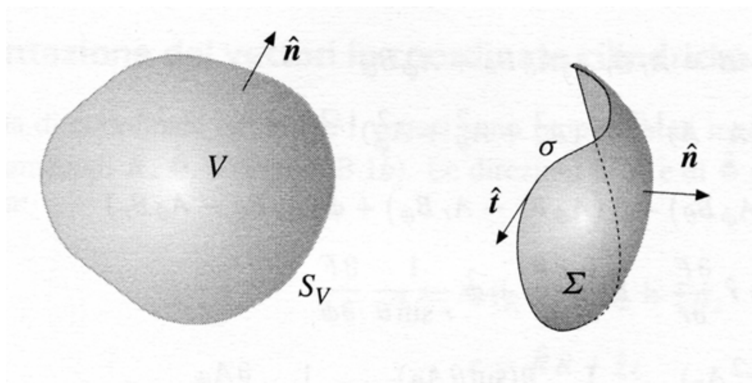
$$\nabla \times \nabla f = 0$$

$$\nabla \cdot \nabla \times \underline{A} = 0$$

$$\nabla \times \nabla \times \underline{A} = \nabla \nabla \cdot \underline{A} - \nabla^2 \underline{A}$$

$$\nabla^2 \left( f\, \underline{A} \right) = \underline{A} \nabla^2 f \qquad (se\ \underline{A}\ \text{è costante})$$

# Identità Integrali




- Teorema della divergenza (Gauss)

$$\int_V \nabla \cdot \underline{A}\, dV = \int_{S_V} \underline{A} \cdot \underline{\hat{n}}\, dS_V$$

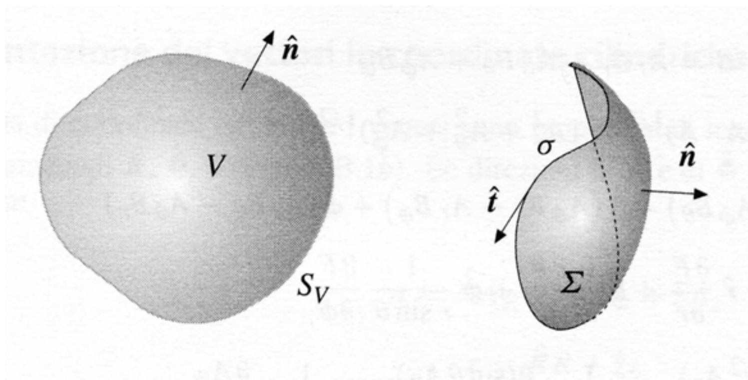
- Teorema del rotore

$$\int_V \nabla \times \underline{A}\, dV = \int_{S_V} \underline{\hat{n}} \times \underline{A}\, dS_V$$

- Formula del gradiente

$$\int_V \nabla f \, dV = \int_{S_V} f\, \underline{\hat{n}}\, dS_V$$

# Identità Integrali



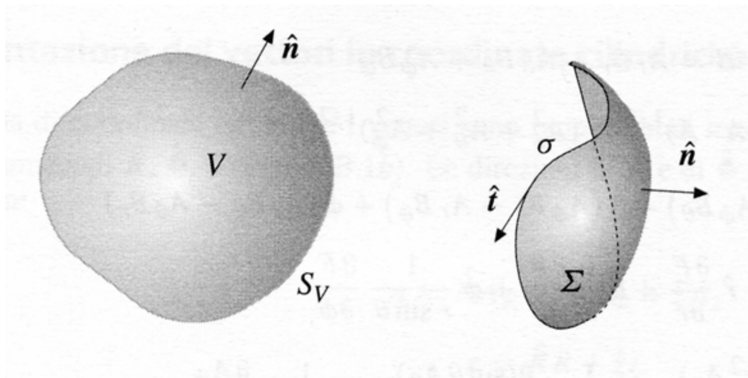

- Integrale di un campo conservativo lungo una linea orientata

$$\int_P^Q \nabla f \cdot \underline{\hat{t}}\, d\ell = f_Q - f_P$$

- Circuitazione di un campo conservativo

$$\oint_\sigma \nabla f \cdot \underline{\hat{t}}\, d\ell = 0$$

# Identità Integrali




- Formula di Stokes

$$\int_S \nabla \times \underline{A} \cdot \underline{\hat{n}}\, dS = \int_\sigma \underline{A} \cdot \underline{d\sigma} = \int_\sigma \underline{A} \cdot \underline{\hat{t}}\, d\ell$$

- Formula di Green

$$\int_V f \nabla^2 G\, dV = \int_{S_V} f \frac{\partial G}{\partial n} dS_V - \int_V \nabla f \cdot \nabla G\, dV$$

ovvero (prima formula di Green):

$$\int_V \left( f \nabla^2 G + \nabla f \cdot \nabla G \right) dV = \int_{S_V} f \frac{\partial G}{\partial n} dS_V$$

- Seconda Formula di Green

$$\int_V \left( G \nabla^2 f - f \nabla^2 G \right) dV = \int_{S_V} \left( G \frac{\partial f}{\partial n} - f \frac{\partial G}{\partial n} \right) dS_V$$

# Formula di Green

integrazione per parti:

$$(fg)' = f'g + fg' \quad \Rightarrow \quad fg = \int f'g + \int fg'$$

consideriamo:

$$\boxed{\int_V f\, \nabla \cdot \underline{A}\, dV}$$

$$\nabla \cdot \left(f\, \underline{A}\right) = f \nabla \cdot \underline{A} + \nabla f \cdot \underline{A} \qquad \text{(vedi lucido 4)}$$

integrando si ottiene:

$$\int_V f\, \nabla \cdot \underline{A}\, dV = \boxed{\int_V \nabla \cdot \left(f\, \underline{A}\right) dV} - \int_V \nabla f \cdot \underline{A}\, dV =$$

(teo. della divergenza) $$= \boxed{\int_{S_V} f\, \underline{A} \cdot \underline{dS}} - \int_V \nabla f \cdot \underline{A}\, dV$$

se $\underline{A} = \nabla G$

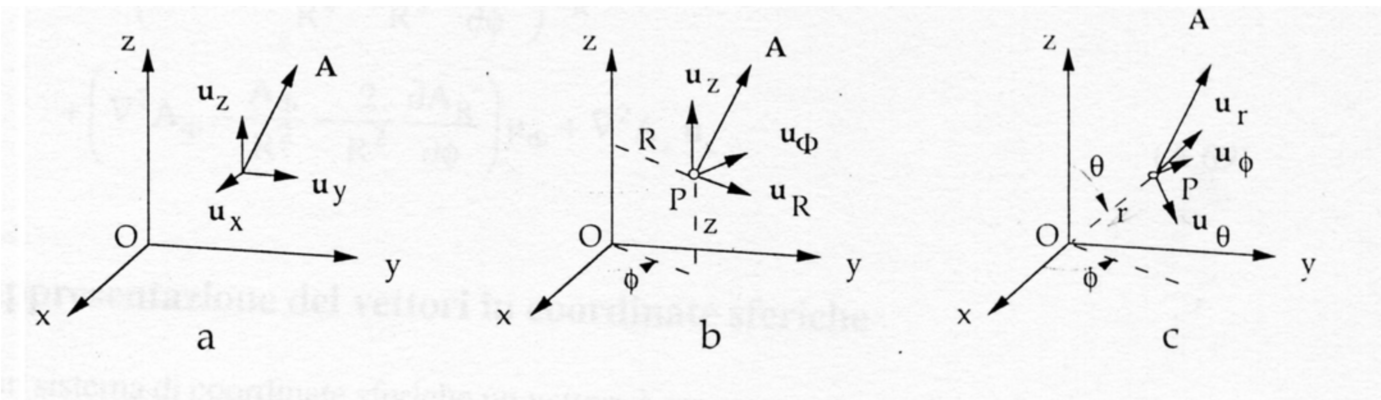
l'ultima identità diventa:

$$\int_V f \nabla \cdot \nabla G\, dV = \int_{S_V} f\, \nabla G \cdot \underline{\hat{n}}\, dS - \int_V \nabla f \cdot \nabla G\, dV$$

ovvero:

$$\boxed{\int_V f \nabla^2 G\, dV = \int_{S_V} f\, \frac{\partial G}{\partial n}\, dS_V - \int_V \nabla f \cdot \nabla G\, dV}$$

# Sistemi di Riferimento

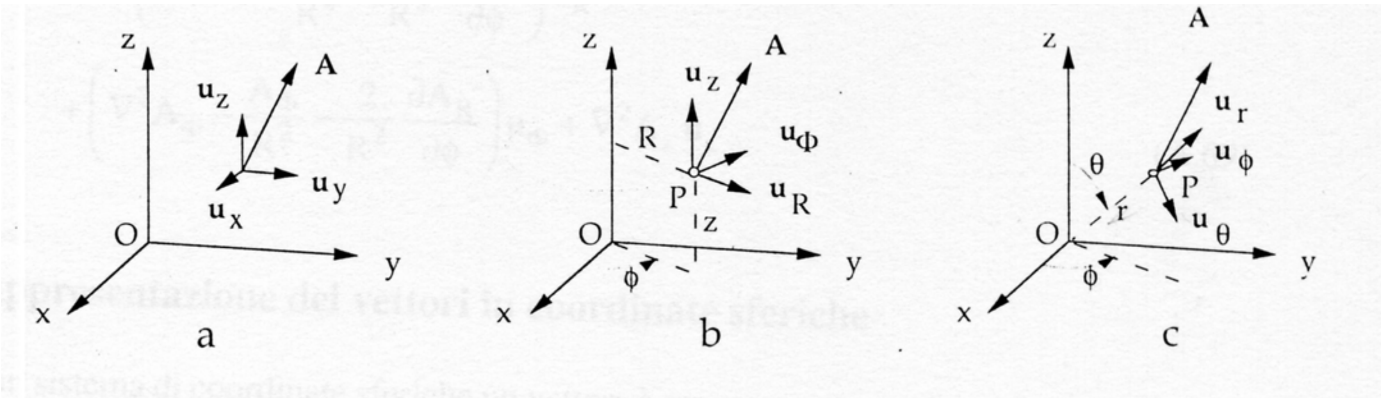



- trasformazione fra coord. cartesiane e coord. cilindriche:

$$\begin{cases} \underline{u}_R = \cos\phi\, \underline{u}_x + \sin\phi\, \underline{u}_y \\ \underline{u}_\phi = \sin\phi\, \underline{u}_x + \cos\phi\, \underline{u}_y \end{cases}$$

$$\begin{cases} \underline{u}_x = \cos\phi\, \underline{u}_R - \sin\phi\, \underline{u}_\phi \\ \underline{u}_y = \sin\phi\, \underline{u}_R + \cos\phi\, \underline{u}_\phi \end{cases}$$

# Sistemi di Riferimento




- trasformazione fra coord. cartesiane e coord. sferiche:

$$\begin{cases} \underline{u}_r = \sin\theta\cos\phi\,\underline{u}_x + \sin\theta\sin\phi\,\underline{u}_y + \cos\theta\,\underline{u}_z \\ \underline{u}_\theta = \cos\theta\cos\phi\,\underline{u}_x + \cos\theta\sin\phi\,\underline{u}_y + \sin\theta\,\underline{u}_z \\ \underline{u}_\phi = \sin\phi\,\underline{u}_x - \cos\phi\,\underline{u}_y \end{cases}$$

$$\begin{cases} \underline{u}_x = \sin\theta\cos\phi\,\underline{u}_R + \cos\theta\cos\phi\,\underline{u}_\theta - \sin\phi\,\underline{u}_\phi \\ \underline{u}_y = \sin\theta\sin\phi\,\underline{u}_R + \cos\theta\sin\phi\,\underline{u}_\theta + \cos\phi\,\underline{u}_\phi \\ \underline{u}_z = \cos\theta\,\underline{u}_R - \sin\theta\,\underline{u}_\theta \end{cases}$$