5 TRIGONOMETRIC FUNCTIONS

DEFINITION OF TRIGONOMETRIC FUNCTIONS FOR A RIGHT TRIANGLE

Triangle ABC has a right angle (90°) at C and sides of length a, b, c. The trigonometric functions of angle A are defined as follows.

5.1 $\quad$ *sine* of A $= \sin A = \dfrac{a}{c} = \dfrac{\text{opposite}}{\text{hypotenuse}}$

5.2 $\quad$ *cosine* of A $= \cos A = \dfrac{b}{c} = \dfrac{\text{adjacent}}{\text{hypotenuse}}$

5.3 $\quad$ *tangent* of A $= \tan A = \dfrac{a}{b} = \dfrac{\text{opposite}}{\text{adjacent}}$

5.4 $\quad$ *cotangent* of A $= \cot A = \dfrac{b}{a} = \dfrac{\text{adjacent}}{\text{opposite}}$

5.5 $\quad$ *secant* of A $= \sec A = \dfrac{c}{b} = \dfrac{\text{hypotenuse}}{\text{adjacent}}$

5.6 $\quad$ *cosecant* of A $= \csc A = \dfrac{c}{a} = \dfrac{\text{hypotenuse}}{\text{opposite}}$

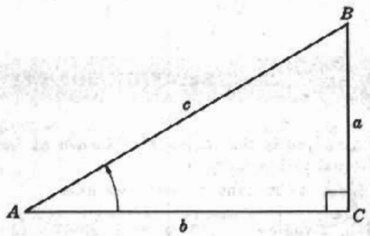

Fig. 5-1

EXTENSIONS TO ANGLES WHICH MAY BE GREATER THAN 90°

Consider an xy coordinate system [see Fig. 5-2 and 5-3 below]. A point P in the xy plane has coordinates (x, y) where x is considered as positive along OX and negative along OX' while y is positive along OY and negative along OY'. The distance from origin O to point P is positive and denoted by $r = \sqrt{x^2 + y^2}$. The angle A described *counterclockwise* from OX is considered *positive*. If it is described *clockwise* from OX it is considered *negative*. We call $X'OX$ and $Y'OY$ the x and y axis respectively.

The various quadrants are denoted by I, II, III and IV called the first, second, third and fourth quadrants respectively. In Fig. 5-2, for example, angle A is in the second quadrant while in Fig. 5-3 angle A is in the third quadrant.

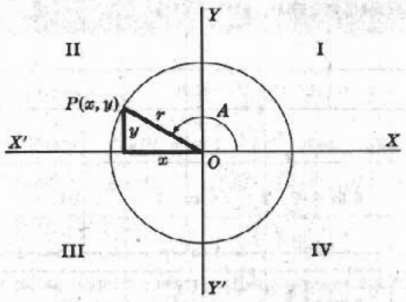

Fig. 5-2

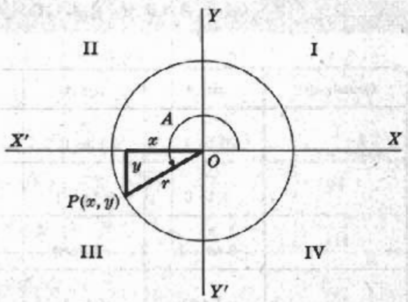

Fig. 5-3

For an angle A in any quadrant the trigonometric functions of A are defined as follows.

5.7 $\quad \sin A = y/r$

5.8 $\quad \cos A = x/r$

5.9 $\quad \tan A = y/x$

5.10 $\quad \cot A = x/y$

5.11 $\quad \sec A = r/x$

5.12 $\quad \csc A = r/y$

RELATIONSHIP BETWEEN DEGREES AND RADIANS

A *radian* is that angle θ subtended at center O of a circle by an arc MN equal to the radius r.

Since 2π radians = 360° we have

5.13 $\quad 1 \text{ radian} = 180°/\pi = 57.29577\ 95130\ 8232\ldots°$

5.14 $\quad 1° = \pi/180 \text{ radians} = 0.01745\ 32925\ 19943\ 29576\ 92\ldots \text{radians}$

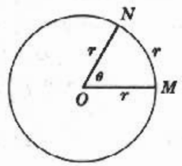

Fig. 5-4

RELATIONSHIPS AMONG TRIGONOMETRIC FUNCTIONS

5.15 $\quad \tan A = \dfrac{\sin A}{\cos A}$

5.16 $\quad \cot A = \dfrac{1}{\tan A} = \dfrac{\cos A}{\sin A}$

5.17 $\quad \sec A = \dfrac{1}{\cos A}$

5.18 $\quad \csc A = \dfrac{1}{\sin A}$

5.19 $\quad \sin^2 A + \cos^2 A = 1$

5.20 $\quad \sec^2 A - \tan^2 A = 1$

5.21 $\quad \csc^2 A - \cot^2 A = 1$

SIGNS AND VARIATIONS OF TRIGONOMETRIC FUNCTIONS

Quadrant	$\sin A$	$\cos A$	$\tan A$	$\cot A$	$\sec A$	$\csc A$
I	+ 0 to 1	+ 1 to 0	+ 0 to ∞	+ ∞ to 0	+ 1 to ∞	+ ∞ to 1
II	+ 1 to 0	− 0 to −1	− −∞ to 0	− 0 to −∞	− −∞ to −1	+ 1 to ∞
III	− 0 to −1	− −1 to 0	+ 0 to ∞	+ ∞ to 0	− −1 to −∞	− −∞ to −1
IV	− −1 to 0	+ 0 to 1	− −∞ to 0	− 0 to −∞	+ ∞ to 1	− −1 to −∞

EXACT VALUES FOR TRIGONOMETRIC FUNCTIONS OF VARIOUS ANGLES

Angle A in degrees	Angle A in radians	$\sin A$	$\cos A$	$\tan A$	$\cot A$	$\sec A$	$\csc A$
0°	0	0	1	0	∞	1	∞
15°	$\pi/12$	$\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$2-\sqrt{3}$	$2+\sqrt{3}$	$\sqrt{6}-\sqrt{2}$	$\sqrt{6}+\sqrt{2}$
30°	$\pi/6$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	2
45°	$\pi/4$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\pi/3$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	2	$\frac{2}{3}\sqrt{3}$
75°	$5\pi/12$	$\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$2+\sqrt{3}$	$2-\sqrt{3}$	$\sqrt{6}+\sqrt{2}$	$\sqrt{6}-\sqrt{2}$
90°	$\pi/2$	1	0	$\pm\infty$	0	$\pm\infty$	1
105°	$7\pi/12$	$\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$-\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$-(2+\sqrt{3})$	$-(2-\sqrt{3})$	$-(\sqrt{6}+\sqrt{2})$	$\sqrt{6}-\sqrt{2}$
120°	$2\pi/3$	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	-2	$\frac{2}{3}\sqrt{3}$
135°	$3\pi/4$	$\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$
150°	$5\pi/6$	$\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	$-\sqrt{3}$	$-\frac{2}{3}\sqrt{3}$	2
165°	$11\pi/12$	$\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$-\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$-(2-\sqrt{3})$	$-(2+\sqrt{3})$	$-(\sqrt{6}-\sqrt{2})$	$\sqrt{6}+\sqrt{2}$
180°	π	0	-1	0	$\mp\infty$	-1	$\pm\infty$
195°	$13\pi/12$	$-\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$-\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$2-\sqrt{3}$	$2+\sqrt{3}$	$-(\sqrt{6}-\sqrt{2})$	$-(\sqrt{6}+\sqrt{2})$
210°	$7\pi/6$	$-\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$	$-\frac{2}{3}\sqrt{3}$	-2
225°	$5\pi/4$	$-\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	1	1	$-\sqrt{2}$	$-\sqrt{2}$
240°	$4\pi/3$	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	-2	$-\frac{2}{3}\sqrt{3}$
255°	$17\pi/12$	$-\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$-\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$2+\sqrt{3}$	$2-\sqrt{3}$	$-(\sqrt{6}+\sqrt{2})$	$-(\sqrt{6}-\sqrt{2})$
270°	$3\pi/2$	-1	0	$\pm\infty$	0	$\mp\infty$	-1
285°	$19\pi/12$	$-\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$-(2+\sqrt{3})$	$-(2-\sqrt{3})$	$\sqrt{6}+\sqrt{2}$	$-(\sqrt{6}-\sqrt{2})$
300°	$5\pi/3$	$-\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	2	$-\frac{2}{3}\sqrt{3}$
315°	$7\pi/4$	$-\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	-1	-1	$\sqrt{2}$	$-\sqrt{2}$
330°	$11\pi/6$	$-\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	$-\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	-2
345°	$23\pi/12$	$-\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$-(2-\sqrt{3})$	$-(2+\sqrt{3})$	$\sqrt{6}-\sqrt{2}$	$-(\sqrt{6}+\sqrt{2})$
360°	2π	0	1	0	$\mp\infty$	1	$\mp\infty$

For tables involving other angles see pages 206-211 and 212-215.

GRAPHS OF TRIGONOMETRIC FUNCTIONS

In each graph x is in radians.

5.22 $y = \sin x$

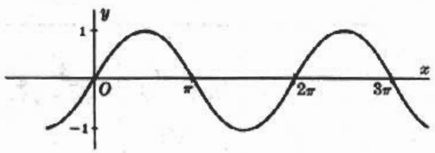

Fig. 5-5

5.23 $y = \cos x$

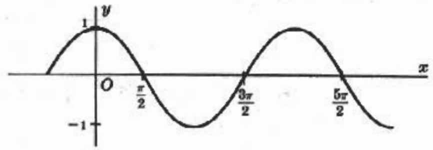

Fig. 5-6

5.24 $y = \tan x$

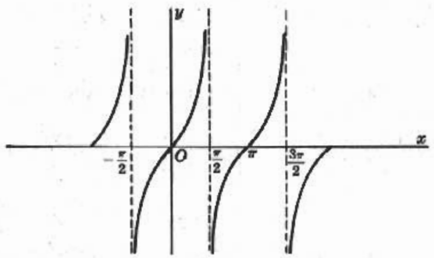

Fig. 5-7

5.25 $y = \cot x$

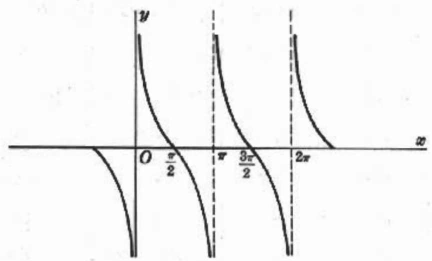

Fig. 5-8

5.26 $y = \sec x$

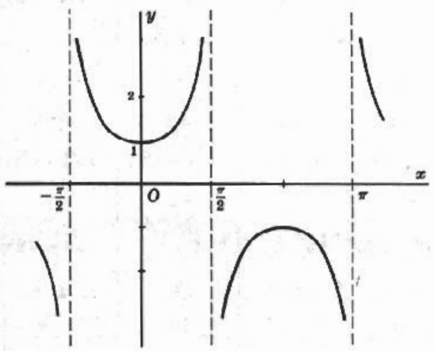

Fig. 5-9

5.27 $y = \csc x$

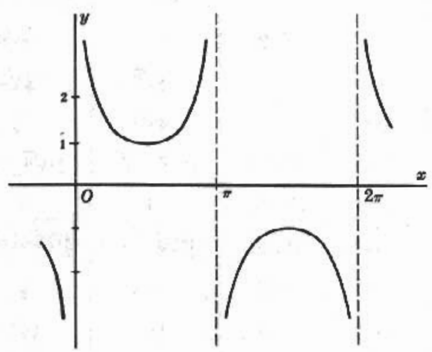

Fig. 5-10

FUNCTIONS OF NEGATIVE ANGLES

5.28 $\sin(-A) = -\sin A$ **5.29** $\cos(-A) = \cos A$ **5.30** $\tan(-A) = -\tan A$

5.31 $\csc(-A) = -\csc A$ **5.32** $\sec(-A) = \sec A$ **5.33** $\cot(-A) = -\cot A$

ADDITION FORMULAS

5.34 $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

5.35 $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

5.36 $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

5.37 $\cot(A \pm B) = \dfrac{\cot A \cot B \mp 1}{\cot B \pm \cot A}$

FUNCTIONS OF ANGLES IN ALL QUADRANTS IN TERMS OF THOSE IN QUADRANT I

	$-A$	$90° \pm A$ $\frac{\pi}{2} \pm A$	$180° \pm A$ $\pi \pm A$	$270° \pm A$ $\frac{3\pi}{2} \pm A$	$k(360°) \pm A$ $2k\pi \pm A$ $k =$ integer
sin	$-\sin A$	$\cos A$	$\mp \sin A$	$-\cos A$	$\pm \sin A$
cos	$\cos A$	$\mp \sin A$	$-\cos A$	$\pm \sin A$	$\cos A$
tan	$-\tan A$	$\mp \cot A$	$\pm \tan A$	$\mp \cot A$	$\pm \tan A$
csc	$-\csc A$	$\sec A$	$\mp \csc A$	$-\sec A$	$\pm \csc A$
sec	$\sec A$	$\mp \csc A$	$-\sec A$	$\pm \csc A$	$\sec A$
cot	$-\cot A$	$\mp \tan A$	$\pm \cot A$	$\mp \tan A$	$\pm \cot A$

RELATIONSHIPS AMONG FUNCTIONS OF ANGLES IN QUADRANT I

	$\sin A = u$	$\cos A = u$	$\tan A = u$	$\cot A = u$	$\sec A = u$	$\csc A = u$
$\sin A$	u	$\sqrt{1-u^2}$	$u/\sqrt{1+u^2}$	$1/\sqrt{1+u^2}$	$\sqrt{u^2-1}/u$	$1/u$
$\cos A$	$\sqrt{1-u^2}$	u	$1/\sqrt{1+u^2}$	$u/\sqrt{1+u^2}$	$1/u$	$\sqrt{u^2-1}/u$
$\tan A$	$u/\sqrt{1-u^2}$	$\sqrt{1-u^2}/u$	u	$1/u$	$\sqrt{u^2-1}$	$1/\sqrt{u^2-1}$
$\cot A$	$\sqrt{1-u^2}/u$	$u/\sqrt{1-u^2}$	$1/u$	u	$1/\sqrt{u^2-1}$	$\sqrt{u^2-1}$
$\sec A$	$1/\sqrt{1-u^2}$	$1/u$	$\sqrt{1+u^2}$	$\sqrt{1+u^2}/u$	u	$u/\sqrt{u^2-1}$
$\csc A$	$1/u$	$1/\sqrt{1-u^2}$	$\sqrt{1+u^2}/u$	$\sqrt{1+u^2}$	$u/\sqrt{u^2-1}$	u

For extensions to other quadrants use appropriate signs as given in the preceding table.

DOUBLE ANGLE FORMULAS

5.38 $\sin 2A = 2 \sin A \cos A$

5.39 $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1$

5.40 $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

HALF ANGLE FORMULAS

5.41 $\sin \dfrac{A}{2} = \pm\sqrt{\dfrac{1-\cos A}{2}}$ $\begin{bmatrix} + \text{ if } A/2 \text{ is in quadrant I or II} \\ - \text{ if } A/2 \text{ is in quadrant III or IV} \end{bmatrix}$

5.42 $\cos \dfrac{A}{2} = \pm\sqrt{\dfrac{1+\cos A}{2}}$ $\begin{bmatrix} + \text{ if } A/2 \text{ is in quadrant I or IV} \\ - \text{ if } A/2 \text{ is in quadrant II or III} \end{bmatrix}$

5.43 $\tan \dfrac{A}{2} = \pm\sqrt{\dfrac{1-\cos A}{1+\cos A}}$ $\begin{bmatrix} + \text{ if } A/2 \text{ is in quadrant I or III} \\ - \text{ if } A/2 \text{ is in quadrant II or IV} \end{bmatrix}$

$$= \frac{\sin A}{1+\cos A} = \frac{1-\cos A}{\sin A} = \csc A - \cot A$$

MULTIPLE ANGLE FORMULAS

5.44 $\sin 3A = 3 \sin A - 4 \sin^3 A$

5.45 $\cos 3A = 4 \cos^3 A - 3 \cos A$

5.46 $\tan 3A = \dfrac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$

5.47 $\sin 4A = 4 \sin A \cos A - 8 \sin^3 A \cos A$

5.48 $\cos 4A = 8 \cos^4 A - 8 \cos^2 A + 1$

5.49 $\tan 4A = \dfrac{4 \tan A - 4 \tan^3 A}{1 - 6 \tan^2 A + \tan^4 A}$

5.50 $\sin 5A = 5 \sin A - 20 \sin^3 A + 16 \sin^5 A$

5.51 $\cos 5A = 16 \cos^5 A - 20 \cos^3 A + 5 \cos A$

5.52 $\tan 5A = \dfrac{\tan^5 A - 10 \tan^3 A + 5 \tan A}{1 - 10 \tan^2 A + 5 \tan^4 A}$

See also formulas 5.68 and 5.69.

POWERS OF TRIGONOMETRIC FUNCTIONS

5.53 $\sin^2 A = \frac{1}{2} - \frac{1}{2}\cos 2A$

5.54 $\cos^2 A = \frac{1}{2} + \frac{1}{2}\cos 2A$

5.55 $\sin^3 A = \frac{3}{4}\sin A - \frac{1}{4}\sin 3A$

5.56 $\cos^3 A = \frac{3}{4}\cos A + \frac{1}{4}\cos 3A$

5.57 $\sin^4 A = \frac{3}{8} - \frac{1}{2}\cos 2A + \frac{1}{8}\cos 4A$

5.58 $\cos^4 A = \frac{3}{8} + \frac{1}{2}\cos 2A + \frac{1}{8}\cos 4A$

5.59 $\sin^5 A = \frac{5}{8}\sin A - \frac{5}{16}\sin 3A + \frac{1}{16}\sin 5A$

5.60 $\cos^5 A = \frac{5}{8}\cos A + \frac{5}{16}\cos 3A + \frac{1}{16}\cos 5A$

See also formulas 5.70 through 5.73.

SUM, DIFFERENCE AND PRODUCT OF TRIGONOMETRIC FUNCTIONS

5.61 $\sin A + \sin B = 2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)$

5.62 $\sin A - \sin B = 2 \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)$

5.63 $\cos A + \cos B = 2 \cos \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)$

5.64 $\cos A - \cos B = 2 \sin \frac{1}{2}(A+B) \sin \frac{1}{2}(B-A)$

5.65 $\sin A \sin B = \frac{1}{2}\{\cos(A-B) - \cos(A+B)\}$

5.66 $\cos A \cos B = \frac{1}{2}\{\cos(A-B) + \cos(A+B)\}$

5.67 $\sin A \cos B = \frac{1}{2}\{\sin(A-B) + \sin(A+B)\}$

GENERAL FORMULAS

5.68 $$\sin nA = \sin A\left\{(2\cos A)^{n-1} - \binom{n-2}{1}(2\cos A)^{n-3} + \binom{n-3}{2}(2\cos A)^{n-5} - \cdots\right\}$$

5.69 $$\cos nA = \frac{1}{2}\left\{(2\cos A)^n - \frac{n}{1}(2\cos A)^{n-2} + \frac{n}{2}\binom{n-3}{1}(2\cos A)^{n-4} - \frac{n}{3}\binom{n-4}{2}(2\cos A)^{n-6} + \cdots\right\}$$

5.70 $$\sin^{2n-1} A = \frac{(-1)^{n-1}}{2^{2n-2}}\left\{\sin(2n-1)A - \binom{2n-1}{1}\sin(2n-3)A + \cdots (-1)^{n-1}\binom{2n-1}{n-1}\sin A\right\}$$

5.71 $$\cos^{2n-1} A = \frac{1}{2^{2n-2}}\left\{\cos(2n-1)A + \binom{2n-1}{1}\cos(2n-3)A + \cdots + \binom{2n-1}{n-1}\cos A\right\}$$

5.72 $$\sin^{2n} A = \frac{1}{2^{2n}}\binom{2n}{n} + \frac{(-1)^n}{2^{2n-1}}\left\{\cos 2nA - \binom{2n}{1}\cos(2n-2)A + \cdots (-1)^{n-1}\binom{2n}{n-1}\cos 2A\right\}$$

5.73 $$\cos^{2n} A = \frac{1}{2^{2n}}\binom{2n}{n} + \frac{1}{2^{2n-1}}\left\{\cos 2nA + \binom{2n}{1}\cos(2n-2)A + \cdots + \binom{2n}{n-1}\cos 2A\right\}$$

INVERSE TRIGONOMETRIC FUNCTIONS

If $x = \sin y$ then $y = \sin^{-1} x$, i.e. *the angle whose sine is x* or *inverse sine of x*, is a many-valued function of x which is a collection of single-valued functions called *branches*. Similarly the other inverse trigonometric functions are multiple-valued.

For many purposes a particular branch is required. This is called the *principal branch* and the values for this branch are called *principal values*.

PRINCIPAL VALUES FOR INVERSE TRIGONOMETRIC FUNCTIONS

Principal values for $x \geqq 0$	Principal values for $x < 0$
$0 \leqq \sin^{-1} x \leqq \pi/2$	$-\pi/2 \leqq \sin^{-1} x < 0$
$0 \leqq \cos^{-1} x \leqq \pi/2$	$\pi/2 < \cos^{-1} x \leqq \pi$
$0 \leqq \tan^{-1} x < \pi/2$	$-\pi/2 < \tan^{-1} x < 0$
$0 < \cot^{-1} x \leqq \pi/2$	$\pi/2 < \cot^{-1} x < \pi$
$0 \leqq \sec^{-1} x < \pi/2$	$\pi/2 < \sec^{-1} x \leqq \pi$
$0 < \csc^{-1} x \leqq \pi/2$	$-\pi/2 \leqq \csc^{-1} x < 0$

RELATIONS BETWEEN INVERSE TRIGONOMETRIC FUNCTIONS

In all cases it is assumed that principal values are used.

5.74 $\sin^{-1} x + \cos^{-1} x = \pi/2$

5.75 $\tan^{-1} x + \cot^{-1} x = \pi/2$

5.76 $\sec^{-1} x + \csc^{-1} x = \pi/2$

5.77 $\csc^{-1} x = \sin^{-1}(1/x)$

5.78 $\sec^{-1} x = \cos^{-1}(1/x)$

5.79 $\cot^{-1} x = \tan^{-1}(1/x)$

5.80 $\sin^{-1}(-x) = -\sin^{-1} x$

5.81 $\cos^{-1}(-x) = \pi - \cos^{-1} x$

5.82 $\tan^{-1}(-x) = -\tan^{-1} x$

5.83 $\cot^{-1}(-x) = \pi - \cot^{-1} x$

5.84 $\sec^{-1}(-x) = \pi - \sec^{-1} x$

5.85 $\csc^{-1}(-x) = -\csc^{-1} x$

GRAPHS OF INVERSE TRIGONOMETRIC FUNCTIONS

In each graph y is in radians. Solid portions of curves correspond to principal values.

5.86 $y = \sin^{-1} x$

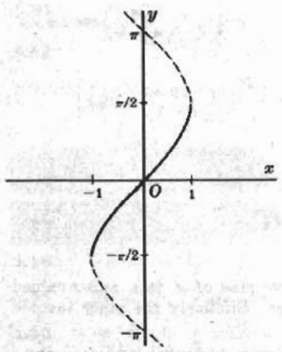

Fig. 5-11

5.87 $y = \cos^{-1} x$

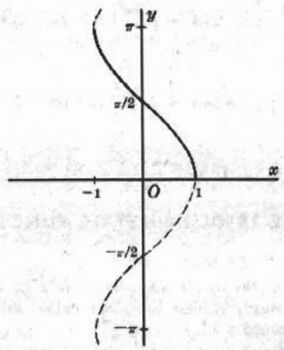

Fig. 5-12

5.88 $y = \tan^{-1} x$

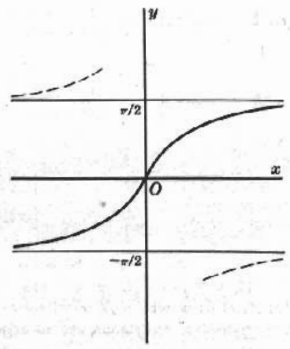

Fig. 5-13

5.89 $y = \cot^{-1} x$

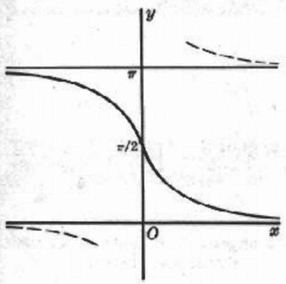

Fig. 5-14

5.90 $y = \sec^{-1} x$

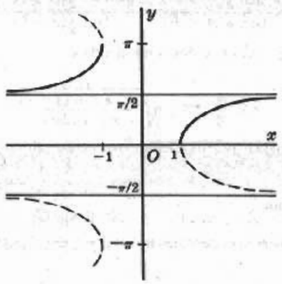

Fig. 5-15

5.91 $y = \csc^{-1} x$

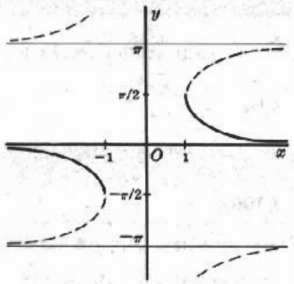

Fig. 5-16

RELATIONSHIPS BETWEEN SIDES AND ANGLES OF A PLANE TRIANGLE

The following results hold for any plane triangle ABC with sides a, b, c and angles A, B, C.

5.92 Law of Sines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

5.93 Law of Cosines

$$c^2 = a^2 + b^2 - 2ab\cos C$$

with similar relations involving the other sides and angles.

5.94 Law of Tangents

$$\frac{a+b}{a-b} = \frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}$$

with similar relations involving the other sides and angles.

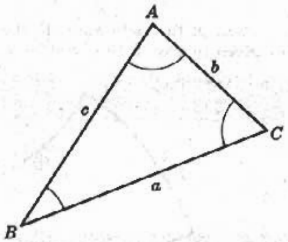

Fig. 5-17

5.95

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a+b+c)$ is the semiperimeter of the triangle. Similar relations involving angles B and C can be obtained.

See also formulas 4.5, page 5; 4.15 and 4.16, page 6.

RELATIONSHIPS BETWEEN SIDES AND ANGLES OF A SPHERICAL TRIANGLE

Spherical triangle ABC is on the surface of a sphere as shown in Fig. 5-18. Sides a, b, c [which are arcs of great circles] are measured by their angles subtended at center O of the sphere. A, B, C are the angles opposite sides a, b, c respectively. Then the following results hold.

5.96 Law of Sines

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

5.97 Law of Cosines

$$\cos a = \cos b\cos c + \sin b\sin c\cos A$$
$$\cos A = -\cos B\cos C + \sin B\sin C\cos a$$

with similar results involving other sides and angles.

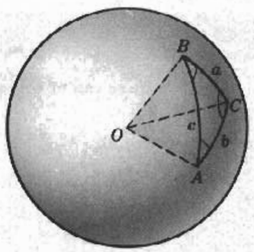

Fig. 5-18

5.98 Law of Tangents

$$\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)} = \frac{\tan\frac{1}{2}(a+b)}{\tan\frac{1}{2}(a-b)}$$

with similar results involving other sides and angles.

5.99

$$\cos\frac{A}{2} = \sqrt{\frac{\sin s\sin(s-c)}{\sin b\sin c}}$$

where $s = \frac{1}{2}(a+b+c)$. Similar results hold for other sides and angles.

5.100

$$\cos\frac{a}{2} = \sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B\sin C}}$$

where $S = \frac{1}{2}(A+B+C)$. Similar results hold for other sides and angles.

See also formula 4.44, page 10.

NAPIER'S RULES FOR RIGHT ANGLED SPHERICAL TRIANGLES

Except for right angle C, there are five parts of spherical triangle ABC which if arranged in the order as given in Fig. 5-19 would be a, b, A, c, B.

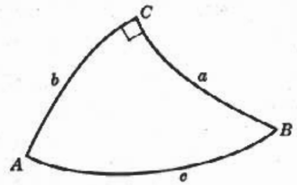

Fig. 5-19

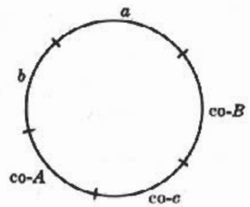

Fig. 5-20

Suppose these quantities are arranged in a circle as in Fig. 5-20 where we attach the prefix co [indicating *complement*] to hypotenuse c and angles A and B.

Any one of the parts of this circle is called a ***middle part***, the two neighboring parts are called ***adjacent parts*** and the two remaining parts are called ***opposite parts***. Then Napier's rules are

5.101 The sine of any middle part equals the product of the tangents of the adjacent parts.

5.102 The sine of any middle part equals the product of the cosines of the opposite parts.

Example: Since co-$A = 90° - A$, co-$B = 90° - B$, we have

$\sin a = \tan b\tan(\text{co-}B)$ or $\sin a = \tan b\cot B$

$\sin(\text{co-}A) = \cos a\cos(\text{co-}B)$ or $\cos A = \cos a\sin B$

These can of course be obtained also from the results 5.97 on page 19.

6 COMPLEX NUMBERS

DEFINITIONS INVOLVING COMPLEX NUMBERS

A *complex number* is generally written as $a + bi$ where a and b are real numbers and i, called the *imaginary unit*, has the property that $i^2 = -1$. The real numbers a and b are called the *real* and *imaginary parts* of $a + bi$ respectively.

The complex numbers $a + bi$ and $a - bi$ are called *complex conjugates* of each other.

EQUALITY OF COMPLEX NUMBERS

6.1 $$a + bi = c + di \quad \text{if and only if} \quad a = c \text{ and } b = d$$

ADDITION OF COMPLEX NUMBERS

6.2 $$(a + bi) + (c + di) = (a + c) + (b + d)i$$

SUBTRACTION OF COMPLEX NUMBERS

6.3 $$(a + bi) - (c + di) = (a - c) + (b - d)i$$

MULTIPLICATION OF COMPLEX NUMBERS

6.4 $$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

DIVISION OF COMPLEX NUMBERS

6.5 $$\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{ac + bd}{c^2 + d^2} + \left(\frac{bc - ad}{c^2 + d^2}\right)i$$

Note that the above operations are obtained by using the ordinary rules of algebra and replacing i^2 by -1 wherever it occurs.

GRAPH OF A COMPLEX NUMBER

A complex number $a + bi$ can be plotted as a point (a, b) on an xy plane called an *Argand diagram* or *Gaussian plane*. For example in Fig. 6-1 P represents the complex number $-3 + 4i$.

A complex number can also be interpreted as a *vector* from O to P.

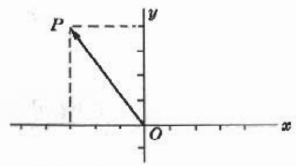

Fig. 6-1

POLAR FORM OF A COMPLEX NUMBER

In Fig. 6-2 point P with coordinates (x, y) represents the complex number $x + iy$. Point P can also be represented by *polar coordinates* (r, θ). Since $x = r\cos\theta$, $y = r\sin\theta$ we have

6.6 $$x + iy = r(\cos\theta + i\sin\theta)$$

called the *polar form* of the complex number. We often call $r = \sqrt{x^2 + y^2}$ the *modulus* and θ the *amplitude* of $x + iy$.

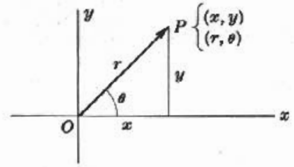

Fig. 6-2

MULTIPLICATION AND DIVISION OF COMPLEX NUMBERS IN POLAR FORM

6.7 $$[r_1(\cos\theta_1 + i\sin\theta_1)][r_2(\cos\theta_2 + i\sin\theta_2)] = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

6.8 $$\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

DE MOIVRE'S THEOREM

If p is any real number, De Moivre's theorem states that

6.9 $$[r(\cos\theta + i\sin\theta)]^p = r^p(\cos p\theta + i\sin p\theta)$$

ROOTS OF COMPLEX NUMBERS

If $p = 1/n$ where n is any positive integer, 6.9 can be written

6.10 $$[r(\cos\theta + i\sin\theta)]^{1/n} = r^{1/n}\left[\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right]$$

where k is any integer. From this the n nth roots of a complex number can be obtained by putting $k = 0, 1, 2, \ldots, n - 1$.

7 EXPONENTIAL and LOGARITHMIC FUNCTIONS

LAWS OF EXPONENTS

In the following p, q are real numbers, a, b are positive numbers and m, n are positive integers.

7.1 $a^p \cdot a^q = a^{p+q}$ **7.2** $a^p/a^q = a^{p-q}$ **7.3** $(a^p)^q = a^{pq}$

7.4 $a^0 = 1,\ a \neq 0$ **7.5** $a^{-p} = 1/a^p$ **7.6** $(ab)^p = a^p b^p$

7.7 $\sqrt[n]{a} = a^{1/n}$ **7.8** $\sqrt[n]{a^m} = a^{m/n}$ **7.9** $\sqrt[n]{a/b} = \sqrt[n]{a}/\sqrt[n]{b}$

In a^p, p is called the *exponent*, a is the *base* and a^p is called the pth *power of a*. The function $y = a^x$ is called an *exponential function.*

LOGARITHMS AND ANTILOGARITHMS

If $a^p = N$ where $a \neq 0$ or 1, then $p = \log_a N$ is called the *logarithm* of N to the base a. The number $N = a^p$ is called the *antilogarithm* of p to the base a, written $\text{antilog}_a\, p$.

Example: Since $3^2 = 9$ we have $\log_3 9 = 2$, $\text{antilog}_3 2 = 9$.

The function $y = \log_a x$ is called a *logarithmic function.*

LAWS OF LOGARITHMS

7.10 $$\log_a MN = \log_a M + \log_a N$$

7.11 $$\log_a \frac{M}{N} = \log_a M - \log_a N$$

7.12 $$\log_a M^p = p \log_a M$$

COMMON LOGARITHMS AND ANTILOGARITHMS

Common logarithms and antilogarithms [also called *Briggsian*] are those in which the base $a = 10$. The common logarithm of N is denoted by $\log_{10} N$ or briefly $\log N$. For tables of common logarithms and antilogarithms, see pages 202-205. For illustrations using these tables see pages 194-196.

NATURAL LOGARITHMS AND ANTILOGARITHMS

Natural logarithms and antilogarithms [also called Napierian] are those in which the base $a = e = 2.71828\,18\ldots$ [see page 1]. The natural logarithm of N is denoted by $\log_e N$ or $\ln N$. For tables of natural logarithms see pages 224-225. For tables of natural antilogarithms [i.e. tables giving e^x for values of x] see pages 226-227. For illustrations using these tables see pages 196 and 200.

CHANGE OF BASE OF LOGARITHMS

The relationship between logarithms of a number N to different bases a and b is given by

7.13 $$\log_a N = \frac{\log_b N}{\log_b a}$$

In particular,

7.14 $$\log_e N = \ln N = 2.30258\,50929\,94\ldots \log_{10} N$$

7.15 $$\log_{10} N = \log N = 0.43429\,44819\,03\ldots \log_e N$$

RELATIONSHIP BETWEEN EXPONENTIAL AND TRIGONOMETRIC FUNCTIONS

7.16 $$e^{i\theta} = \cos\theta + i\sin\theta, \qquad e^{-i\theta} = \cos\theta - i\sin\theta$$

These are called *Euler's identities*. Here i is the imaginary unit [see page 21].

7.17 $$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

7.18 $$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

7.19 $$\tan\theta = \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})} = -i\left(\frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}}\right)$$

7.20 $$\cot\theta = i\left(\frac{e^{i\theta} + e^{-i\theta}}{e^{i\theta} - e^{-i\theta}}\right)$$

7.21 $$\sec\theta = \frac{2}{e^{i\theta} + e^{-i\theta}}$$

7.22 $$\csc\theta = \frac{2i}{e^{i\theta} - e^{-i\theta}}$$

PERIODICITY OF EXPONENTIAL FUNCTIONS

7.23 $$e^{i(\theta + 2k\pi)} = e^{i\theta} \qquad k = \text{integer}$$

From this it is seen that e^x has period $2\pi i$.

POLAR FORM OF COMPLEX NUMBERS EXPRESSED AS AN EXPONENTIAL

The polar form of a complex number $x + iy$ can be written in terms of exponentials [see 6.6, page 22] as

7.24 $$x + iy \;=\; r(\cos\theta + i\sin\theta) \;=\; re^{i\theta}$$

OPERATIONS WITH COMPLEX NUMBERS IN POLAR FORM

Formulas 6.7 through 6.10 on page 22 are equivalent to the following.

7.25 $$(r_1e^{i\theta_1})(r_2e^{i\theta_2}) \;=\; r_1r_2e^{i(\theta_1+\theta_2)}$$

7.26 $$\frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} \;=\; \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$$

7.27 $$(re^{i\theta})^p \;=\; r^pe^{ip\theta} \qquad \text{[De Moivre's theorem]}$$

7.28 $$(re^{i\theta})^{1/n} \;=\; [re^{i(\theta+2k\pi)}]^{1/n} \;=\; r^{1/n}e^{i(\theta+2k\pi)/n}$$

LOGARITHM OF A COMPLEX NUMBER

7.29 $$\ln(re^{i\theta}) \;=\; \ln r + i\theta + 2k\pi i \qquad k = \text{integer}$$

8 HYPERBOLIC FUNCTIONS

DEFINITION OF HYPERBOLIC FUNCTIONS

8.1 *Hyperbolic sine of x* $\;=\; \sinh x \;=\; \dfrac{e^x - e^{-x}}{2}$

8.2 *Hyperbolic cosine of x* $\;=\; \cosh x \;=\; \dfrac{e^x + e^{-x}}{2}$

8.3 *Hyperbolic tangent of x* $\;=\; \tanh x \;=\; \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

8.4 *Hyperbolic cotangent of x* $\;=\; \coth x \;=\; \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

8.5 *Hyperbolic secant of x* $\;=\; \operatorname{sech} x \;=\; \dfrac{2}{e^x + e^{-x}}$

8.6 *Hyperbolic cosecant of x* $\;=\; \operatorname{csch} x \;=\; \dfrac{2}{e^x - e^{-x}}$

RELATIONSHIPS AMONG HYPERBOLIC FUNCTIONS

8.7 $$\tanh x \;=\; \frac{\sinh x}{\cosh x}$$

8.8 $$\coth x \;=\; \frac{1}{\tanh x} \;=\; \frac{\cosh x}{\sinh x}$$

8.9 $$\operatorname{sech} x \;=\; \frac{1}{\cosh x}$$

8.10 $$\operatorname{csch} x \;=\; \frac{1}{\sinh x}$$

8.11 $$\cosh^2 x - \sinh^2 x \;=\; 1$$

8.12 $$\operatorname{sech}^2 x + \tanh^2 x \;=\; 1$$

8.13 $$\coth^2 x - \operatorname{csch}^2 x \;=\; 1$$

FUNCTIONS OF NEGATIVE ARGUMENTS

8.14 $\sinh(-x) = -\sinh x$ **8.15** $\cosh(-x) = \cosh x$ **8.16** $\tanh(-x) = -\tanh x$

8.17 $\operatorname{csch}(-x) = -\operatorname{csch} x$ **8.18** $\operatorname{sech}(-x) = \operatorname{sech} x$ **8.19** $\coth(-x) = -\coth x$

ADDITION FORMULAS

8.20 $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$

8.21 $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

8.22 $\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$

8.23 $\coth(x \pm y) = \dfrac{\coth x \coth y \pm 1}{\coth y \pm \coth x}$

DOUBLE ANGLE FORMULAS

8.24 $\sinh 2x = 2 \sinh x \cosh x$

8.25 $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$

8.26 $\tanh 2x = \dfrac{2 \tanh x}{1 + \tanh^2 x}$

HALF ANGLE FORMULAS

8.27 $\sinh \dfrac{x}{2} = \pm\sqrt{\dfrac{\cosh x - 1}{2}}$ $\quad [+ \text{ if } x > 0,\ - \text{ if } x < 0]$

8.28 $\cosh \dfrac{x}{2} = \sqrt{\dfrac{\cosh x + 1}{2}}$

8.29 $\tanh \dfrac{x}{2} = \pm\sqrt{\dfrac{\cosh x - 1}{\cosh x + 1}}$ $\quad [+ \text{ if } x > 0,\ - \text{ if } x < 0]$

$= \dfrac{\sinh x}{\cosh x + 1} = \dfrac{\cosh x - 1}{\sinh x}$

MULTIPLE ANGLE FORMULAS

8.30 $\sinh 3x = 3 \sinh x + 4 \sinh^3 x$

8.31 $\cosh 3x = 4 \cosh^3 x - 3 \cosh x$

8.32 $\tanh 3x = \dfrac{3 \tanh x + \tanh^3 x}{1 + 3 \tanh^2 x}$

8.33 $\sinh 4x = 8 \sinh^3 x \cosh x + 4 \sinh x \cosh x$

8.34 $\cosh 4x = 8 \cosh^4 x - 8 \cosh^2 x + 1$

8.35 $\tanh 4x = \dfrac{4 \tanh x + 4 \tanh^3 x}{1 + 6 \tanh^2 x + \tanh^4 x}$

POWERS OF HYPERBOLIC FUNCTIONS

8.36 $\sinh^2 x = \frac{1}{2} \cosh 2x - \frac{1}{2}$

8.37 $\cosh^2 x = \frac{1}{2} \cosh 2x + \frac{1}{2}$

8.38 $\sinh^3 x = \frac{1}{4} \sinh 3x - \frac{3}{4} \sinh x$

8.39 $\cosh^3 x = \frac{1}{4} \cosh 3x + \frac{3}{4} \cosh x$

8.40 $\sinh^4 x = \frac{3}{8} - \frac{1}{2} \cosh 2x + \frac{1}{8} \cosh 4x$

8.41 $\cosh^4 x = \frac{3}{8} + \frac{1}{2} \cosh 2x + \frac{1}{8} \cosh 4x$

SUM, DIFFERENCE AND PRODUCT OF HYPERBOLIC FUNCTIONS

8.42 $\sinh x + \sinh y = 2 \sinh \frac{1}{2}(x+y) \cosh \frac{1}{2}(x-y)$

8.43 $\sinh x - \sinh y = 2 \cosh \frac{1}{2}(x+y) \sinh \frac{1}{2}(x-y)$

8.44 $\cosh x + \cosh y = 2 \cosh \frac{1}{2}(x+y) \cosh \frac{1}{2}(x-y)$

8.45 $\cosh x - \cosh y = 2 \sinh \frac{1}{2}(x+y) \sinh \frac{1}{2}(x-y)$

8.46 $\sinh x \sinh y = \frac{1}{2}\{\cosh(x+y) - \cosh(x-y)\}$

8.47 $\cosh x \cosh y = \frac{1}{2}\{\cosh(x+y) + \cosh(x-y)\}$

8.48 $\sinh x \cosh y = \frac{1}{2}\{\sinh(x+y) + \sinh(x-y)\}$

EXPRESSION OF HYPERBOLIC FUNCTIONS IN TERMS OF OTHERS

In the following we assume $x > 0$. If $x < 0$ use the appropriate sign as indicated by formulas 8.14 to 8.19.

	$\sinh x = u$	$\cosh x = u$	$\tanh x = u$	$\coth x = u$	$\operatorname{sech} x = u$	$\operatorname{csch} x = u$
$\sinh x$	u	$\sqrt{u^2-1}$	$u/\sqrt{1-u^2}$	$1/\sqrt{u^2-1}$	$\sqrt{1-u^2}/u$	$1/u$
$\cosh x$	$\sqrt{1+u^2}$	u	$1/\sqrt{1-u^2}$	$u/\sqrt{u^2-1}$	$1/u$	$\sqrt{1+u^2}/u$
$\tanh x$	$u/\sqrt{1+u^2}$	$\sqrt{u^2-1}/u$	u	$1/u$	$\sqrt{1-u^2}$	$1/\sqrt{1+u^2}$
$\coth x$	$\sqrt{u^2+1}/u$	$u/\sqrt{u^2-1}$	$1/u$	u	$1/\sqrt{1-u^2}$	$\sqrt{1+u^2}$
$\operatorname{sech} x$	$1/\sqrt{1+u^2}$	$1/u$	$\sqrt{1-u^2}$	$\sqrt{u^2-1}/u$	u	$u/\sqrt{1+u^2}$
$\operatorname{csch} x$	$1/u$	$1/\sqrt{u^2-1}$	$\sqrt{1-u^2}/u$	$\sqrt{u^2-1}$	$u/\sqrt{1-u^2}$	u

GRAPHS OF HYPERBOLIC FUNCTIONS

8.49 $y = \sinh x$

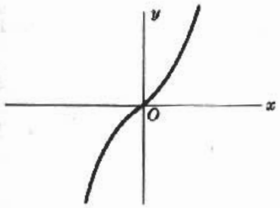

Fig. 8-1

8.50 $y = \cosh x$

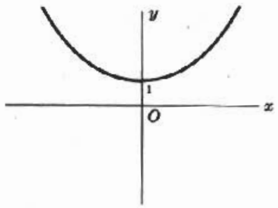

Fig. 8-2

8.51 $y = \tanh x$

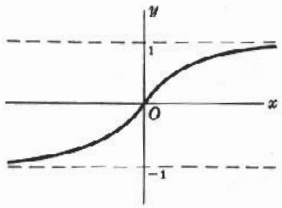

Fig. 8-3

8.52 $y = \coth x$

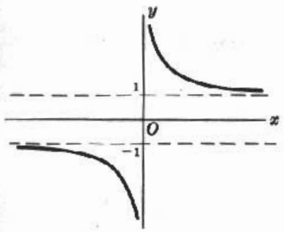

Fig. 8-4

8.53 $y = \operatorname{sech} x$

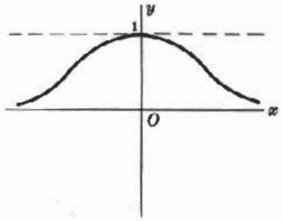

Fig. 8-5

8.54 $y = \operatorname{csch} x$

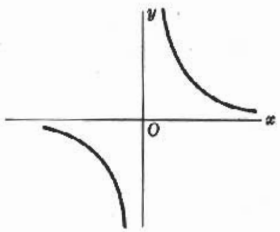

Fig. 8-6

INVERSE HYPERBOLIC FUNCTIONS

If $x = \sinh y$, then $y = \sinh^{-1} x$ is called the *inverse hyperbolic sine* of x. Similarly we define the other inverse hyperbolic functions. The inverse hyperbolic functions are multiple-valued and as in the case of inverse trigonometric functions [see page 17] we restrict ourselves to principal values for which they can be considered as single-valued.

The following list shows the principal values [unless otherwise indicated] of the inverse hyperbolic functions expressed in terms of logarithmic functions which are taken as real valued.

8.55 $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$ $\quad -\infty < x < \infty$

8.56 $\cosh^{-1} x = \ln(x + \sqrt{x^2-1})$ $\quad x \geqq 1$ $\quad [\cosh^{-1} x > 0$ is principal value$]$

8.57 $\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ $\quad -1 < x < 1$

8.58 $\coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$ $\quad x > 1$ or $x < -1$

8.59 $\operatorname{sech}^{-1} x = \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1}\right)$ $\quad 0 < x \leqq 1$ $\quad [\operatorname{sech}^{-1} x > 0$ is principal value$]$

8.60 $\operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1}\right)$ $\quad x \neq 0$

RELATIONS BETWEEN INVERSE HYPERBOLIC FUNCTIONS

8.61 $\operatorname{csch}^{-1} x = \sinh^{-1}(1/x)$

8.62 $\operatorname{sech}^{-1} x = \cosh^{-1}(1/x)$

8.63 $\coth^{-1} x = \tanh^{-1}(1/x)$

8.64 $\sinh^{-1}(-x) = -\sinh^{-1} x$

8.65 $\tanh^{-1}(-x) = -\tanh^{-1} x$

8.66 $\coth^{-1}(-x) = -\coth^{-1} x$

8.67 $\operatorname{csch}^{-1}(-x) = -\operatorname{csch}^{-1} x$

GRAPHS OF INVERSE HYPERBOLIC FUNCTIONS

8.68 $y = \sinh^{-1} x$

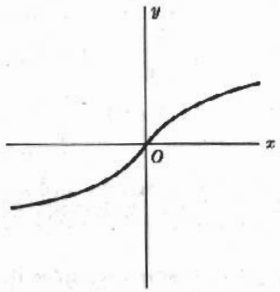

Fig. 8-7

8.69 $y = \cosh^{-1} x$

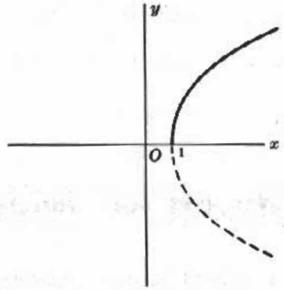

Fig. 8-8

8.70 $y = \tanh^{-1} x$

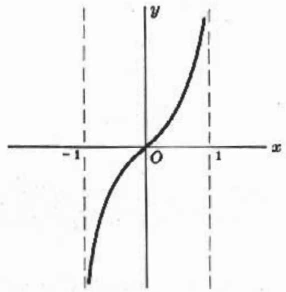

Fig. 8-9

8.71 $y = \coth^{-1} x$

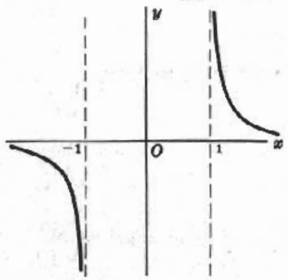

Fig. 8-10

8.72 $y = \operatorname{sech}^{-1} x$

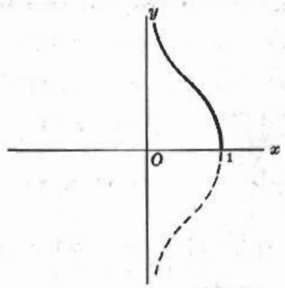

Fig. 8-11

8.73 $y = \operatorname{csch}^{-1} x$

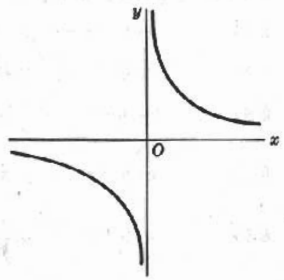

Fig. 8-12

RELATIONSHIP BETWEEN HYPERBOLIC AND TRIGONOMETRIC FUNCTIONS

8.74 $\sin(ix) = i \sinh x$ **8.75** $\cos(ix) = \cosh x$ **8.76** $\tan(ix) = i \tanh x$

8.77 $\csc(ix) = -i \operatorname{csch} x$ **8.78** $\sec(ix) = \operatorname{sech} x$ **8.79** $\cot(ix) = -i \coth x$

8.80 $\sinh(ix) = i \sin x$ **8.81** $\cosh(ix) = \cos x$ **8.82** $\tanh(ix) = i \tan x$

8.83 $\operatorname{csch}(ix) = -i \csc x$ **8.84** $\operatorname{sech}(ix) = \sec x$ **8.85** $\coth(ix) = -i \cot x$

PERIODICITY OF HYPERBOLIC FUNCTIONS

In the following k is any integer.

8.86 $\sinh(x + 2k\pi i) = \sinh x$ **8.87** $\cosh(x + 2k\pi i) = \cosh x$ **8.88** $\tanh(x + k\pi i) = \tanh x$

8.89 $\operatorname{csch}(x + 2k\pi i) = \operatorname{csch} x$ **8.90** $\operatorname{sech}(x + 2k\pi i) = \operatorname{sech} x$ **8.91** $\coth(x + k\pi i) = \coth x$

RELATIONSHIP BETWEEN INVERSE HYPERBOLIC AND INVERSE TRIGONOMETRIC FUNCTIONS

8.92 $\sin^{-1}(ix) = i \sinh^{-1} x$ **8.93** $\sinh^{-1}(ix) = i \sin^{-1} x$

8.94 $\cos^{-1} x = \pm i \cosh^{-1} x$ **8.95** $\cosh^{-1} x = \pm i \cos^{-1} x$

8.96 $\tan^{-1}(ix) = i \tanh^{-1} x$ **8.97** $\tanh^{-1}(ix) = i \tan^{-1} x$

8.98 $\cot^{-1}(ix) = -i \coth^{-1} x$ **8.99** $\coth^{-1}(ix) = -i \cot^{-1} x$

8.100 $\sec^{-1} x = \pm i \operatorname{sech}^{-1} x$ **8.101** $\operatorname{sech}^{-1} x = \pm i \sec^{-1} x$

8.102 $\csc^{-1}(ix) = -i \operatorname{csch}^{-1} x$ **8.103** $\operatorname{csch}^{-1}(ix) = -i \csc^{-1} x$

13 DERIVATIVES

DEFINITION OF A DERIVATIVE

If $y = f(x)$, the derivative of y or $f(x)$ with respect to x is defined as

13.1 $$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

where $h = \Delta x$. The derivative is also denoted by y', df/dx or $f'(x)$. The process of taking a derivative is called *differentiation.*

GENERAL RULES OF DIFFERENTIATION

In the following, u, v, w are functions of x; a, b, c, n are constants [restricted if indicated]; $e = 2.71828\ldots$ is the natural base of logarithms; $\ln u$ is the natural logarithm of u [i.e. the logarithm to the base e] where it is assumed that $u > 0$ and all angles are in radians.

13.2 $\frac{d}{dx}(c) = 0$

13.3 $\frac{d}{dx}(cx) = c$

13.4 $\frac{d}{dx}(cx^n) = ncx^{n-1}$

13.5 $\frac{d}{dx}(u \pm v \pm w \pm \cdots) = \frac{du}{dx} \pm \frac{dv}{dx} \pm \frac{dw}{dx} \pm \cdots$

13.6 $\frac{d}{dx}(cu) = c\frac{du}{dx}$

13.7 $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

13.8 $\frac{d}{dx}(uvw) = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$

13.9 $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v(du/dx) - u(dv/dx)}{v^2}$

13.10 $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$

13.11 $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ (Chain rule)

13.12 $\frac{du}{dx} = \frac{1}{dx/du}$

13.13 $\frac{dy}{dx} = \frac{dy/du}{dx/du}$

DERIVATIVES OF TRIGONOMETRIC AND INVERSE TRIGONOMETRIC FUNCTIONS

13.14 $\frac{d}{dx}\sin u = \cos u \frac{du}{dx}$

13.15 $\frac{d}{dx}\cos u = -\sin u \frac{du}{dx}$

13.16 $\frac{d}{dx}\tan u = \sec^2 u \frac{du}{dx}$

13.17 $\frac{d}{dx}\cot u = -\csc^2 u \frac{du}{dx}$

13.18 $\frac{d}{dx}\sec u = \sec u \tan u \frac{du}{dx}$

13.19 $\frac{d}{dx}\csc u = -\csc u \cot u \frac{du}{dx}$

13.20 $\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$ $\quad\left[-\frac{\pi}{2} < \sin^{-1} u < \frac{\pi}{2}\right]$

13.21 $\frac{d}{dx}\cos^{-1} u = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$ $\quad[0 < \cos^{-1} u < \pi]$

13.22 $\frac{d}{dx}\tan^{-1} u = \frac{1}{1+u^2}\frac{du}{dx}$ $\quad\left[-\frac{\pi}{2} < \tan^{-1} u < \frac{\pi}{2}\right]$

13.23 $\frac{d}{dx}\cot^{-1} u = \frac{-1}{1+u^2}\frac{du}{dx}$ $\quad[0 < \cot^{-1} u < \pi]$

13.24 $\frac{d}{dx}\sec^{-1} u = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} = \frac{\pm 1}{u\sqrt{u^2-1}}\frac{du}{dx}$ $\quad\begin{bmatrix} + \text{ if } 0 < \sec^{-1} u < \pi/2 \\ - \text{ if } \pi/2 < \sec^{-1} u < \pi \end{bmatrix}$

13.25 $\frac{d}{dx}\csc^{-1} u = \frac{-1}{|u|\sqrt{u^2-1}}\frac{du}{dx} = \frac{\mp 1}{u\sqrt{u^2-1}}\frac{du}{dx}$ $\quad\begin{bmatrix} - \text{ if } 0 < \csc^{-1} u < \pi/2 \\ + \text{ if } -\pi/2 < \csc^{-1} u < 0 \end{bmatrix}$

DERIVATIVES OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

13.26 $\frac{d}{dx}\log_a u = \frac{\log_a e}{u}\frac{du}{dx} \qquad a \neq 0, 1$

13.27 $\frac{d}{dx}\ln u = \frac{d}{dx}\log_e u = \frac{1}{u}\frac{du}{dx}$

13.28 $\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}$

13.29 $\frac{d}{dx}e^u = e^u \frac{du}{dx}$

13.30 $\frac{d}{dx}u^v = \frac{d}{dx}e^{v\ln u} = e^{v \ln u}\frac{d}{dx}[v \ln u] = vu^{v-1}\frac{du}{dx} + u^v \ln u \frac{dv}{dx}$

DERIVATIVES OF HYPERBOLIC AND INVERSE HYPERBOLIC FUNCTIONS

13.31 $\frac{d}{dx}\sinh u = \cosh u \frac{du}{dx}$

13.32 $\frac{d}{dx}\cosh u = \sinh u \frac{du}{dx}$

13.33 $\frac{d}{dx}\tanh u = \operatorname{sech}^2 u \frac{du}{dx}$

13.34 $\frac{d}{dx}\coth u = -\operatorname{csch}^2 u \frac{du}{dx}$

13.35 $\frac{d}{dx}\operatorname{sech} u = -\operatorname{sech} u \tanh u \frac{du}{dx}$

13.36 $\frac{d}{dx}\operatorname{csch} u = -\operatorname{csch} u \coth u \frac{du}{dx}$

13.37 $\frac{d}{dx}\sinh^{-1} u = \frac{1}{\sqrt{u^2+1}}\frac{du}{dx}$

13.38 $\frac{d}{dx}\cosh^{-1} u = \frac{\pm 1}{\sqrt{u^2-1}}\frac{du}{dx}$ $\quad\begin{bmatrix} + \text{ if } \cosh^{-1} u > 0,\ u > 1 \\ - \text{ if } \cosh^{-1} u < 0,\ u > 1 \end{bmatrix}$

13.39 $\frac{d}{dx}\tanh^{-1} u = \frac{1}{1-u^2}\frac{du}{dx}$ $\quad[-1 < u < 1]$

13.40 $\frac{d}{dx}\coth^{-1} u = \frac{1}{1-u^2}\frac{du}{dx}$ $\quad[u > 1 \text{ or } u < -1]$

13.41 $\frac{d}{dx}\operatorname{sech}^{-1} u = \frac{\mp 1}{u\sqrt{1-u^2}}\frac{du}{dx}$ $\quad\begin{bmatrix} - \text{ if } \operatorname{sech}^{-1} u > 0,\ 0 < u < 1 \\ + \text{ if } \operatorname{sech}^{-1} u < 0,\ 0 < u < 1 \end{bmatrix}$

13.42 $\frac{d}{dx}\operatorname{csch}^{-1} u = \frac{-1}{|u|\sqrt{1+u^2}}\frac{du}{dx} = \frac{\mp 1}{u\sqrt{1+u^2}}\frac{du}{dx}$ $\quad[-\text{ if } u > 0,\ + \text{ if } u < 0]$

HIGHER DERIVATIVES

The second, third and higher derivatives are defined as follows.

13.43 Second derivative $= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x) = y''$

13.44 Third derivative $= \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = f'''(x) = y'''$

13.45 nth derivative $= \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^ny}{dx^n} = f^{(n)}(x) = y^{(n)}$

LEIBNITZ'S RULE FOR HIGHER DERIVATIVES OF PRODUCTS

Let D^p stand for the operator $\frac{d^p}{dx^p}$ so that $D^pu = \frac{d^pu}{dx^p}$ = the pth derivative of u. Then

13.46 $$D^n(uv) = uD^nv + \binom{n}{1}(Du)(D^{n-1}v) + \binom{n}{2}(D^2u)(D^{n-2}v) + \cdots + vD^nu$$

where $\binom{n}{1}, \binom{n}{2}, \ldots$ are the binomial coefficients [page 3].

As special cases we have

13.47 $$\frac{d^2}{dx^2}(uv) = u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2}$$

13.48 $$\frac{d^3}{dx^3}(uv) = u\frac{d^3v}{dx^3} + 3\frac{du}{dx}\frac{d^2v}{dx^2} + 3\frac{d^2u}{dx^2}\frac{dv}{dx} + v\frac{d^3u}{dx^3}$$

DIFFERENTIALS

Let $y = f(x)$ and $\Delta y = f(x + \Delta x) - f(x)$. Then

13.49 $$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x) + \epsilon = \frac{dy}{dx} + \epsilon$$

where $\epsilon \to 0$ as $\Delta x \to 0$. Thus

13.50 $$\Delta y = f'(x)\,\Delta x + \epsilon\,\Delta x$$

If we call $\Delta x = dx$ the differential of x, then we define the differential of y to be

13.51 $$dy = f'(x)\,dx$$

RULES FOR DIFFERENTIALS

The rules for differentials are exactly analogous to those for derivatives. As examples we observe that

13.52 $$d(u \pm v \pm w \pm \cdots) \;=\; du \pm dv \pm dw \pm \cdots$$

13.53 $$d(uv) \;=\; u\,dv + v\,du$$

13.54 $$d\left(\frac{u}{v}\right) \;=\; \frac{v\,du - u\,dv}{v^2}$$

13.55 $$d(u^n) \;=\; nu^{n-1}\,du$$

13.56 $$d(\sin u) \;=\; \cos u\,du$$

13.57 $$d(\cos u) \;=\; -\sin u\,du$$

PARTIAL DERIVATIVES

Let $f(x,y)$ be a function of the two variables x and y. Then we define the partial derivative of $f(x,y)$ with respect to x, keeping y constant, to be

13.58 $$\frac{\partial f}{\partial x} \;=\; \lim_{\Delta x\to 0}\frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

Similarly the partial derivative of $f(x,y)$ with respect to y, keeping x constant, is defined to be

13.59 $$\frac{\partial f}{\partial y} \;=\; \lim_{\Delta y\to 0}\frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

Partial derivatives of higher order can be defined as follows.

13.60 $$\frac{\partial^2 f}{\partial x^2} \;=\; \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \qquad \frac{\partial^2 f}{\partial y^2} \;=\; \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

13.61 $$\frac{\partial^2 f}{\partial x\,\partial y} \;=\; \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \qquad \frac{\partial^2 f}{\partial y\,\partial x} \;=\; \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

The results in 13.61 will be equal if the function and its partial derivatives are continuous, i.e. in such case the order of differentiation makes no difference.

The differential of $f(x,y)$ is defined as

13.62 $$df \;=\; \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

where $dx = \Delta x$ and $dy = \Delta y$.

Extension to functions of more than two variables are exactly analogous.

14 INDEFINITE INTEGRALS

DEFINITION OF AN INDEFINITE INTEGRAL

If $\dfrac{dy}{dx} = f(x)$, then y is the function whose derivative is $f(x)$ and is called the *anti-derivative* of $f(x)$ or the *indefinite integral* of $f(x)$, denoted by $\int f(x)\,dx$. Similarly if $y = \int f(u)\,du$, then $\dfrac{dy}{du} = f(u)$. Since the derivative of a constant is zero, all indefinite integrals differ by an arbitrary constant.

For the definition of a definite integral, see page 94. The process of finding an integral is called *integration*.

GENERAL RULES OF INTEGRATION

In the following, u, v, w are functions of x; a, b, p, q, n any constants, restricted if indicated; $e = 2.71828\ldots$ is the natural base of logarithms; $\ln u$ denotes the natural logarithm of u where it is assumed that $u > 0$ [in general, to extend formulas to cases where $u < 0$ as well, replace $\ln u$ by $\ln|u|$]; all angles are in radians; all constants of integration are omitted but implied.

14.1 $$\int a\,dx \;=\; ax$$

14.2 $$\int af(x)\,dx \;=\; a\int f(x)\,dx$$

14.3 $$\int (u \pm v \pm w \pm \cdots)\,dx \;=\; \int u\,dx \pm \int v\,dx \pm \int w\,dx \pm \cdots$$

14.4 $$\int u\,dv \;=\; uv - \int v\,du \qquad \text{[Integration by parts]}$$

For generalized integration by parts, see 14.48.

14.5 $$\int f(ax)\,dx \;=\; \frac{1}{a}\int f(u)\,du$$

14.6 $$\int F\{f(x)\}\,dx \;=\; \int F(u)\frac{dx}{du}\,du \;=\; \int \frac{F(u)}{f'(x)}\,du \quad \text{where } u = f(x)$$

14.7 $$\int u^n\,du \;=\; \frac{u^{n+1}}{n+1}, \quad n \neq -1 \qquad \text{[For } n = -1\text{, see 14.8]}$$

14.8 $$\int \frac{du}{u} \;=\; \ln u \ \text{ if } u > 0 \text{ or } \ln(-u) \text{ if } u < 0$$
$$=\; \ln|u|$$

14.9 $$\int e^u\,du \;=\; e^u$$

14.10 $$\int a^u\,du \;=\; \int e^{u\ln a}\,du \;=\; \frac{e^{u\ln a}}{\ln a} \;=\; \frac{a^u}{\ln a}, \quad a > 0,\ a \neq 1$$

14.11 $\int \sin u\,du = -\cos u$

14.12 $\int \cos u\,du = \sin u$

14.13 $\int \tan u\,du = \ln \sec u = -\ln \cos u$

14.14 $\int \cot u\,du = \ln \sin u$

14.15 $\int \sec u\,du = \ln(\sec u + \tan u) = \ln \tan\left(\frac{u}{2} + \frac{\pi}{4}\right)$

14.16 $\int \csc u\,du = \ln(\csc u - \cot u) = \ln \tan\frac{u}{2}$

14.17 $\int \sec^2 u\,du = \tan u$

14.18 $\int \csc^2 u\,du = -\cot u$

14.19 $\int \tan^2 u\,du = \tan u - u$

14.20 $\int \cot^2 u\,du = -\cot u - u$

14.21 $\int \sin^2 u\,du = \frac{u}{2} - \frac{\sin 2u}{4} = \frac{1}{2}(u - \sin u \cos u)$

14.22 $\int \cos^2 u\,du = \frac{u}{2} + \frac{\sin 2u}{4} = \frac{1}{2}(u + \sin u \cos u)$

14.23 $\int \sec u \tan u\,du = \sec u$

14.24 $\int \csc u \cot u\,du = -\csc u$

14.25 $\int \sinh u\,du = \cosh u$

14.26 $\int \cosh u\,du = \sinh u$

14.27 $\int \tanh u\,du = \ln \cosh u$

14.28 $\int \coth u\,du = \ln \sinh u$

14.29 $\int \operatorname{sech} u\,du = \sin^{-1}(\tanh u)$ or $2\tan^{-1} e^u$

14.30 $\int \operatorname{csch} u\,du = \ln \tanh\frac{u}{2}$ or $-\coth^{-1} e^u$

14.31 $\int \operatorname{sech}^2 u\,du = \tanh u$

14.32 $\int \operatorname{csch}^2 u\,du = -\coth u$

14.33 $\int \tanh^2 u\,du = u - \tanh u$

14.34 $\int \coth^2 u\,du = u - \coth u$

14.35 $\int \sinh^2 u\,du = \frac{\sinh 2u}{4} - \frac{u}{2} = \frac{1}{2}(\sinh u \cosh u - u)$

14.36 $\int \cosh^2 u\,du = \frac{\sinh 2u}{4} + \frac{u}{2} = \frac{1}{2}(\sinh u \cosh u + u)$

14.37 $\int \operatorname{sech} u \tanh u\,du = -\operatorname{sech} u$

14.38 $\int \operatorname{csch} u \coth u\,du = -\operatorname{csch} u$

14.39 $\int \frac{du}{u^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{u}{a}$

14.40 $\int \frac{du}{u^2 - a^2} = \frac{1}{2a}\ln\left(\frac{u-a}{u+a}\right) = -\frac{1}{a}\coth^{-1}\frac{u}{a} \qquad u^2 > a^2$

14.41 $\int \frac{du}{a^2 - u^2} = \frac{1}{2a}\ln\left(\frac{a+u}{a-u}\right) = \frac{1}{a}\tanh^{-1}\frac{u}{a} \qquad u^2 < a^2$

14.42 $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a}$

14.43 $\int \frac{du}{\sqrt{u^2 + a^2}} = \ln(u + \sqrt{u^2 + a^2})$ or $\sinh^{-1}\frac{u}{a}$

14.44 $\int \frac{du}{\sqrt{u^2 - a^2}} = \ln(u + \sqrt{u^2 - a^2})$

14.45 $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right|$

14.46 $\int \frac{du}{u\sqrt{u^2 + a^2}} = -\frac{1}{a}\ln\left(\frac{a + \sqrt{u^2 + a^2}}{u}\right)$

14.47 $\int \frac{du}{u\sqrt{a^2 - u^2}} = -\frac{1}{a}\ln\left(\frac{a + \sqrt{a^2 - u^2}}{u}\right)$

14.48 $\int f^{(n)} g\,dx = f^{(n-1)} g - f^{(n-2)} g' + f^{(n-3)} g'' - \cdots (-1)^n \int f g^{(n)}\,dx$

This is called *generalized integration* by parts.

IMPORTANT TRANSFORMATIONS

Often in practice an integral can be simplified by using an appropriate transformation or substitution and formula 14.6, page 57. The following list gives some transformations and their effects.

14.49 $\int F(ax + b)\,dx = \frac{1}{a}\int F(u)\,du$ where $u = ax + b$

14.50 $\int F(\sqrt{ax + b})\,dx = \frac{2}{a}\int u\,F(u)\,du$ where $u = \sqrt{ax + b}$

14.51 $\int F(\sqrt[n]{ax + b})\,dx = \frac{n}{a}\int u^{n-1} F(u)\,du$ where $u = \sqrt[n]{ax + b}$

14.52 $\int F(\sqrt{a^2 - x^2})\,dx = a\int F(a\cos u)\cos u\,du$ where $x = a\sin u$

14.53 $\int F(\sqrt{x^2 + a^2})\,dx = a\int F(a\sec u)\sec^2 u\,du$ where $x = a\tan u$

14.335 $\displaystyle\int \frac{x^{p-1}\,dx}{x^{2m}+a^{2m}} = \frac{1}{ma^{2m-p}}\sum_{k=1}^{m} \sin\frac{(2k-1)p\pi}{2m}\tan^{-1}\left(\frac{x + a\cos[(2k-1)\pi/2m]}{a\sin[(2k-1)\pi/2m]}\right)$

$\displaystyle\qquad - \frac{1}{2ma^{2m-p}}\sum_{k=1}^{m}\cos\frac{(2k-1)p\pi}{2m}\ln\left(x^2 + 2ax\cos\frac{(2k-1)\pi}{2m} + a^2\right)$

where $0 < p \leqq 2m$.

14.336 $\displaystyle\int \frac{x^{p-1}\,dx}{x^{2m}-a^{2m}} = \frac{1}{2ma^{2m-p}}\sum_{k=1}^{m-1}\cos\frac{kp\pi}{m}\ln\left(x^2 - 2ax\cos\frac{k\pi}{m} + a^2\right)$

$\displaystyle\qquad - \frac{1}{ma^{2m-p}}\sum_{k=1}^{m-1}\sin\frac{kp\pi}{m}\tan^{-1}\left(\frac{x - a\cos(k\pi/m)}{a\sin(k\pi/m)}\right)$

$\displaystyle\qquad + \frac{1}{2ma^{2m-p}}\{\ln(x-a) + (-1)^p\ln(x+a)\}$

where $0 < p \leqq 2m$.

14.337 $\displaystyle\int \frac{x^{p-1}\,dx}{x^{2m+1}+a^{2m+1}}$

$\displaystyle\qquad = \frac{2(-1)^{p-1}}{(2m+1)a^{2m-p+1}}\sum_{k=1}^{m}\sin\frac{2kp\pi}{2m+1}\tan^{-1}\left(\frac{x + a\cos[2k\pi/(2m+1)]}{a\sin[2k\pi/(2m+1)]}\right)$

$\displaystyle\qquad - \frac{(-1)^{p-1}}{(2m+1)a^{2m-p+1}}\sum_{k=1}^{m}\cos\frac{2kp\pi}{2m+1}\ln\left(x^2 + 2ax\cos\frac{2k\pi}{2m+1} + a^2\right)$

$\displaystyle\qquad + \frac{(-1)^{p-1}\ln(x+a)}{(2m+1)a^{2m-p+1}}$

where $0 < p \leqq 2m+1$.

14.338 $\displaystyle\int \frac{x^{p-1}\,dx}{x^{2m+1}-a^{2m+1}}$

$\displaystyle\qquad = \frac{-2}{(2m+1)a^{2m-p+1}}\sum_{k=1}^{m}\sin\frac{2kp\pi}{2m+1}\tan^{-1}\left(\frac{x - a\cos[2k\pi/(2m+1)]}{a\sin[2k\pi/(2m+1)]}\right)$

$\displaystyle\qquad + \frac{1}{(2m+1)a^{2m-p+1}}\sum_{k=1}^{m}\cos\frac{2kp\pi}{2m+1}\ln\left(x^2 - 2ax\cos\frac{2k\pi}{2m+1} + a^2\right)$

$\displaystyle\qquad + \frac{\ln(x-a)}{(2m+1)a^{2m-p+1}}$

where $0 < p \leqq 2m+1$.

INTEGRALS INVOLVING $\sin ax$

14.339 $\displaystyle\int \sin ax\,dx = -\frac{\cos ax}{a}$

14.340 $\displaystyle\int x\sin ax\,dx = \frac{\sin ax}{a^2} - \frac{x\cos ax}{a}$

14.341 $\displaystyle\int x^2\sin ax\,dx = \frac{2x}{a^2}\sin ax + \left(\frac{2}{a^3} - \frac{x^2}{a}\right)\cos ax$

14.342 $\displaystyle\int x^3\sin ax\,dx = \left(\frac{3x^2}{a^2} - \frac{6}{a^4}\right)\sin ax + \left(\frac{6x}{a^3} - \frac{x^3}{a}\right)\cos ax$

14.343 $\displaystyle\int \frac{\sin ax}{x}\,dx = ax - \frac{(ax)^3}{3\cdot 3!} + \frac{(ax)^5}{5\cdot 5!} - \cdots$

14.344 $\displaystyle\int \frac{\sin ax}{x^2}\,dx = -\frac{\sin ax}{x} + a\int\frac{\cos ax}{x}\,dx$ [see 14.373]

14.345 $\displaystyle\int \frac{dx}{\sin ax} = \frac{1}{a}\ln(\csc ax - \cot ax) = \frac{1}{a}\ln\tan\frac{ax}{2}$

14.346 $\displaystyle\int \frac{x\,dx}{\sin ax} = \frac{1}{a^2}\left\{ax + \frac{(ax)^3}{18} + \frac{7(ax)^5}{1800} + \cdots + \frac{2(2^{2n-1}-1)B_n(ax)^{2n+1}}{(2n+1)!} + \cdots\right\}$

14.347 $\displaystyle\int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$

14.348 $\displaystyle\int x\sin^2 ax\,dx = \frac{x^2}{4} - \frac{x\sin 2ax}{4a} - \frac{\cos 2ax}{8a^2}$

14.349 $\displaystyle\int \sin^3 ax\,dx = -\frac{\cos ax}{a} + \frac{\cos^3 ax}{3a}$

14.350 $\displaystyle\int \sin^4 ax\,dx = \frac{3x}{8} - \frac{\sin 2ax}{4a} + \frac{\sin 4ax}{32a}$

14.351 $\displaystyle\int \frac{dx}{\sin^2 ax} = -\frac{1}{a}\cot ax$

14.352 $\displaystyle\int \frac{dx}{\sin^3 ax} = -\frac{\cos ax}{2a\sin^2 ax} + \frac{1}{2a}\ln\tan\frac{ax}{2}$

14.353 $\displaystyle\int \sin px\sin qx\,dx = \frac{\sin(p-q)x}{2(p-q)} - \frac{\sin(p+q)x}{2(p+q)}$ [If $p = \pm q$, see 14.368.]

14.354 $\displaystyle\int \frac{dx}{1-\sin ax} = \frac{1}{a}\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$

14.355 $\displaystyle\int \frac{x\,dx}{1-\sin ax} = \frac{x}{a}\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right) + \frac{2}{a^2}\ln\sin\left(\frac{\pi}{4} - \frac{ax}{2}\right)$

14.356 $\displaystyle\int \frac{dx}{1+\sin ax} = -\frac{1}{a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right)$

14.357 $\displaystyle\int \frac{x\,dx}{1+\sin ax} = -\frac{x}{a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) + \frac{2}{a^2}\ln\sin\left(\frac{\pi}{4} + \frac{ax}{2}\right)$

14.358 $\displaystyle\int \frac{dx}{(1-\sin ax)^2} = \frac{1}{2a}\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right) + \frac{1}{6a}\tan^3\left(\frac{\pi}{4} + \frac{ax}{2}\right)$

14.359 $\displaystyle\int \frac{dx}{(1+\sin ax)^2} = -\frac{1}{2a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) - \frac{1}{6a}\tan^3\left(\frac{\pi}{4} - \frac{ax}{2}\right)$

14.360 $\displaystyle\int \frac{dx}{p+q\sin ax} = \begin{cases} \dfrac{2}{a\sqrt{p^2-q^2}}\tan^{-1}\dfrac{p\tan\frac{1}{2}ax + q}{\sqrt{p^2-q^2}} \\[2ex] \dfrac{1}{a\sqrt{q^2-p^2}}\ln\left(\dfrac{p\tan\frac{1}{2}ax + q - \sqrt{q^2-p^2}}{p\tan\frac{1}{2}ax + q + \sqrt{q^2-p^2}}\right) \end{cases}$

If $p = \pm q$ see 14.354 and 14.356.

14.361 $\displaystyle\int \frac{dx}{(p+q\sin ax)^2} = \frac{q\cos ax}{a(p^2-q^2)(p+q\sin ax)} + \frac{p}{p^2-q^2}\int\frac{dx}{p+q\sin ax}$

If $p = \pm q$ see 14.358 and 14.359.

14.362 $\displaystyle\int \frac{dx}{p^2+q^2\sin^2 ax} = \frac{1}{ap\sqrt{p^2+q^2}}\tan^{-1}\frac{\sqrt{p^2+q^2}\tan ax}{p}$

14.363 $\displaystyle\int \frac{dx}{p^2-q^2\sin^2 ax} = \begin{cases} \dfrac{1}{ap\sqrt{p^2-q^2}}\tan^{-1}\dfrac{\sqrt{p^2-q^2}\tan ax}{p} \\[2ex] \dfrac{1}{2ap\sqrt{q^2-p^2}}\ln\left(\dfrac{\sqrt{q^2-p^2}\tan ax + p}{\sqrt{q^2-p^2}\tan ax - p}\right) \end{cases}$

14.364 $\displaystyle\int x^m\sin ax\,dx = -\frac{x^m\cos ax}{a} + \frac{mx^{m-1}\sin ax}{a^2} - \frac{m(m-1)}{a^2}\int x^{m-2}\sin ax\,dx$

14.365 $\displaystyle\int \frac{\sin ax}{x^n}\,dx = -\frac{\sin ax}{(n-1)x^{n-1}} + \frac{a}{n-1}\int\frac{\cos ax}{x^{n-1}}\,dx$ [see 14.395]

14.366 $\displaystyle\int \sin^n ax\,dx = -\frac{\sin^{n-1}ax\cos ax}{an} + \frac{n-1}{n}\int\sin^{n-2}ax\,dx$

14.367 $\displaystyle\int \frac{dx}{\sin^n ax} = \frac{-\cos ax}{a(n-1)\sin^{n-1}ax} + \frac{n-2}{n-1}\int\frac{dx}{\sin^{n-2}ax}$

14.368 $\displaystyle\int \frac{x\,dx}{\sin^n ax} = \frac{-x\cos ax}{a(n-1)\sin^{n-1}ax} - \frac{1}{a^2(n-1)(n-2)\sin^{n-2}ax} + \frac{n-2}{n-1}\int\frac{x\,dx}{\sin^{n-2}ax}$

INTEGRALS INVOLVING cos ax

14.369 $$\int \cos ax\,dx = \frac{\sin ax}{a}$$

14.370 $$\int x\cos ax\,dx = \frac{\cos ax}{a^2} + \frac{x\sin ax}{a}$$

14.371 $$\int x^2\cos ax\,dx = \frac{2x}{a^2}\cos ax + \left(\frac{x^2}{a} - \frac{2}{a^3}\right)\sin ax$$

14.372 $$\int x^3\cos ax\,dx = \left(\frac{3x^2}{a^2} - \frac{6}{a^4}\right)\cos ax + \left(\frac{x^3}{a} - \frac{6x}{a^3}\right)\sin ax$$

14.373 $$\int \frac{\cos ax}{x}\,dx = \ln x - \frac{(ax)^2}{2\cdot 2!} + \frac{(ax)^4}{4\cdot 4!} - \frac{(ax)^6}{6\cdot 6!} + \cdots$$

14.374 $$\int \frac{\cos ax}{x^2}\,dx = -\frac{\cos ax}{x} - a\int\frac{\sin ax}{x}\,dx$$ [See 14.343]

14.375 $$\int \frac{dx}{\cos ax} = \frac{1}{a}\ln(\sec ax + \tan ax) = \frac{1}{a}\ln\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$$

14.376 $$\int \frac{x\,dx}{\cos ax} = \frac{1}{a^2}\left\{\frac{(ax)^2}{2} + \frac{(ax)^4}{8} + \frac{5(ax)^6}{144} + \cdots + \frac{E_n(ax)^{2n+2}}{(2n+2)(2n)!} + \cdots\right\}$$

14.377 $$\int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

14.378 $$\int x\cos^2 ax\,dx = \frac{x^2}{4} + \frac{x\sin 2ax}{4a} + \frac{\cos 2ax}{8a^2}$$

14.379 $$\int \cos^3 ax\,dx = \frac{\sin ax}{a} - \frac{\sin^3 ax}{3a}$$

14.380 $$\int \cos^4 ax\,dx = \frac{3x}{8} + \frac{\sin 2ax}{4a} + \frac{\sin 4ax}{32a}$$

14.381 $$\int \frac{dx}{\cos^2 ax} = \frac{\tan ax}{a}$$

14.382 $$\int \frac{dx}{\cos^3 ax} = \frac{\sin ax}{2a\cos^2 ax} + \frac{1}{2a}\ln\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$$

14.383 $$\int \cos ax\cos px\,dx = \frac{\sin(a-p)x}{2(a-p)} + \frac{\sin(a+p)x}{2(a+p)}$$ [If $a = \pm p$, see 14.377.]

14.384 $$\int \frac{dx}{1-\cos ax} = -\frac{1}{a}\cot\frac{ax}{2}$$

14.385 $$\int \frac{x\,dx}{1-\cos ax} = -\frac{x}{a}\cot\frac{ax}{2} + \frac{2}{a^2}\ln\sin\frac{ax}{2}$$

14.386 $$\int \frac{dx}{1+\cos ax} = \frac{1}{a}\tan\frac{ax}{2}$$

14.387 $$\int \frac{x\,dx}{1+\cos ax} = \frac{x}{a}\tan\frac{ax}{2} + \frac{2}{a^2}\ln\cos\frac{ax}{2}$$

14.388 $$\int \frac{dx}{(1-\cos ax)^2} = -\frac{1}{2a}\cot\frac{ax}{2} - \frac{1}{6a}\cot^3\frac{ax}{2}$$

14.389 $$\int \frac{dx}{(1+\cos ax)^2} = \frac{1}{2a}\tan\frac{ax}{2} + \frac{1}{6a}\tan^3\frac{ax}{2}$$

14.390 $$\int \frac{dx}{p+q\cos ax} = \begin{cases} \dfrac{2}{a\sqrt{p^2-q^2}}\tan^{-1}\sqrt{(p-q)/(p+q)}\,\tan\tfrac{1}{2}ax \\[2ex] \dfrac{1}{a\sqrt{q^2-p^2}}\ln\left(\dfrac{\tan\frac{1}{2}ax + \sqrt{(q+p)/(q-p)}}{\tan\frac{1}{2}ax - \sqrt{(q+p)/(q-p)}}\right) \end{cases}$$ [If $p = \pm q$ see 14.384 and 14.386.]

14.391 $$\int \frac{dx}{(p+q\cos ax)^2} = \frac{q\sin ax}{a(q^2-p^2)(p+q\cos ax)} - \frac{p}{q^2-p^2}\int\frac{dx}{p+q\cos ax}$$ [If $p = \pm q$ see 14.388 and 14.389.]

14.392 $$\int \frac{dx}{p^2+q^2\cos^2 ax} = \frac{1}{ap\sqrt{p^2+q^2}}\tan^{-1}\frac{p\tan ax}{\sqrt{p^2+q^2}}$$

14.393 $$\int \frac{dx}{p^2-q^2\cos^2 ax} = \begin{cases} \dfrac{1}{ap\sqrt{p^2-q^2}}\tan^{-1}\dfrac{p\tan ax}{\sqrt{p^2-q^2}} \\[2ex] \dfrac{1}{2ap\sqrt{q^2-p^2}}\ln\left(\dfrac{p\tan ax - \sqrt{q^2-p^2}}{p\tan ax + \sqrt{q^2-p^2}}\right) \end{cases}$$

14.394 $$\int x^m\cos ax\,dx = \frac{x^m\sin ax}{a} + \frac{mx^{m-1}}{a^2}\cos ax - \frac{m(m-1)}{a^2}\int x^{m-2}\cos ax\,dx$$

14.395 $$\int \frac{\cos ax}{x^n}\,dx = -\frac{\cos ax}{(n-1)x^{n-1}} - \frac{a}{n-1}\int\frac{\sin ax}{x^{n-1}}\,dx$$ [See 14.365]

14.396 $$\int \cos^n ax\,dx = \frac{\sin ax\cos^{n-1} ax}{an} + \frac{n-1}{n}\int\cos^{n-2} ax\,dx$$

14.397 $$\int \frac{dx}{\cos^n ax} = \frac{\sin ax}{a(n-1)\cos^{n-1} ax} + \frac{n-2}{n-1}\int\frac{dx}{\cos^{n-2} ax}$$

14.398 $$\int \frac{x\,dx}{\cos^n ax} = \frac{x\sin ax}{a(n-1)\cos^{n-1} ax} - \frac{1}{a^2(n-1)(n-2)\cos^{n-2} ax} + \frac{n-2}{n-1}\int\frac{x\,dx}{\cos^{n-2} ax}$$

INTEGRALS INVOLVING sin ax AND cos ax

14.399 $$\int \sin ax\cos ax\,dx = \frac{\sin^2 ax}{2a}$$

14.400 $$\int \sin px\cos qx\,dx = -\frac{\cos(p-q)x}{2(p-q)} - \frac{\cos(p+q)x}{2(p+q)}$$

14.401 $$\int \sin^n ax\cos ax\,dx = \frac{\sin^{n+1} ax}{(n+1)a}$$ [If $n = -1$, see 14.440.]

14.402 $$\int \cos^n ax\sin ax\,dx = -\frac{\cos^{n+1} ax}{(n+1)a}$$ [If $n = -1$, see 14.429.]

14.403 $$\int \sin^2 ax\cos^2 ax\,dx = \frac{x}{8} - \frac{\sin 4ax}{32a}$$

14.404 $$\int \frac{dx}{\sin ax\cos ax} = \frac{1}{a}\ln\tan ax$$

14.405 $$\int \frac{dx}{\sin^2 ax\cos ax} = \frac{1}{a}\ln\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right) - \frac{1}{a\sin ax}$$

14.406 $$\int \frac{dx}{\sin ax\cos^2 ax} = \frac{1}{a}\ln\tan\frac{ax}{2} + \frac{1}{a\cos ax}$$

14.407 $$\int \frac{dx}{\sin^2 ax\cos^2 ax} = -\frac{2\cot 2ax}{a}$$

14.408 $$\int \frac{\sin^2 ax}{\cos ax}\,dx = -\frac{\sin ax}{a} + \frac{1}{a}\ln\tan\left(\frac{ax}{2}+\frac{\pi}{4}\right)$$

14.409 $$\int \frac{\cos^2 ax}{\sin ax}\,dx = \frac{\cos ax}{a} + \frac{1}{a}\ln\tan\frac{ax}{2}$$

14.410 $$\int \frac{dx}{\cos ax(1 \pm \sin ax)} = \mp\frac{1}{2a(1 \pm \sin ax)} + \frac{1}{2a}\ln\tan\left(\frac{ax}{2}+\frac{\pi}{4}\right)$$

14.411 $$\int \frac{dx}{\sin ax(1 \pm \cos ax)} = \pm\frac{1}{2a(1 \pm \cos ax)} + \frac{1}{2a}\ln\tan\frac{ax}{2}$$

14.412 $$\int \frac{dx}{\sin ax \pm \cos ax} = \frac{1}{a\sqrt{2}}\ln\tan\left(\frac{ax}{2} \pm \frac{\pi}{8}\right)$$

14.413 $$\int \frac{\sin ax\,dx}{\sin ax \pm \cos ax} = \frac{x}{2} \mp \frac{1}{2a}\ln(\sin ax \pm \cos ax)$$

14.414 $$\int \frac{\cos ax\,dx}{\sin ax \pm \cos ax} = \pm\frac{x}{2} + \frac{1}{2a}\ln(\sin ax \pm \cos ax)$$

14.415 $$\int \frac{\sin ax\,dx}{p + q\cos ax} = -\frac{1}{aq}\ln(p + q\cos ax)$$

14.416 $$\int \frac{\cos ax\,dx}{p + q\sin ax} = \frac{1}{aq}\ln(p + q\sin ax)$$

14.417 $$\int \frac{\sin ax\,dx}{(p + q\cos ax)^n} = \frac{1}{aq(n-1)(p + q\cos ax)^{n-1}}$$

14.418 $$\int \frac{\cos ax\,dx}{(p + q\sin ax)^n} = \frac{-1}{aq(n-1)(p + q\sin ax)^{n-1}}$$

14.419 $$\int \frac{dx}{p\sin ax + q\cos ax} = \frac{1}{a\sqrt{p^2+q^2}}\ln\tan\left(\frac{ax + \tan^{-1}(q/p)}{2}\right)$$

14.420 $$\int \frac{dx}{p\sin ax + q\cos ax + r} = \begin{cases} \dfrac{2}{a\sqrt{r^2-p^2-q^2}}\tan^{-1}\left(\dfrac{p + (r-q)\tan(ax/2)}{\sqrt{r^2-p^2-q^2}}\right) \\ \dfrac{1}{a\sqrt{p^2+q^2-r^2}}\ln\left(\dfrac{p - \sqrt{p^2+q^2-r^2} + (r-q)\tan(ax/2)}{p + \sqrt{p^2+q^2-r^2} + (r-q)\tan(ax/2)}\right) \end{cases}$$

If $r = q$ see 14.421. If $r^2 = p^2 + q^2$ see 14.422.

14.421 $$\int \frac{dx}{p\sin ax + q(1 + \cos ax)} = \frac{1}{ap}\ln\left(q + p\tan\frac{ax}{2}\right)$$

14.422 $$\int \frac{dx}{p\sin ax + q\cos ax \pm \sqrt{p^2+q^2}} = \frac{-1}{a\sqrt{p^2+q^2}}\tan\left(\frac{\pi}{4} \mp \frac{ax + \tan^{-1}(q/p)}{2}\right)$$

14.423 $$\int \frac{dx}{p^2\sin^2 ax + q^2\cos^2 ax} = \frac{1}{apq}\tan^{-1}\left(\frac{p\tan ax}{q}\right)$$

14.424 $$\int \frac{dx}{p^2\sin^2 ax - q^2\cos^2 ax} = \frac{1}{2apq}\ln\left(\frac{p\tan ax - q}{p\tan ax + q}\right)$$

14.425 $$\int \sin^m ax\cos^n ax\,dx = \begin{cases} -\dfrac{\sin^{m-1} ax\cos^{n+1} ax}{a(m+n)} + \dfrac{m-1}{m+n}\displaystyle\int \sin^{m-2} ax\cos^n ax\,dx \\ \dfrac{\sin^{m+1} ax\cos^{n-1} ax}{a(m+n)} + \dfrac{n-1}{m+n}\displaystyle\int \sin^m ax\cos^{n-2} ax\,dx \end{cases}$$

14.426 $$\int \frac{\sin^m ax}{\cos^n ax}\,dx = \begin{cases} \dfrac{\sin^{m-1} ax}{a(n-1)\cos^{n-1} ax} - \dfrac{m-1}{n-1}\displaystyle\int \frac{\sin^{m-2} ax}{\cos^{n-2} ax}\,dx \\ \dfrac{\sin^{m+1} ax}{a(n-1)\cos^{n-1} ax} - \dfrac{m-n+2}{n-1}\displaystyle\int \frac{\sin^m ax}{\cos^{n-2} ax}\,dx \\ \dfrac{-\sin^{m-1} ax}{a(m-n)\cos^{n-1} ax} + \dfrac{m-1}{m-n}\displaystyle\int \frac{\sin^{m-2} ax}{\cos^n ax}\,dx \end{cases}$$

14.427 $$\int \frac{\cos^m ax}{\sin^n ax}\,dx = \begin{cases} \dfrac{-\cos^{m-1} ax}{a(n-1)\sin^{n-1} ax} - \dfrac{m-1}{n-1}\displaystyle\int \frac{\cos^{m-2} ax}{\sin^{n-2} ax}\,dx \\ \dfrac{-\cos^{m+1} ax}{a(n-1)\sin^{n-1} ax} - \dfrac{m-n+2}{n-1}\displaystyle\int \frac{\cos^m ax}{\sin^{n-2} ax}\,dx \\ \dfrac{\cos^{m-1} ax}{a(m-n)\sin^{n-1} ax} + \dfrac{m-1}{m-n}\displaystyle\int \frac{\cos^{m-2} ax}{\sin^n ax}\,dx \end{cases}$$

14.428 $$\int \frac{dx}{\sin^m ax\cos^n ax} = \begin{cases} \dfrac{1}{a(n-1)\sin^{m-1} ax\cos^{n-1} ax} + \dfrac{m+n-2}{n-1}\displaystyle\int \frac{dx}{\sin^m ax\cos^{n-2} ax} \\ \dfrac{-1}{a(m-1)\sin^{m-1} ax\cos^{n-1} ax} + \dfrac{m+n-2}{m-1}\displaystyle\int \frac{dx}{\sin^{m-2} ax\cos^n ax} \end{cases}$$

INTEGRALS INVOLVING $\tan ax$

14.429 $$\int \tan ax\,dx = -\frac{1}{a}\ln\cos ax = \frac{1}{a}\ln\sec ax$$

14.430 $$\int \tan^2 ax\,dx = \frac{\tan ax}{a} - x$$

14.431 $$\int \tan^3 ax\,dx = \frac{\tan^2 ax}{2a} + \frac{1}{a}\ln\cos ax$$

14.432 $$\int \tan^n ax\sec^2 ax\,dx = \frac{\tan^{n+1} ax}{(n+1)a}$$

14.433 $$\int \frac{\sec^2 ax}{\tan ax}\,dx = \frac{1}{a}\ln\tan ax$$

14.434 $$\int \frac{dx}{\tan ax} = \frac{1}{a}\ln\sin ax$$

14.435 $$\int x\tan ax\,dx = \frac{1}{a^2}\left\{\frac{(ax)^3}{3} + \frac{(ax)^5}{15} + \frac{2(ax)^7}{105} + \cdots + \frac{2^{2n}(2^{2n}-1)B_n(ax)^{2n+1}}{(2n+1)!} + \cdots\right\}$$

14.436 $$\int \frac{\tan ax}{x}\,dx = ax + \frac{(ax)^3}{9} + \frac{2(ax)^5}{75} + \cdots + \frac{2^{2n}(2^{2n}-1)B_n(ax)^{2n-1}}{(2n-1)(2n)!} + \cdots$$

14.437 $$\int x\tan^2 ax\,dx = \frac{x\tan ax}{a} + \frac{1}{a^2}\ln\cos ax - \frac{x^2}{2}$$

14.438 $$\int \frac{dx}{p + q\tan ax} = \frac{px}{p^2+q^2} + \frac{q}{a(p^2+q^2)}\ln(q\sin ax + p\cos ax)$$

14.439 $$\int \tan^n ax\,dx = \frac{\tan^{n-1} ax}{(n-1)a} - \int \tan^{n-2} ax\,dx$$